Introduction to Quantum Error Correction

IQC 2024 Lecture 25

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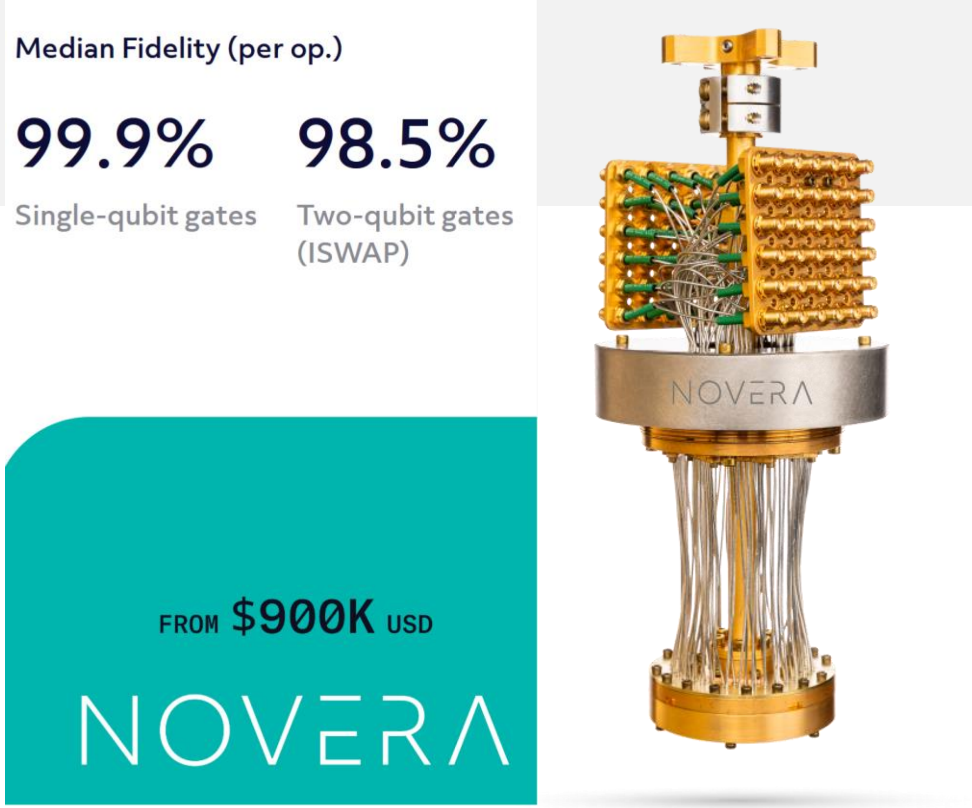
Quantum gate error rates

QuEra Aquila: 256 qubit neutral-atom quantum computer [1]

Two-qubit gate fidelity: **99.5%** [2]

[1] https://quera.com
[2] Nature **volume 622**, pages 268–272 (2023)

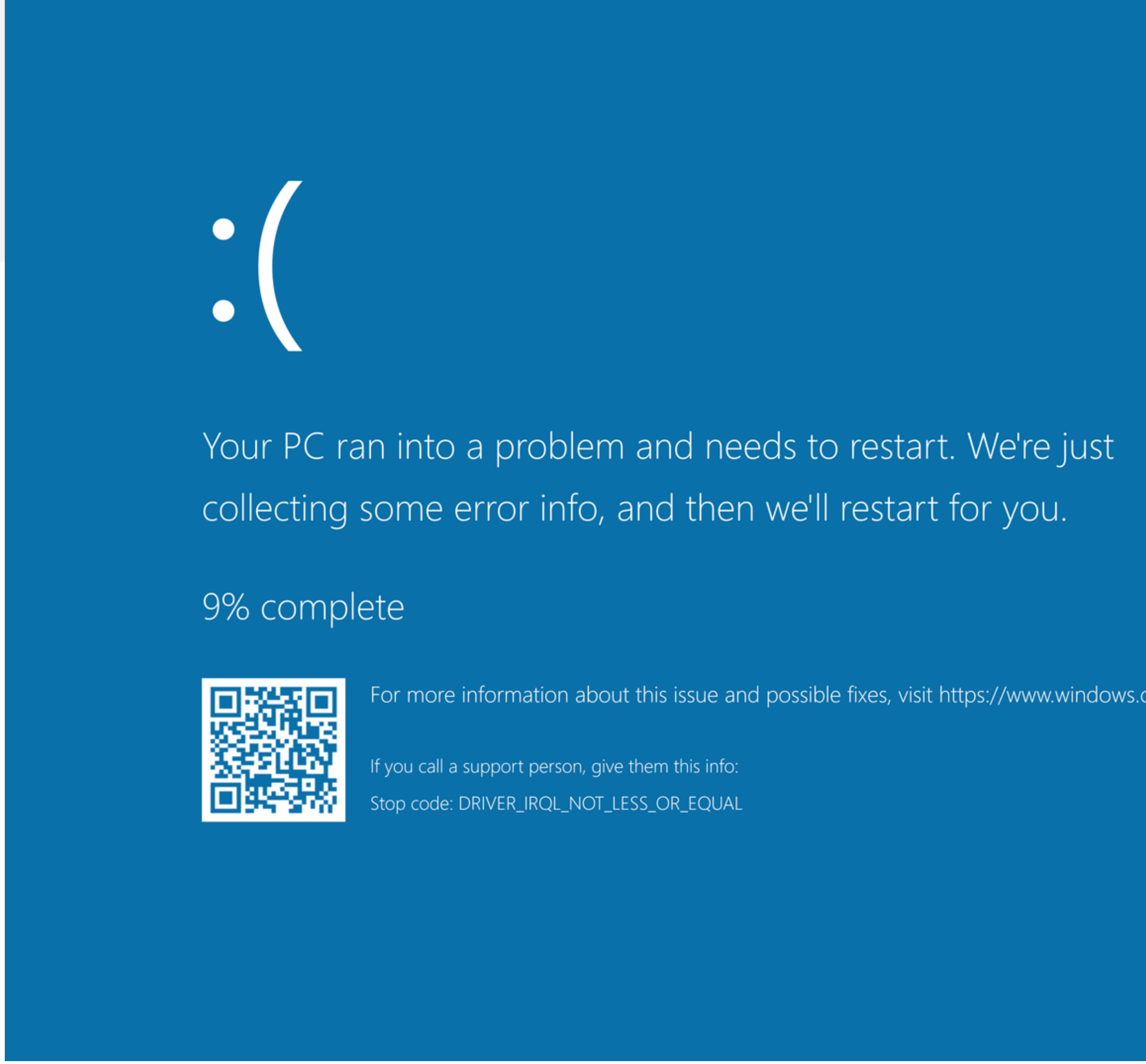


Rigetti Novera superconducting qubit quantum computer [4]

[3] https://www.rigetti.com/novera

Classical vs. Quantum Computing

- Transitor gates in classical CPUs are extremely robust.
- Failure rates $p <\approx 1 \times 10^{-15}$ [Shivakumar et al. 2002].
- Classical gates are over a **trillion** times more reliable than qubit gates!



The dreaded "Blue Screen of Death". Faults such as these are due to software errors rather than hardware in classical CPUs.

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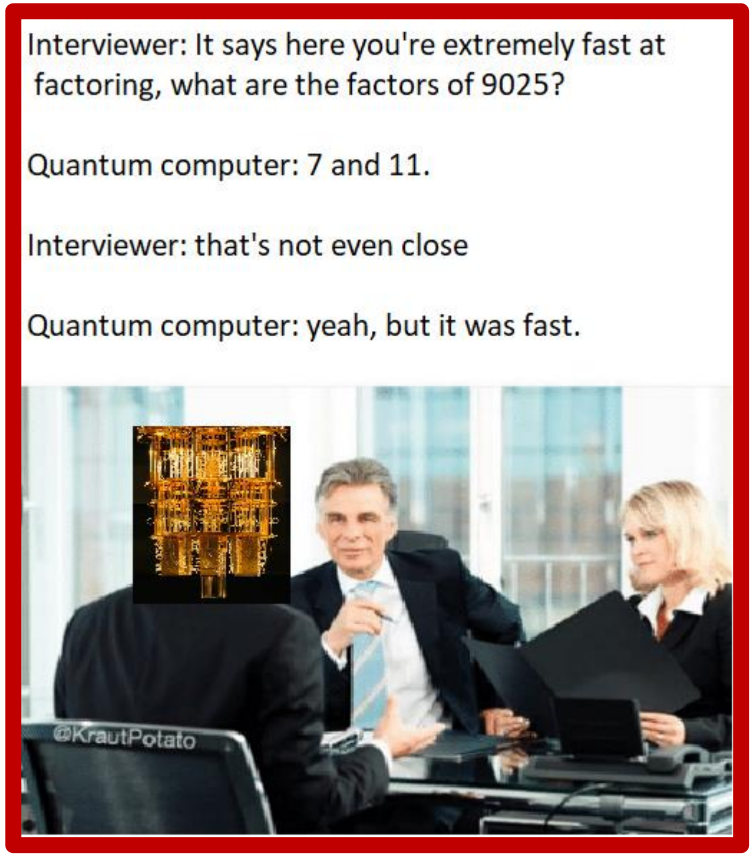
Quantum Error Correction

Quantum error correction describes a family of system-level techniques that allow quantum computers to be built **fault-tolerantly** using noisy qubits.



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Classical error correction

- Raw binary encodings have **zero** redundancy. E.g.

bin(42) ➞101010

- Applying a single bit flip to our binary encoding completely changes its meaning.

dec(100010)=34

- **How do we make our encoding more fault tolerant?**

The Classical Repetition Code

In repetition code protocols redundancy is introduced by duplicating each bit. E.g. applying the 3-bit repetition code protocol to our binary encoding gives:

101010 ➞ (111)(000)(111)(000)(111)(000)

We can now detect and correct single-bit faults through a majority vote:

(101)(100)(011)(000)(111)(010) ➞

(111)(000)(111)(000)(111)(000)

The Classical Repetition Code

3-bit repetition code. Binary symbols mapped to 3-bit codewords.

$$\{0,1\} \rightarrow \{000, 111\}$$

- Single-bit errors can be corrected via majority vote. E.g., $000 \rightarrow 010 \rightarrow 000$
- Two-bit errors can be detected, but are incorrectly corrected via majority vote. E.g. $000 \rightarrow 011 \rightarrow 111$
- Three-bit errors are **undetectable** via majority vote: $000 \rightarrow 111$

Code distance: The code distance is the minimum Hamming-weight of an undetectable error. E.g., distance d=3 for the 3-bit repetition code.

An error correction code can correct t errors, where:

$$t = \frac{d-1}{2}$$

[n,k,d] Notation

n: codeword length

k: encoded message length

d: code distance

E.g. 3-bit repetition code has parameters: $[n = 3, k = 1, d = 3]$

The Challenges of Quantum Error Correction

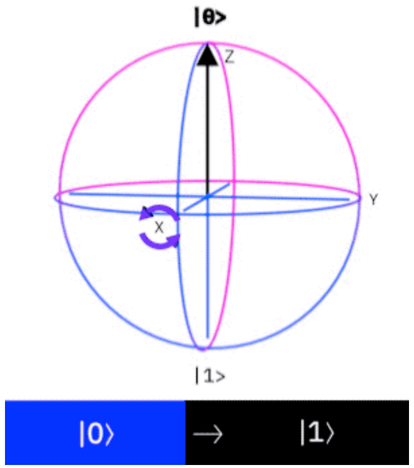
- **More complicated error channels**. In classical error correction we only need to worry about bit flips. In quantum error correction there are phase-flips too:

Bit flips: $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$

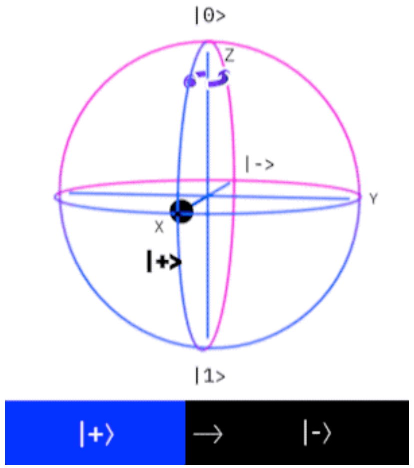
Phase flips: $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$

- **The No-Cloning Theorem**: This prevents us from arbitrarily duplicating data as we do for classical repetition codes

- **Wavefunction collapse**: How do we check for errors in a quantum state without collapsing the encoded quantum information.



Evolution on the Bloch Sphere due to **bit-flip** (X-Pauli error)



Evolution on the Bloch Sphere due to **phase-flip** (Z-Pauli error)

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The No-Cloning Theorem

Q: Can we create a quantum repetition code by duplicating states?

$$|\psi\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle.$$

A: No! This is prohibited by the No-Cloning Theorem.

Dolly would not have existed had she been a Quantum Sheep. Image source: National Museum of Scotland

No-Cloning Theorem Derivation

For cloning, we require a unitary U that duplicates quantum information as follows:

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle.$$

The cloning unitary should apply to any state:

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$
$$U(|\phi\rangle \otimes |0\rangle) = |\phi\rangle \otimes |\phi\rangle$$

Unitary operation preserve the inner product. Taking the inner product of the above gives:

$$\langle\psi|\phi\rangle = (\langle\psi|\phi\rangle)^2$$

- There are only two solutions to the above. Either:

$$|\psi\rangle = |\phi\rangle \quad \text{or} \quad \langle\psi|\phi\rangle = 0$$

- Therefore, U only exists for states that are orthogonal.
- **There is no unitary U that can clone arbitrary states.**

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The two-bit repetition code: redundancy without cloning

The two-qubit repetition code

We can circumvent the No-Cloning theorem and redundantly encode quantum information using *entanglement*.

The two-qubit repetition code maps the computational basis $\{|0\rangle, |1\rangle\}$, to the **Logical basis states:**

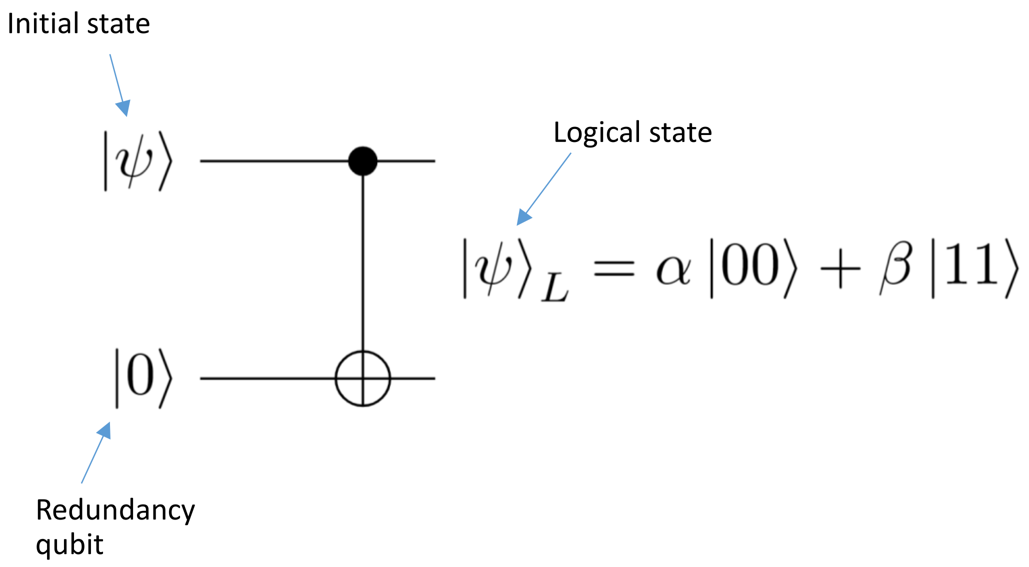
$$|0\rangle_L = |00\rangle \quad \text{and} \quad |1\rangle_L = |11\rangle$$

Example: Consider the following qubit state.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The two-qubit repetition encoder creates the following **state**:

$$|\psi\rangle_L = \alpha|00\rangle + \beta|11\rangle$$



Note. This is not cloning:

$$|\psi\rangle_L \neq |\psi\rangle \otimes |\psi\rangle$$

The two-bit repetition code: partitioning the Hilbert space

Partitioning the Hilbert Space

Prior to encoding, the initial state *exists* within a two-dimensional Hilbert space.

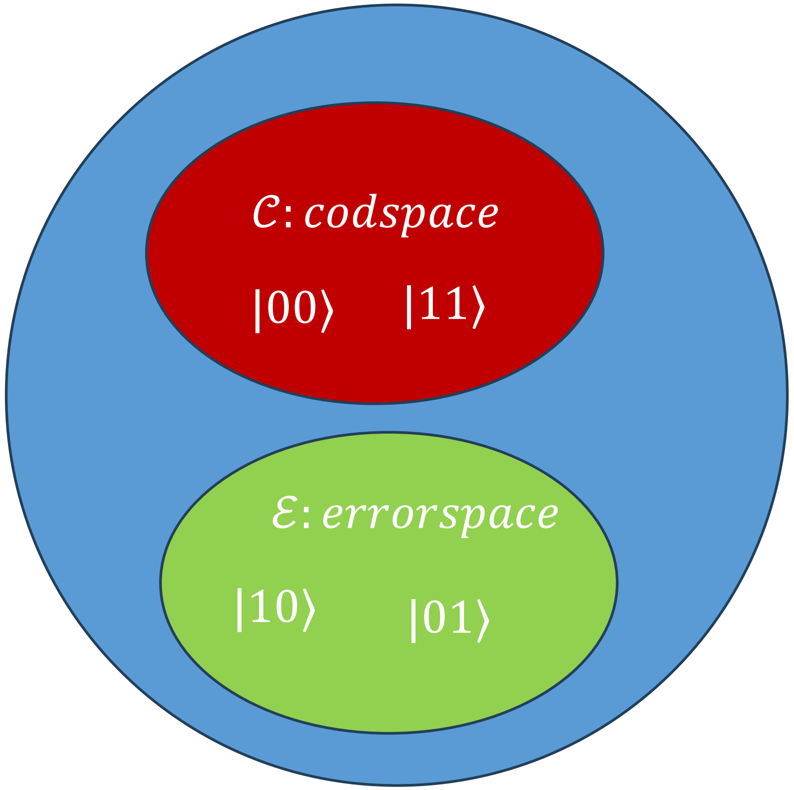
$$|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \in \mathcal{H}_2 = \text{span}(|0\rangle, |1\rangle)$$

After encoding, the logical state exists within a four-dimensional Hilbert space:

$$|\psi\rangle_L = (\alpha|00\rangle + \beta|11\rangle) \in \mathcal{H}_4 = \text{span}(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$$

We can partition $\mathcal{H}_4$ into two orthogonal subspaces

- The code-space: $\mathcal{C} = \text{span}(|00\rangle, |11\rangle)$
- The error-space: $\mathcal{E} = \text{span}(|01\rangle, |10\rangle)$



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Errors map $\mathcal{C} \rightarrow \mathcal{E}$

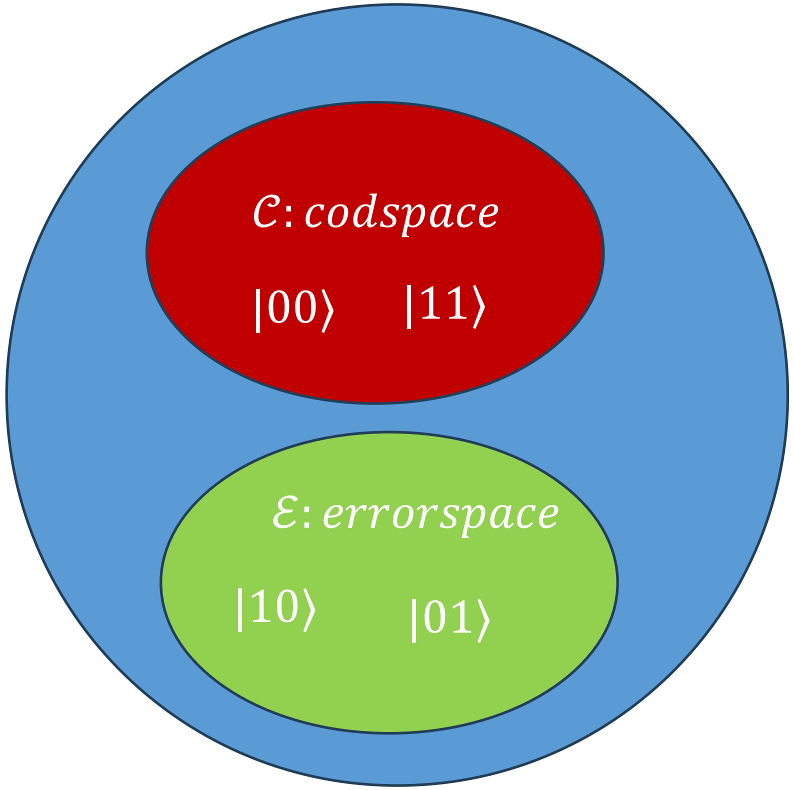
If the logical state is un-errored, it in the codespace

$$|\psi\rangle_L = (\alpha|00\rangle + \beta|11\rangle) \in \mathcal{C} \subset \mathcal{H}_4$$

If it is subject to a single-qubit Pauli-X error, the state is rotated into the error space. E.g.,

$$X_1|\psi\rangle_L = (\alpha|10\rangle + \beta|01\rangle) \in \mathcal{E} \subset \mathcal{H}_4$$

We can detect the occurrence of a single-qubit X-error by performing a measurement to determine which subspace the logical qubit is in.



Detecting errors via stabiliser measurement

The two-qubit code partitions the Hilbert space into a codespace and an errorspace:

The code-space: $\mathcal{C} = \text{span}(|00\rangle, |11\rangle)$

The error-space: $\mathcal{E} = \text{span}(|01\rangle, |10\rangle)$

We can differentiate between the codespace and the error space using a **Hadamard test** (recall Lecture 16). The projector onto the codespace is:

$$\Pi_{\mathcal{C}} = |00\rangle\langle 00| + |11\rangle\langle 11|$$

The projector on the errorspace is:

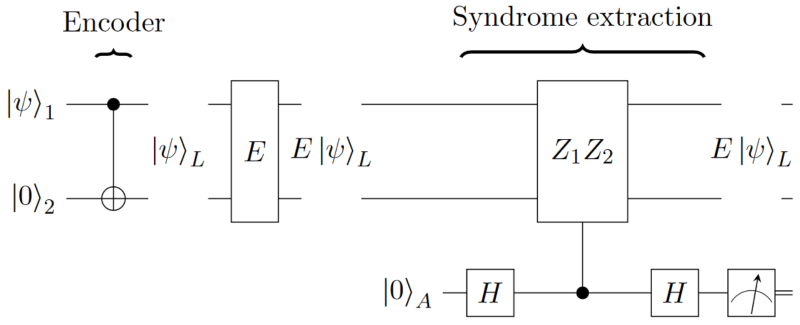
$$\Pi_{\mathcal{E}} = |01\rangle\langle 01| + |10\rangle\langle 10|$$

The following unitary operator has eigenvalues ± 1 depending upon whether it is applied to state in the codespace or the error space:

$$\Pi_S = \Pi_{\mathcal{C}} - \Pi_{\mathcal{E}} = Z_1 Z_2$$

The above operator is referred to as a **stabiliser** as it acts as the identity on the logical state:

$$Z_1 Z_2 |\psi\rangle_L = Z_1 Z_2(\alpha|00\rangle + \beta|11\rangle) = (+1)|\psi\rangle_L$$



The Hadamard test operator $Z_1 Z_1$ has ± 1 eigenvalues.

If the state is in the codespace, we measure the (+1) eigenvalue.

$$Z_1 Z_2 |\psi\rangle_L = Z_1 Z_2(\alpha|00\rangle + \beta|11\rangle) = (+1)|\psi\rangle_L$$

If the state is in the errorspace, we measure the (-1) eigenvalue.

$$Z_1 Z_2 (X_1|\psi\rangle_L) = Z_1 Z_2(\alpha|10\rangle + \beta|01\rangle) = (-1)E|\psi\rangle_L$$

This enables us to detect errors without destroying the superposition.

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Error detection in the 2-qubit code

1. **Initial circuit state**: $|\psi\rangle_1|0\rangle_2|0\rangle_A$
2. **After encoding**: $|\psi\rangle_L|0\rangle_A$
3. **Two qubit state after the Hadamard test (immediately before measurement of qubit A)**:

$$\frac{1}{2}(I+Z_1Z_2)\mathrm{E}|\psi\rangle_L|0\rangle_A+\frac{1}{2}(I-Z_1Z_2)\mathrm{E}|\psi\rangle_L|1\rangle_A$$

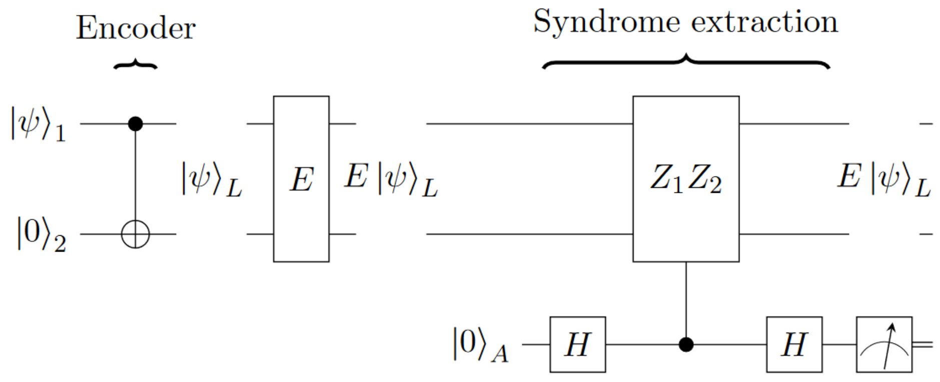
Examples

- If $E=I_1I_2$ (no error case) we measure the `0` syndrome.

$$\frac{1}{2}(I+Z_1Z_2)I_1I_2|\psi\rangle_L|0\rangle_A+\frac{1}{2}(I-Z_1Z_2)I_1I_2|\psi\rangle_L|1\rangle_A$$
$$=\frac{1}{2}(I+Z_1Z_2)I_1I_2|\psi\rangle_L|0\rangle_A=|\psi\rangle_L|0\rangle_A$$

- If $E=X_1I_2$ we measure the `1` syndrome.

$$\frac{1}{2}(I+Z_1Z_2)X_1I_2|\psi\rangle_L|0\rangle_A+\frac{1}{2}(I-Z_1Z_2)X_1I_2|\psi\rangle_L|1\rangle_A$$
$$=\frac{1}{2}(I-Z_1Z_2)X_1I_2|\psi\rangle_L|1\rangle_A=|\psi\rangle_L|1\rangle_A$$



1. **Encoder**. Maps state $|\psi\rangle$ to the logical state $|\psi\rangle_L$.
2. **Error channel:** we assume that the two-qubit state is subject to some Pauli-X error in the region marked by the gate E
3. **Stabiliser measurement and syndrome extraction.** A Hadamard test can be performed to measure the operator Z_1Z_2 and determine whether an error has occurred. The binary outcome of the measurement on auxiliary qubit A is called the *syndrome*.

Error detection in the 2-qubit code

Pauli-operator commutation: Any pair of Pauli operators P_i, P_j either commute or anti-commute with one another $[P_i, P_j] = 0$ or $\{P_i, P_j\} = 0$.

1. **After encoding**: $|\psi\rangle_L |0\rangle_A$
2. **After the Hadamard test (immediately before measurement of qubit A)**:

$$\frac{1}{2}(I + Z_1Z_2)\mathrm{E}|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - Z_1Z_2)\mathrm{E}|\psi\rangle_L|1\rangle_A$$

Assume that E is a Pauli operator.

If error commutes with stabiliser, $[E, Z_1Z_2] = 0$, then we are in the codespace and we measure the `0` syndrome.

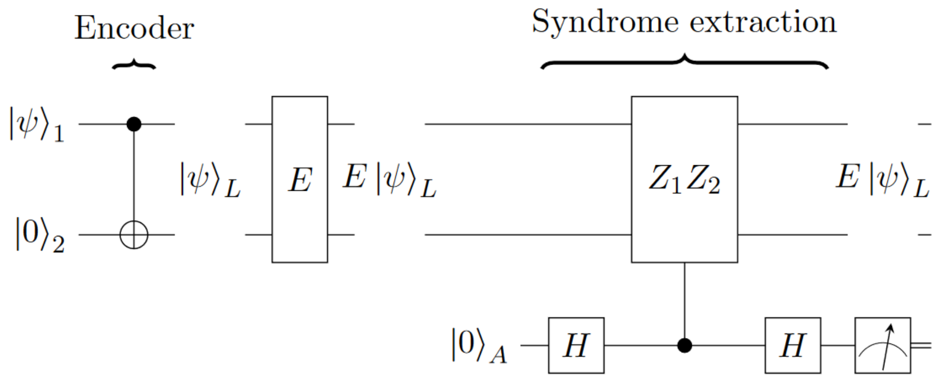
$$\frac{1}{2}(I + Z_1Z_2)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - Z_1Z_2)E|\psi\rangle_L|1\rangle_A$$

$$= \frac{1}{2}E(I + Z_1Z_2)|\psi\rangle_L|0\rangle_A + \frac{1}{2}E(I - Z_1Z_2)|\psi\rangle_L|1\rangle_A = E|\psi\rangle_L|0\rangle_A$$

If error anti-commutes with the stabiliser, $\{E, Z_1Z_2\} = 0$, then we are in the the errorspace and we measure the `1` syndrome.

$$\frac{1}{2}(I + Z_1Z_2)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - Z_1Z_2)E|\psi\rangle_L|1\rangle_A$$

$$= \frac{1}{2}E(I + Z_1Z_2)|\psi\rangle_L|0\rangle_A + \frac{1}{2}E(I - Z_1Z_2)|\psi\rangle_L|1\rangle_A = E|\psi\rangle_L|1\rangle_A$$



Pauli Error, E	Syndrome readout, A
$I_1 \otimes I_2$	0
$X_1 \otimes I_2$	1
$I_1 \otimes X_2$	1
$X_1 \otimes X_2$	0

The syndrome measurement A depends upon whether the error E commutes or anti-comutes with the stabiliser Z_1Z_2

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Pauli commutation rules

All single qubit Pauli operators X, Y, Z anti-commute with one another.

$$X_1 Z_1 = -Z_1 X_1$$
$$X_1 Y_1 = -Y_1 X_1$$
$$Z_1 Y_1 = -Y_1 Z_1$$

Multi-qubit Pauli operators anti-commute if they **non-trivially intersect** on an odd number of qubits.

An intersection is *trivial* if a Pauli operator of type $\lambda \in (X, Y, Z)$ intersects with another Pauli of type Lambda or the identity.

Example 1: $X_1 Z_2$ and $Z_1 Z_2$

- Non-trivially intersect on qubit 1: *Anti-commute*.

Example 2: $X_1 I_2$ and $I_1 Z_2$

- Do not intersect: *Commute*.

Example 3: $Z_1 X_2$ and $X_1 Z_2$

- Non-trivially intersect on qubits 1 and 2. Even number of non-trivial intersections: *Commute*

Example 4: $Z_1 I_2 Z_3 Y_4$ and $X_1 X_2 X_3 X_4$.

- Non-trivially intersect on qubits 1, 3 and 4. Odd number of intersections: *Anti-commute.*

Example 5: $Z_1 X_2 I_3$ and $Z_1 I_2 X_3$

- All intersections are trivial: *commute*.

The three-qubit repetition code

To detect **and** correct errors, we require a larger Hilbert space and multiple overlapping stabiliser measurements. E.g. the three-bit repetition code:

$$|\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$$

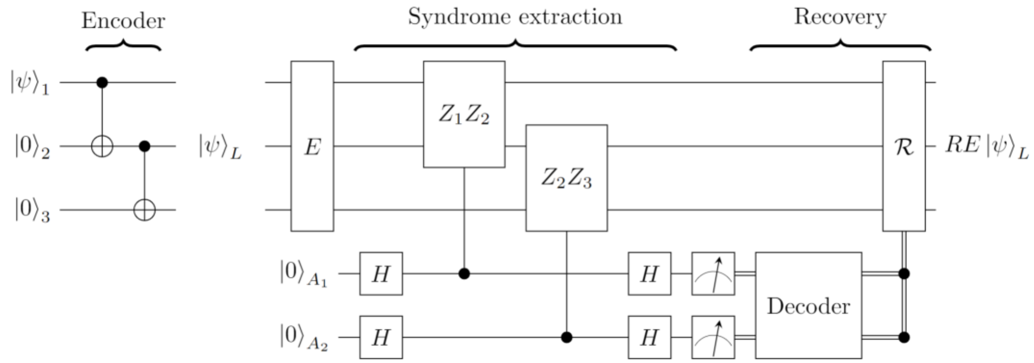
with logical basis states:

$$|0\rangle_L = |000\rangle, \qquad |1\rangle_L = |111\rangle$$

This code has two independent stabilisers: Z_1Z_2 and Z_2Z_3

$$Z_1Z_2|00\rangle_L = |00\rangle_L \quad \text{and} \quad Z_1Z_2|11\rangle_L = |11\rangle_L$$
$$Z_2Z_3|00\rangle_L = |00\rangle_L \quad \text{and} \quad Z_2Z_3|11\rangle_L = |11\rangle_L$$

Each single-qubit X-error maps to a unique syndrome. The three-qubit code is therefore a **correction** code (the two-qubit code is a detection code). Recovery operations can be applied by consulting a look-up table.



Pauli Error, E	Syndrome readout, A_1A_2	Recovery operation
$I_1 \otimes I_2 \otimes I_3$	00	$I_1 \otimes I_2 \otimes I_3$
$I_1 \otimes I_2 \otimes X_3$	01	$I_1 \otimes I_2 \otimes X_3$
$I_1 \otimes X_2 \otimes I_3$	11	$I_1 \otimes X_2 \otimes I_3$
$X_1 \otimes I_2 \otimes I_3$	10	$X_1 \otimes I_2 \otimes I_3$
$I_1 \otimes X_2 \otimes X_3$	10	$I_1 \otimes X_2 \otimes X_3$
$X_1 \otimes X_2 \otimes I_3$	01	$X_1 \otimes X_2 \otimes I_3$
$X_1 \otimes I_2 \otimes X_3$	11	$X_1 \otimes I_2 \otimes X_3$

The digitisation of the error

So far, we've only considered Pauli-X type errors. We now consider the other error types qubits can be subject to.

A general qubit state can be represented as point on a Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Coherent errors can be described a unitary that rotates the state from one point on the Bloch sphere to another:

$$U(\delta\theta,\delta\phi)|\psi\rangle = \cos\left(\frac{\theta+\delta\theta}{2}\right)|0\rangle + e^{i(\phi+\delta\phi)}sin\left(\frac{\theta+\delta\theta}{2}\right)|1\rangle$$

In classical error correction, we only need to worry about one type of error: bit-flips. For qubits, we have an infinite number of errors $U(\delta\theta,\delta\phi)$ corresponding to arbitrary rotations around the Bloch sphere.

This is problematic, as error correction is not generally possible for analogue encodings

Q: Is error correction still possible?

A: Yes!

The coherent error $U(\delta\theta,\delta\phi)$ is a unitary 2x2 matrix.

Any 2x2 matrix can be expanded in terms of a Pauli basis $\{I, X, Z, Y\}$.

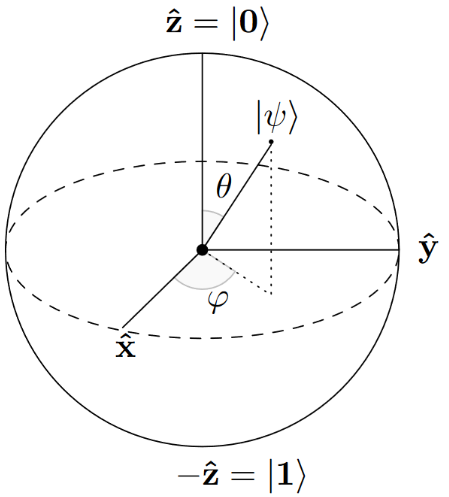
$$U(\delta\theta,\delta\phi)|\psi\rangle = \alpha_I|\psi\rangle + \alpha_X X|\psi\rangle + \alpha_Y Y|\psi\rangle + \alpha_Z Z|\psi\rangle$$

Recall that $Y \propto XZ$. The above can therefore be written.

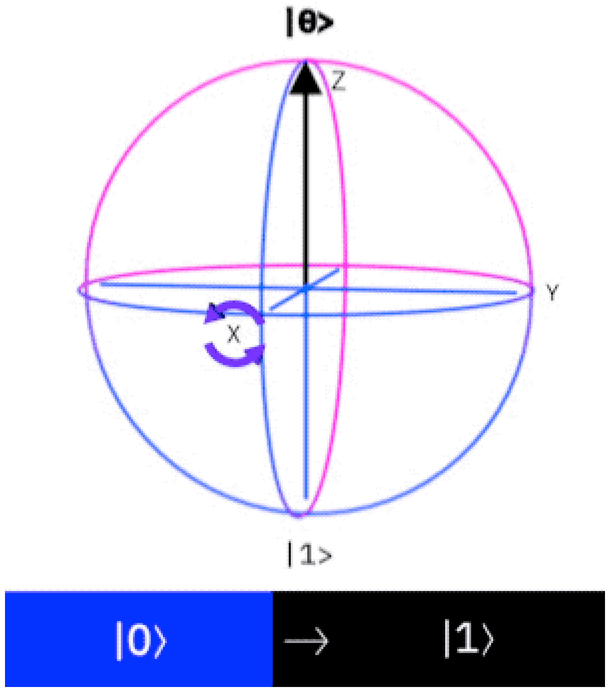
$$U(\delta\theta,\delta\phi)|\psi\rangle = \alpha_I|\psi\rangle + \alpha_X X|\psi\rangle + \alpha_{XZ} XZ|\psi\rangle + \alpha_Z Z|\psi\rangle$$

The ability to correction X-errors (bit-flips) and Z-flips (phase-flips) is sufficient to correct any coherent error.

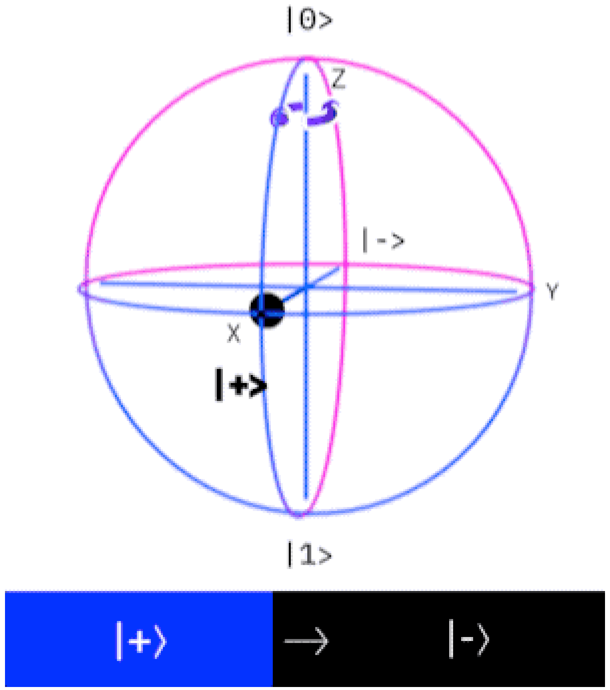
This effect is known as the **digitisation of the error.**



Bit- and Phase-Flips



Evolution on the Bloch Sphere due to **bit-flip** (X-Pauli error)



Evolution on the Bloch Sphere due to **phase-flip** (Z-Pauli error)

Logical operators and code distance

Three-qubit code:

$$|0\rangle_L = |000\rangle, \quad |1\rangle_L = |111\rangle$$

Logical X-operator X_L performs the mapping

$$X_L|0\rangle_L = |1\rangle_L \quad \text{and} \quad X_L|1\rangle_L = |0\rangle$$

For the three-qubit code:

$$X_L = X_1X_2X_3$$

Logical operators always commute with all the stabilisers. E.g., for the three-qubit code with stablisers Z_1Z_2 and Z_2Z_3

$$[X_1X_2X_3, Z_1Z_2] = 0 \quad \text{and} \quad [X_1X_2X_3, Z_2Z_3] = 0$$

$\Rightarrow$ Logical operators always yield the \`0\` syndrome. I.e., they are undetectable.

The **code distance** is the equal to the minimum Hamming-weight of a logical operator.

For the three-qubit code, the minimum-weight X-logical is $X_1X_2X_3$. The distance for X-errors is therefore $d_X = 3$.

However, we also have to consider Z-type logical operators.

Logical operators and code distance

Logical X-operator X_L performs the mapping

$$X_L|0\rangle_L = |1\rangle_L \quad \text{and} \quad X_L|1\rangle_L = |0\rangle$$

Logical Z-operator $\boldsymbol{Z_L}$ performs the mapping

$$Z_L|+\rangle_L = |-\rangle_L \quad \text{and} \quad Z_L|-\rangle_L = |+\rangle_L$$

where $|+\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L)$ and $|-\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L - |1\rangle_L)$

The Z_L **and** X_L **logical operators always anti-commute**:

$$\{X_L, Z_L\} = 0$$

For the three-qubit code

$$|+\rangle_L = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad \text{and} \quad |-\rangle_L = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$$

A choice of logical operator for Z_L is

$$Z_L = I_1 I_2 Z_3$$

$$Z_L\,|+\rangle_L = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) = |-\rangle_L$$

The Z_L operator commutes with both of the stabilisers Z_1Z_2 and Z_2Z_3. It therefore maps to the `00` syndrome.

The **distance** of the three bit-repetition code is therefore $d = 1$. It can detect X-errors, but not Z-errors.

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Detecting both X- and Z-Pauli errors

To detect both bit and phase-type errors, we require an encoding with stabilisers that anti-commute with both error types.

Example. The [4,2,2] code encodes two logical qubits in four physical qubits and has the following logical basis states

$$|00\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

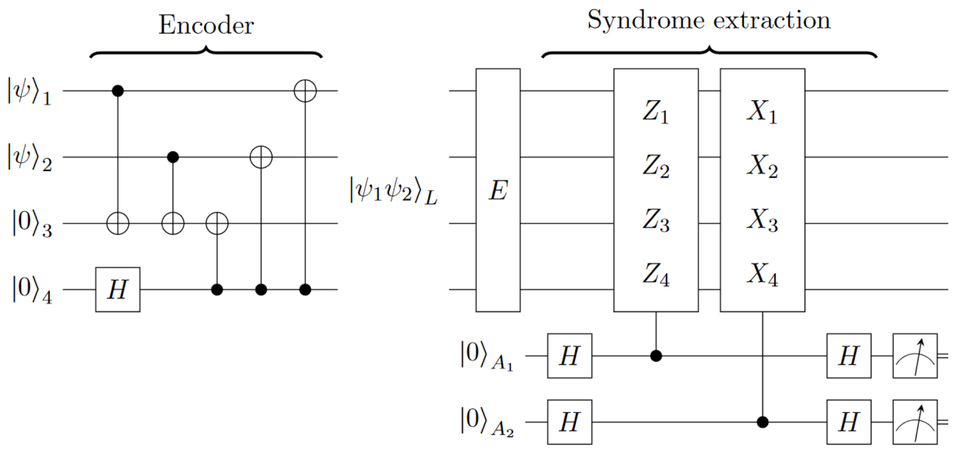
$$|01\rangle_L = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

$$|10\rangle_L = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)$$

$$|11\rangle_L = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle)$$

The above basis states are simultaneously stabilised by the following generators:

$$X_1X_2X_3X_4$$
$$Z_1Z_2Z_3Z_4$$



Error	Syndrome, S	Error	Syndrome, S	Error	Syndrome, S
X_1	10	Z_1	01	Y_1	11
X_2	10	Z_2	01	Y_2	11
X_3	10	Z_3	01	Y_3	11
X_4	10	Z_4	01	Y_4	11

The syndrome table for the [4,2,2] detection code. Each single-qubit X/Y/Z error yields a non-zero syndrome. However, some syndromes are shared. -> This is a **detection** code not a **correction code**.

The Distance of the [4,2,2] Code

The [[4,2,2]] detection code has **two** logical qubits. There are therefore two independent logical X-type operators: X_{L1} and X_{L2}:

$$X_{L1} = X_1 I_2 X_3 I_4$$

$$X_{L2} = X_1 X_2 I_2 I_4$$

We can obtain all of the basis states from the original $|00\rangle_L$ state using the logical operators

$$|00\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

$$X_{L2}|00\rangle_L = |01\rangle_L = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

$$X_{L1}|00\rangle_L = |10\rangle_L = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)$$

$$X_{L1}X_{L2}|00\rangle_L = |11\rangle_L = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle)$$

To find the Z-logical operators, we need to find operators Z_{L1} and Z_{L2} that:

1. Commute with all the stabilisers
2. Anti-commute with X_{L1} and X_{L2}

Two logical operators that satisfy these conditions are:

$$Z_{L1} = Z_1 Z_2 I_3 I_4$$
$$Z_{L2} = Z_1 I_2 Z_3 I_4$$

Distance of the [4,2,2] code

- The logical operators are: $X_1 I_2 X_3 I_4$, $X_1 X_2 I_2 I_4$, $Z_1 Z_2 I_3 I_4$, $Z_1 I_2 Z_3 I_4$
- There are no-single qubit errors that yield the `00` syndrome when the stabilisers are measured.
- Therefore, the minimum-weight undetectable error has weight 2
- The distance of the code is $d = 2$. It is a valid detection code for both Pauli-X and Pauli-Z type errors.

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The [[n,k,d]] notation

Quantum codes are usually labelled using the $[[n, k, d]]$ notation. This represents the following parameters.

- n: the number of physical qubits
- k: the number of logical qubits
- d: the code distance

Example 1: the detection code on the previous slide has parameters [[n=4,k=2,d=2]].

Example 2: the three-bit repetition code has parameters [[3,1,1]].

Detection codes vs. correction codes

Detection codes have $d = 2$

Correction codes (next lecture) have $d \geq 3$.

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