Quantum Error Correction: Surface Codes

IQC 2024 Lecture 29

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Recap: Stabiliser Codes

The Pauli Group: $\mathcal{P}^{\otimes n}$ is the Pauli-group $(\pm I, \pm iI, \pm X, \pm Y, \pm Z, \pm iX, \pm iY, \pm iZ)$ over n qubits. E.g., $X_1 I_2 Y_3 \in \mathcal{P}^{\otimes 3}$.

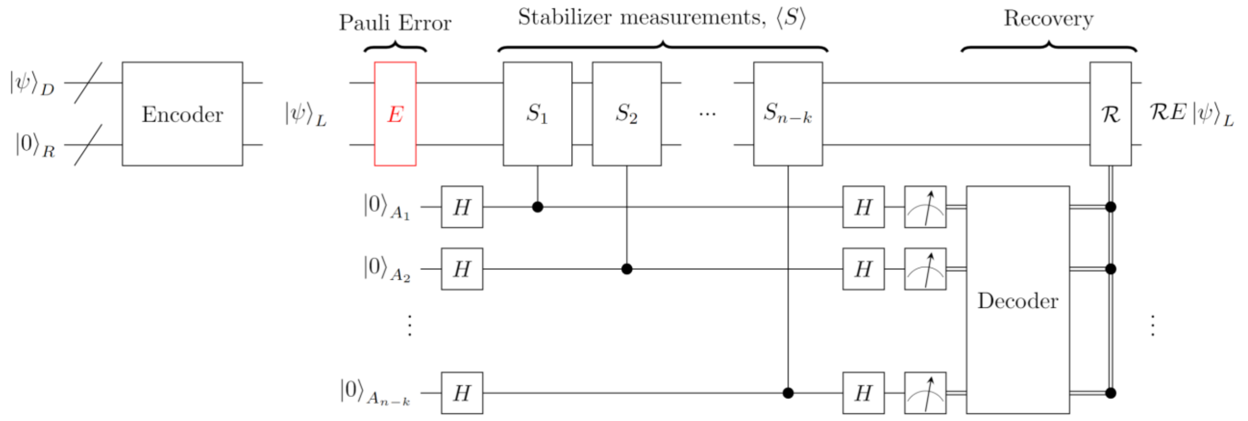
The stabilisers $\mathcal{S} \subset \mathcal{P}^{\otimes N}$ of a quantum error correction code simultaneously act as the identity on the logical state $|\psi\rangle_L$ such that:

$$G_i|\psi\rangle_L = (+1)|\psi\rangle_L \quad \text{for all} \quad G_i \in \mathcal{S}$$

To detect errors, a generating set $\langle S\rangle \subset \mathcal{S}$ is measured using the Hadamard test syndrome extraction gadgets (as described in Lecture 25).

Minimal generating set of stabilisers

A generating $\langle S\rangle$ in minimal if all operators in S are independent. E.g., $[Z_1Z_2,\ \ Z_2Z_3,$ $Z_1Z_3]$ is not a generating set, as $Z_1Z_3 = (Z_1Z_2)(Z_2Z_3)$.



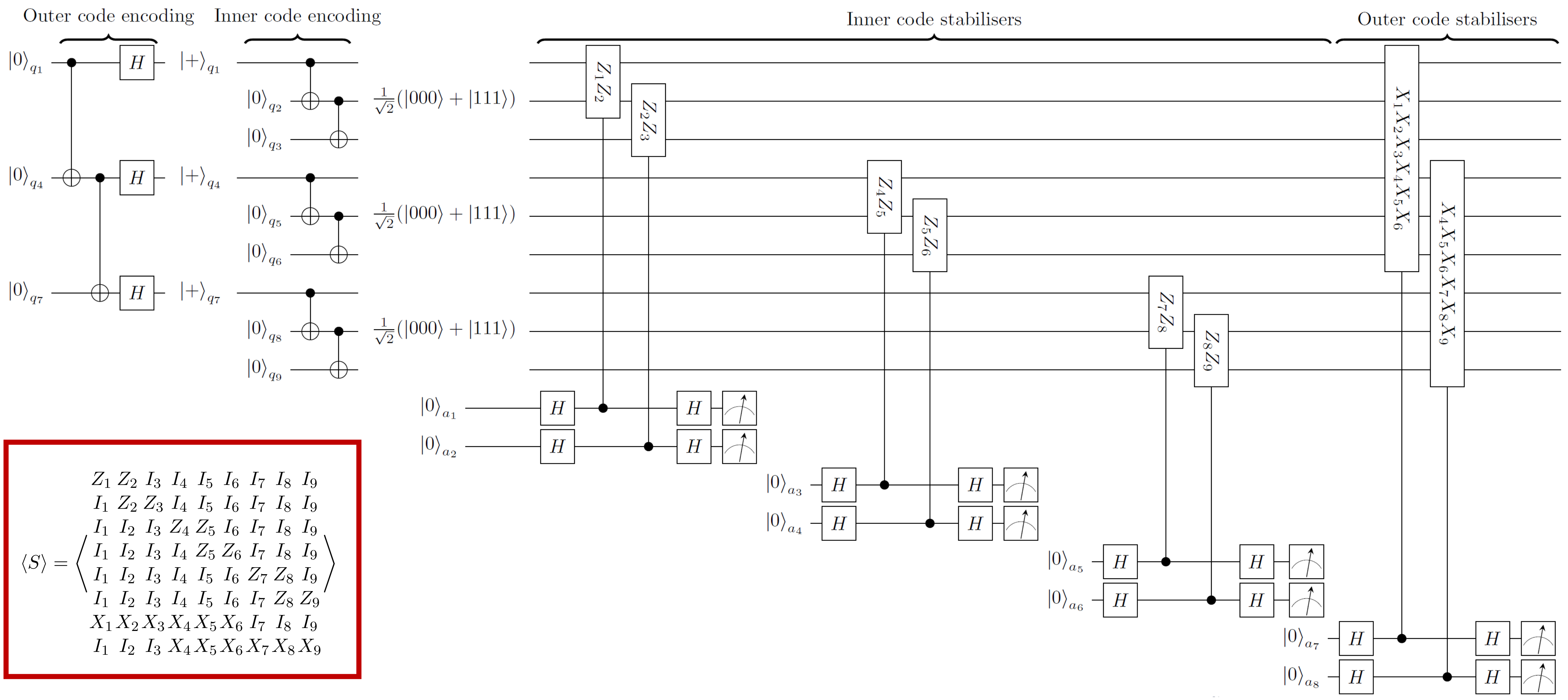
Stabiliser code syndromes

Measurement of generator S_i will yield outcome:

- $\mathrm{s_i} = 0$: if the error commutes with the generator, $[E, S_i] = 0$
- $\mathrm{s_i} = 1$: if the error anti-commutes with the generator, $\{\mathrm{E}, S_i\} = 0$

The **syndrome** is the binary string obtained by concatenating all of the generator measurement outcomes: $\mathrm{s} = \mathrm{s_1 s_2 \ldots s_m}$, where m is the size of the generating set, $m = |S|$.

Recap: The [9,1,3] Shor Code



$$\langle S \rangle = \left\langle \begin{array}{ccccccccc} Z_1 & Z_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 \\ I_1 & Z_2 & Z_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & Z_4 & Z_5 & I_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & I_4 & Z_5 & Z_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & Z_7 & Z_8 & I_9 \\ I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & Z_8 & Z_9 \\ X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \end{array} \right\rangle$$

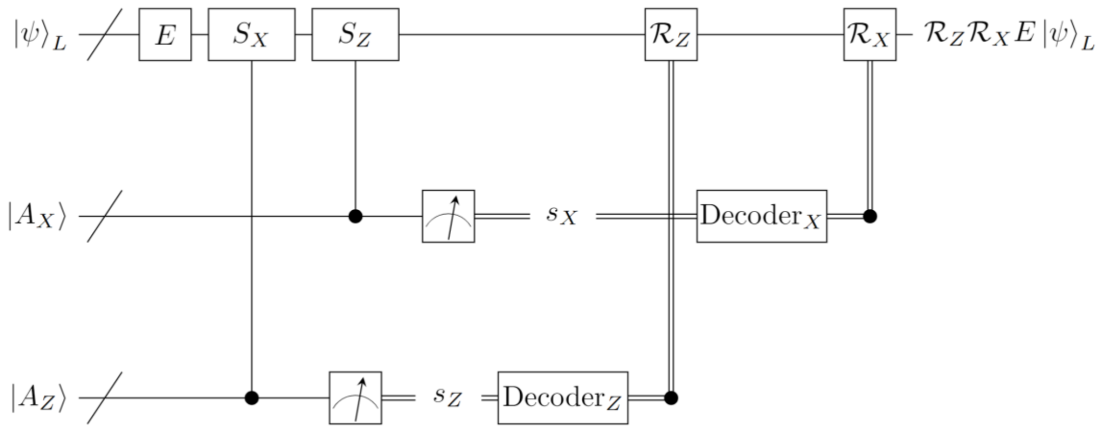
Calderbank-Shor-Steane (CSS) Codes

Note that the stabilisers for the nine qubit Shor code can be partitioned into two types:

- X-type stabilisers S_X that detect phase-flips
- Z-type stabilisers S_Z that detect bit-flips.

$$\langle S \rangle = \left\langle \begin{array}{ccccccccc} Z_1 & Z_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 \\ I_1 & Z_2 & Z_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & Z_4 & Z_5 & I_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & I_4 & Z_5 & Z_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & Z_7 & Z_8 & I_9 \\ I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & Z_8 & Z_9 \\ X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & I_7 & I_8 & I_9 \\ I_1 & I_2 & I_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \end{array} \right\rangle$$

Quantum error correction codes that partition stabilisers in this way are referred to as **Calderbank-Shor-Steane** (CSS) codes. The [[4,2,2]] code, the [[7,1,3]] Steane code are also examples of CSS-type stabiliser codes.



We think of a CSS code as a combination of two separate error correction protocols: one for bit-flips and one for phase-flips. The full stabiliser group is generated by:

$$\mathcal{S} = \mathcal{S}_X \cup \mathcal{S}_Z = \langle \mathcal{S}_X \cup \mathcal{S}_Z \rangle$$

Quantum Error Correction Code Constructions

The Challenge of QEC Code design

Quantum code design for CSS codes is complicated by the fact that the stabiliser group must be Abelian. I.e.,

$$[G_i^X, G_j^Z] = 0 \quad \text{for all } G_i^X \in \mathcal{S}_X \text{ and } G_j^Z \in \mathcal{S}_Z$$

Finding generating sets of stabilisers S_X and S_Z is not straightforward. E.g., we can't just translate classical error correction codes into the S_X and S_Z of a quantum error correction codes.

Up to this point, all the of the code constructions we have looked at – the [[4,2,2]] code, the Steane code, and the Shor Code – have been *hand crafted*.

The goal is now to develop **systematics procedures** for designing large-scale CSS stabiliser codes.

Requirements for a Quantum Code Construction

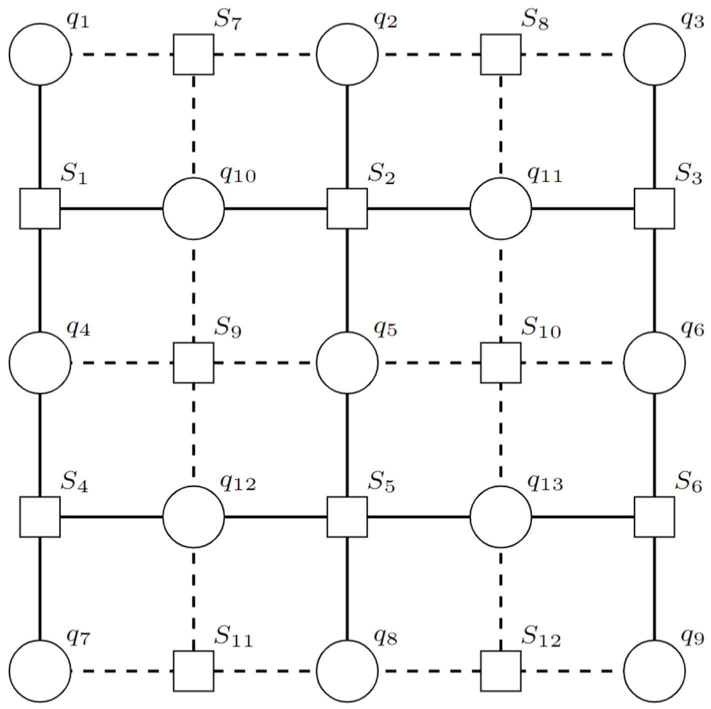
A systematic approach for CSS code design must address the following:

- **Generating Stabiliser Sets**: Provide a method to construct S_X and S_Z such that they are guaranteed to commute.
- **Scalability**: Ensure that the code distance d increases with the size of the system.

The Surface Code

The **surface** code is the leading construction for experimental quantum error correction. It has two major advantages:

- It can be implemented using nearest-neighbour CNOT gates between qubits arrange in a two-dimensional grid. This is a particularly useful for many qubit hardware types – e.g. superconducting qubits – where long-range gates are difficult to implement with high-fidelity.
- The distance can be straightforwardly scaled by increasing the size of the qubit grid.



The Tanner (connectivity) graph for the $d = 3$ surface code.

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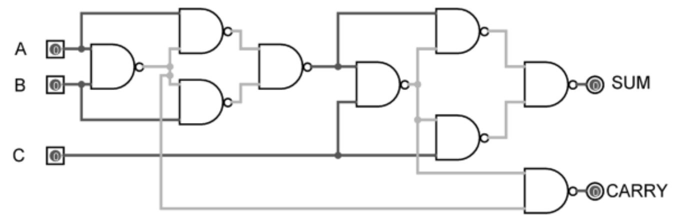
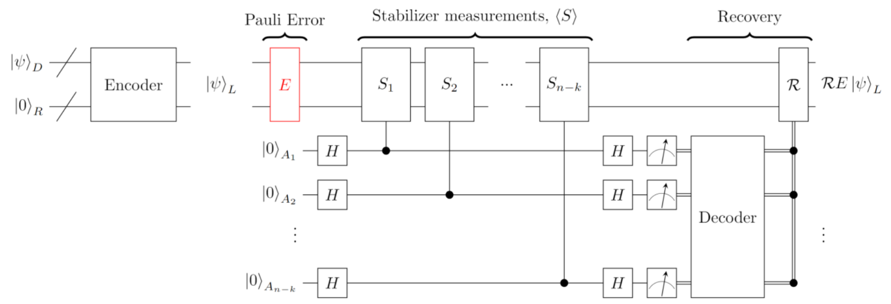
Classical vs. Quantum Circuits





Classical circuit

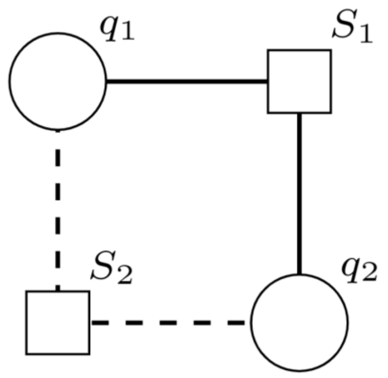
- Provides information about what operations are performed.
- Tells us exactly how to wire together the circuit.

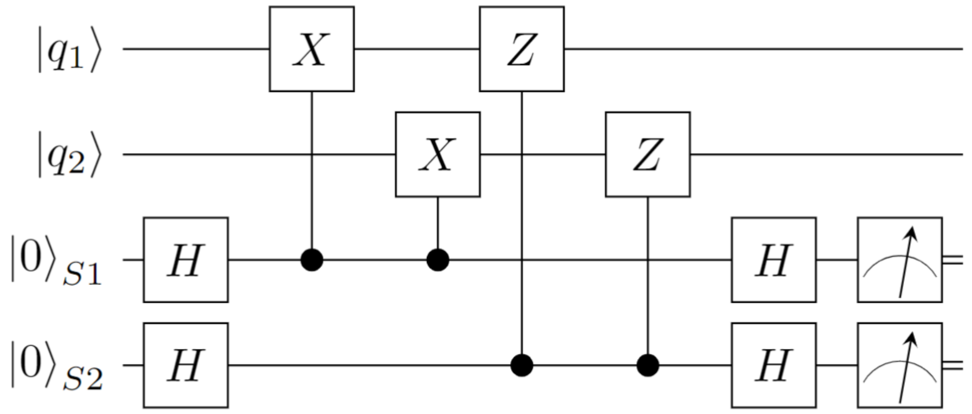
Quantum Circuit

- Provides information about what operations are performed.
- Wires represent the passage of time. I.e., they do not actually correspond to wires and tell us nothing about the qubit connectivity.

The Tanner Graph Representation of QEC

Tanner graphs provide a graphical representation of quantum error correction that allows us to design codes in a way that is *connectivity aware*.





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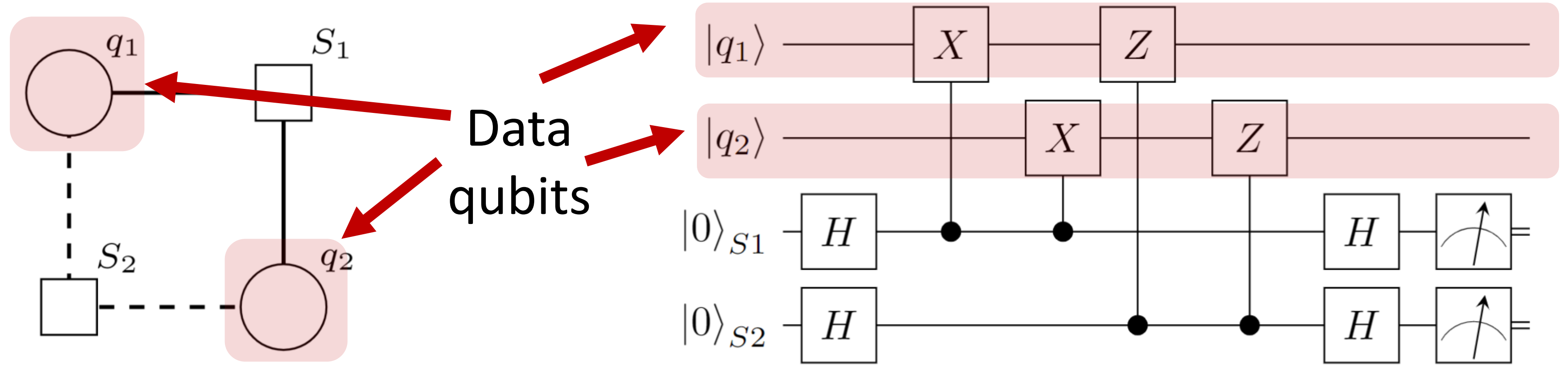
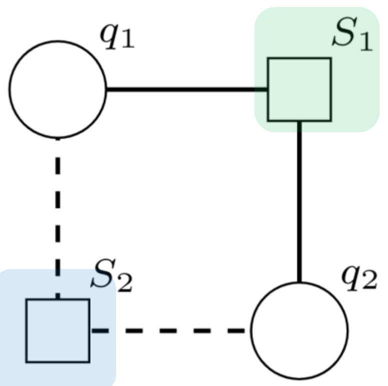
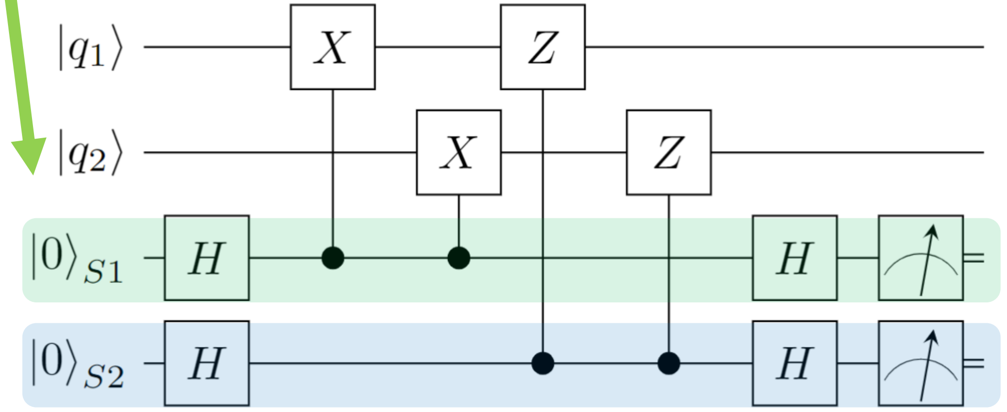


The Tanner Graph Representation of QEC

Tanner graphs provide a graphical representation of quantum error correction that allows us to design codes in a way that is *connectivity aware*.

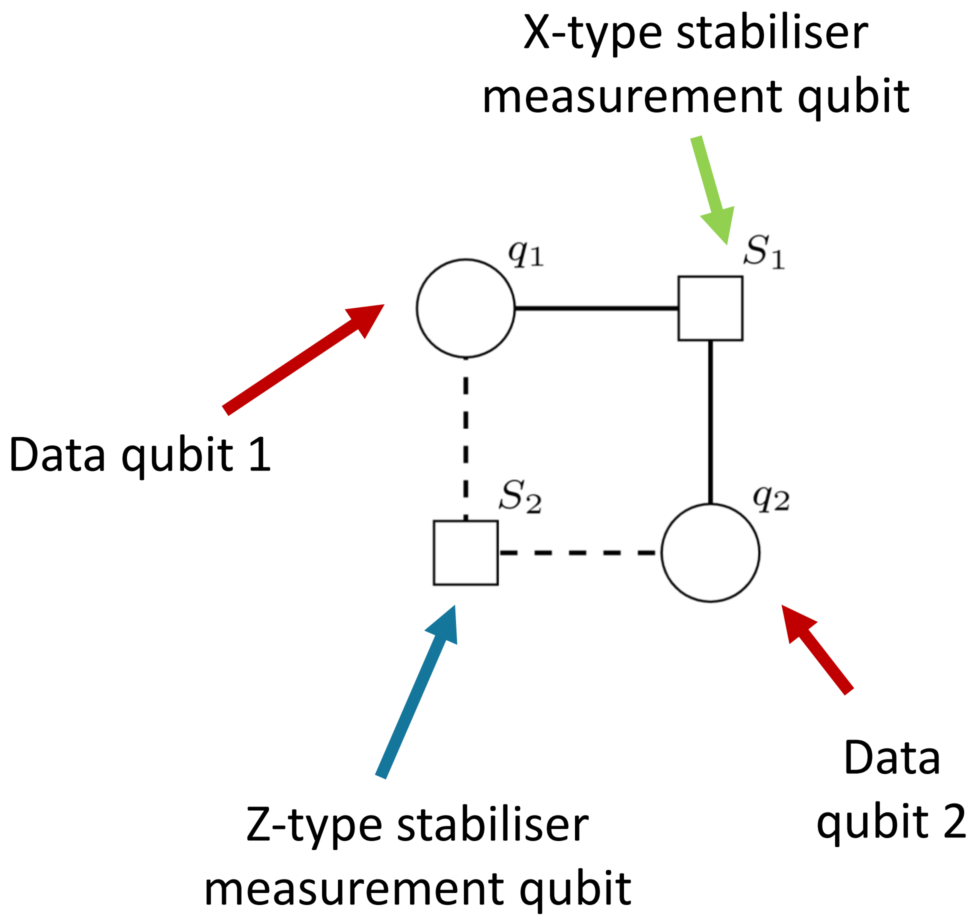
X-type stabiliser measurement qubit





Z-type stabiliser measurement qubit

The Surface Code 4-Cycle


X-type stabiliser measurement qubit
q_1
S_1
Data qubit 1
S_2
q_2
Data qubit 2
Z-type stabiliser measurement qubit

Stabilisers:

$$\langle S \rangle = \langle S_X \cup S_Z \rangle = \langle S_1 \cup S_2 \rangle = \langle Z_1 Z_2, X_1 X_2 \rangle$$

Both stabilisers intersect on two-qubits: they commute!

Logical qubit count

$$k = n - |S| = 2 - 2 = 0$$

The surface code four-cycle does not encode information!

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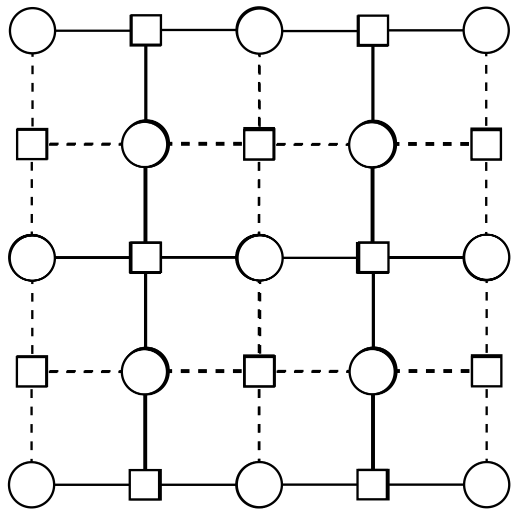
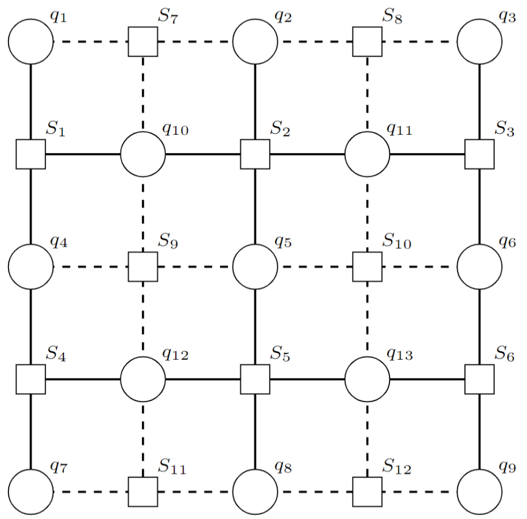
The Surface Code



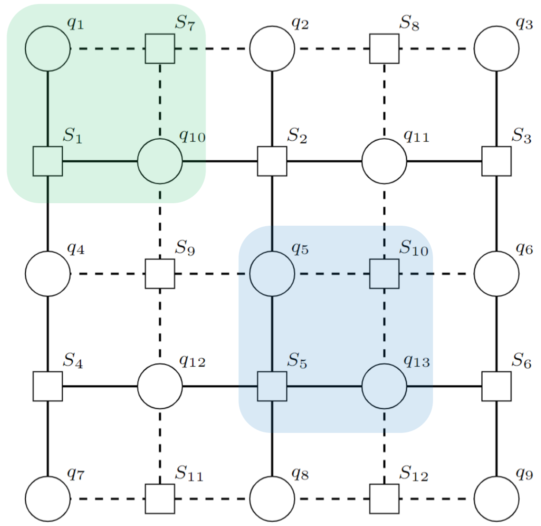
Larger surface codes can be constructed by tiling four-cycles together.

The Surface Code



Larger surface codes can be constructed by tiling four-cycles together.

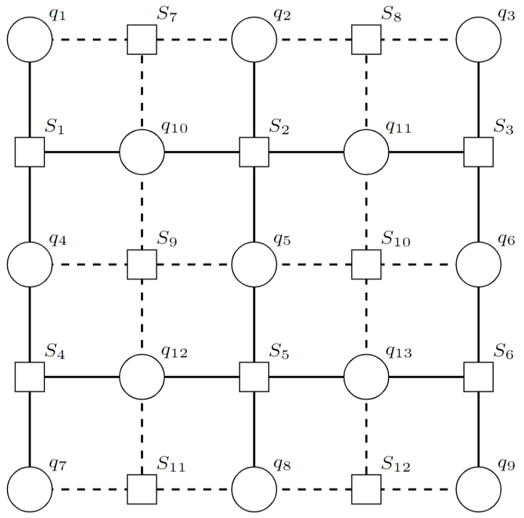
The $d = 3$ Surface Code



Larger surface codes can be constructed by tiling four-cycles together.

The construction guarantees that any two X- and Z-type stabiliser always intersect non-trivially on an even number of qubits. => The stabilisers always commute!

Surface codes: logical qubit count



This surface code has $n = 13$ physical (data) qubits

There are 12 auxiliary qubits measuring 12 independent stabiliser generators. => $rank(\mathcal{S}) = |S| = 12$

Logical qubit count:

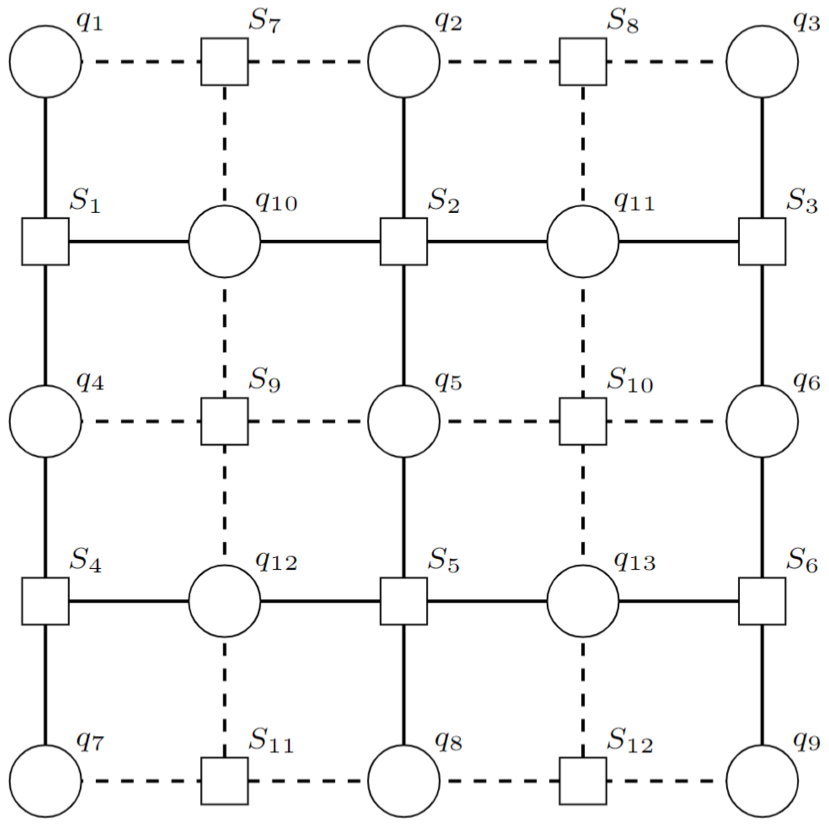
$$k = n - |S| = 1$$

For a surface patch of any size, the number of stabiliser generators is $|S| = n - 1$.

This means all surface codes (other than the surface code four-cycle) encode $k = 1$ logical qubits

Surface codes: error detection

Surface codes are CSS codes: X- and Z-type errors are detected by different stabilisers.



Surface codes: error detection

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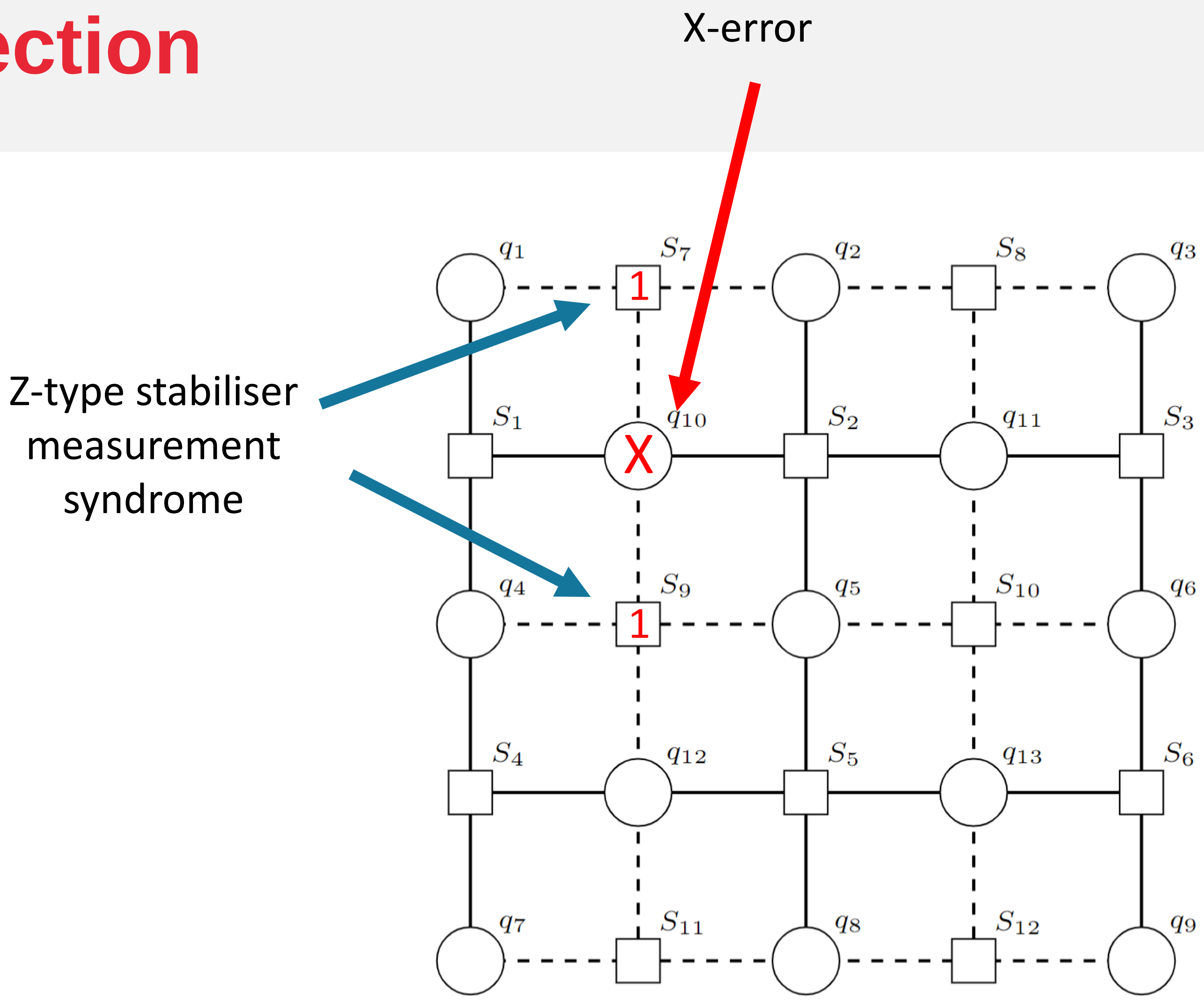

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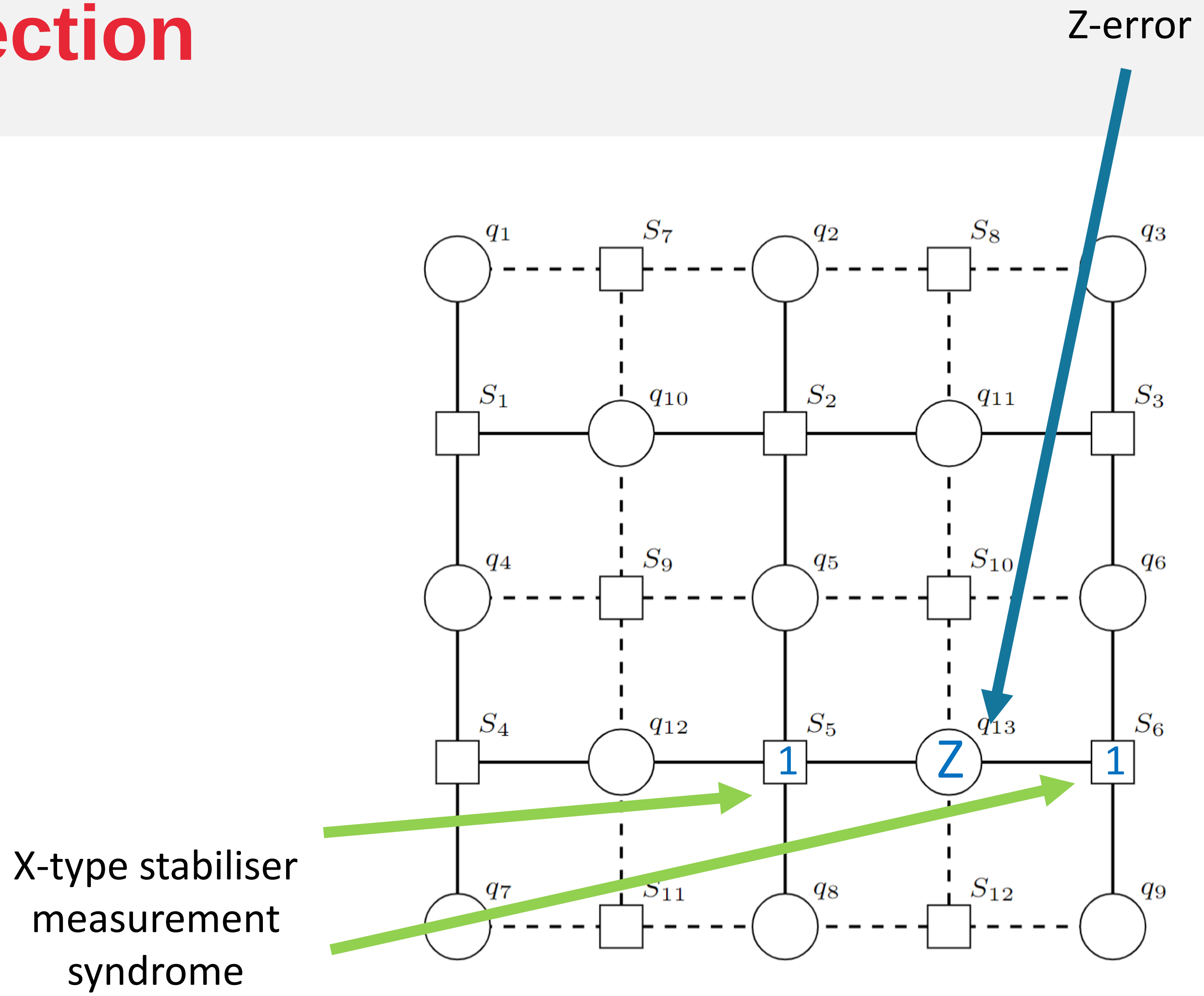

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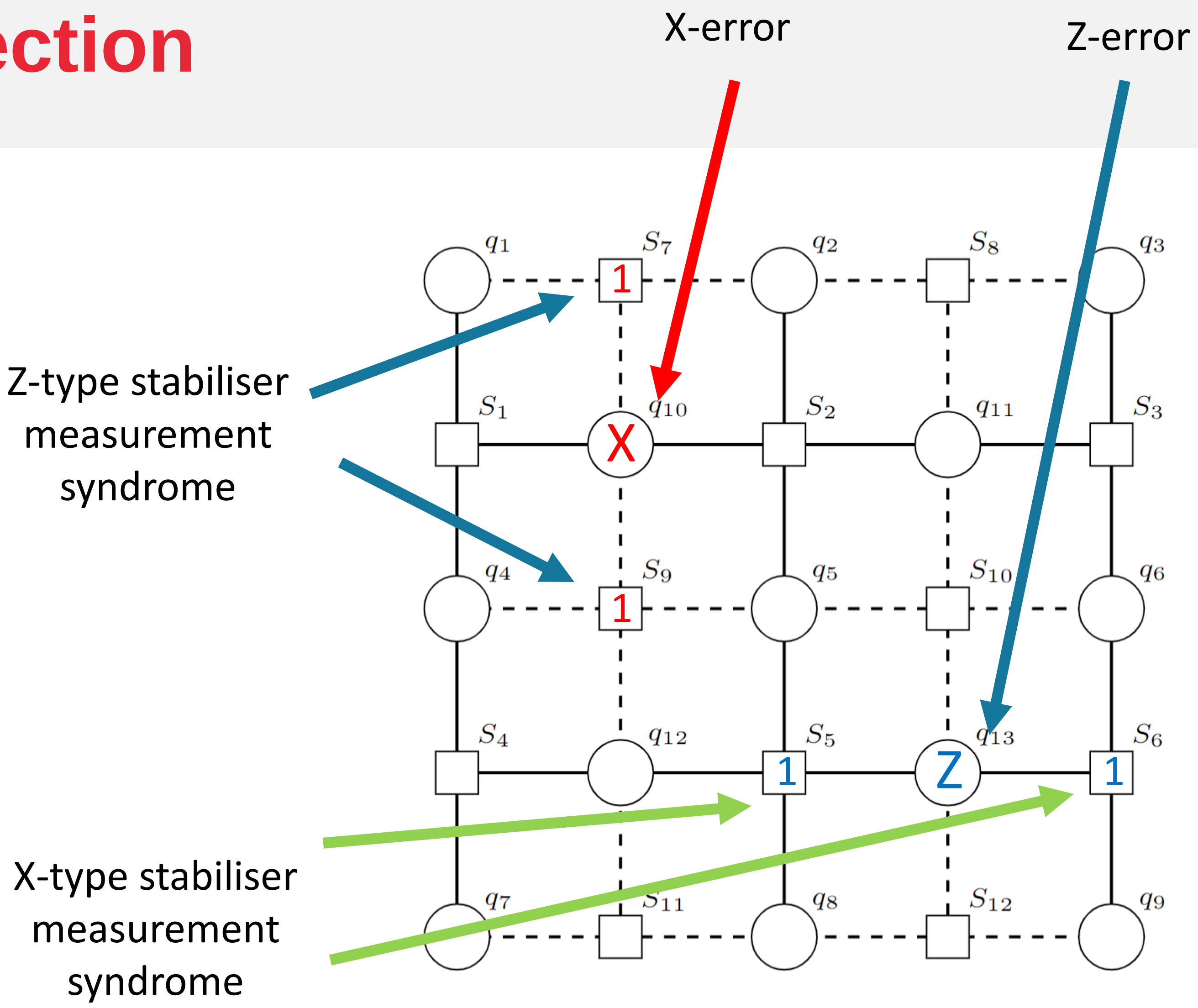
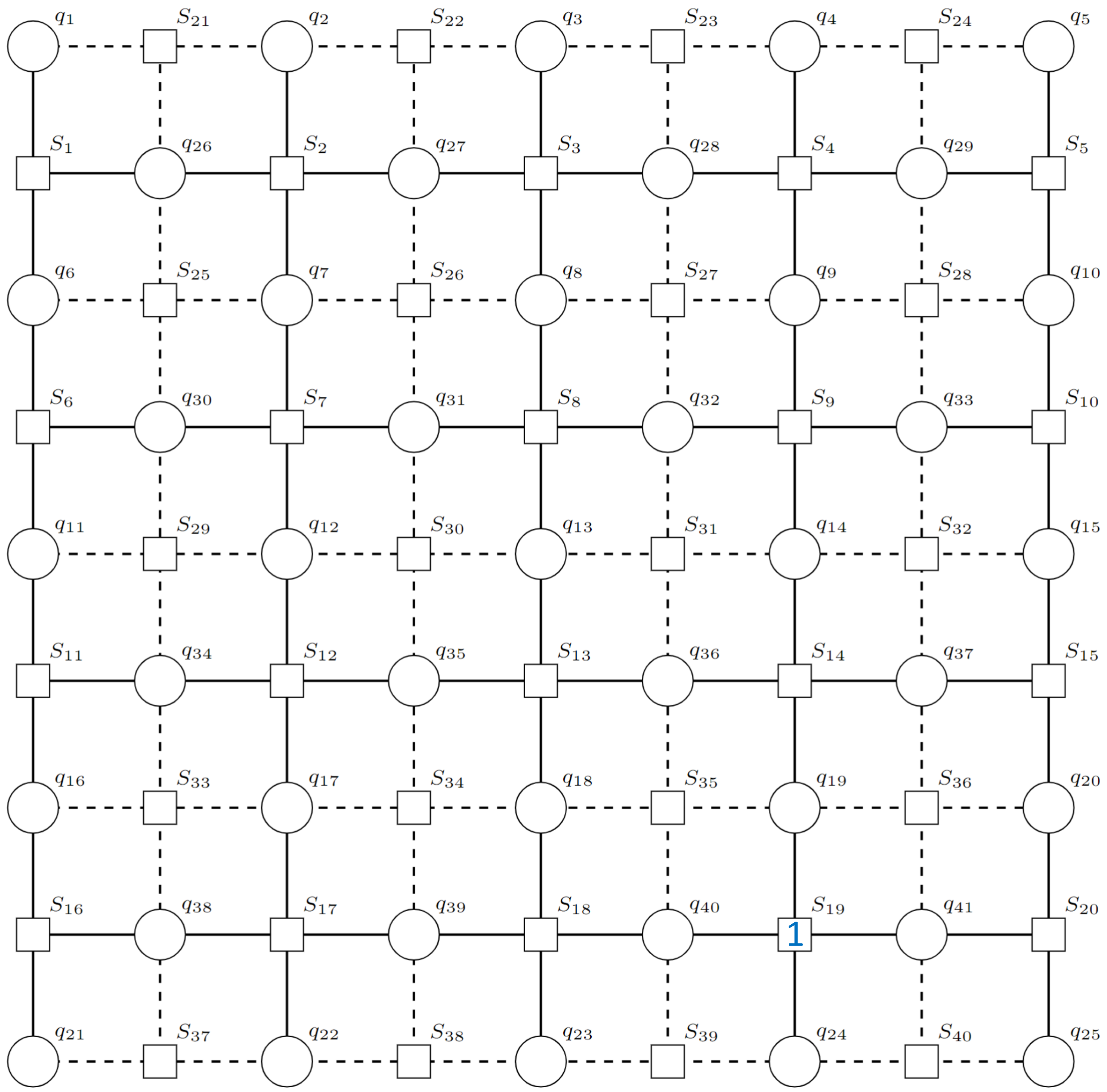


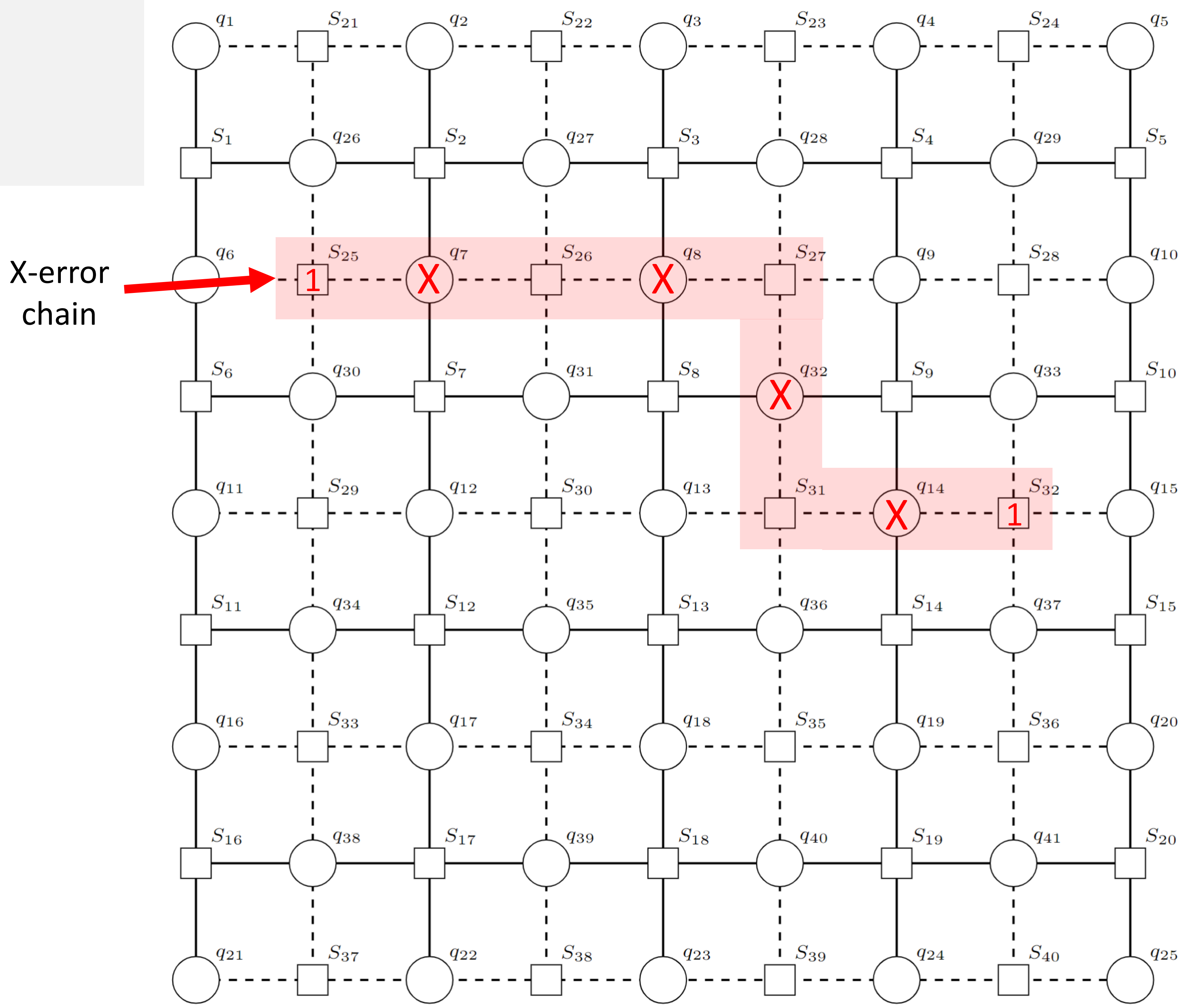
Surface code: error chains



q_1 S_{21} q_2 S_{22} q_3 S_{23} q_4 S_{24} q_5
S_1 q_{26} S_2 q_{27} S_3 q_{28} S_4 q_{29} S_5
q_6 S_{25} q_7 S_{26} q_8 S_{27} q_9 S_{28} q_{10}
S_6 q_{30} S_7 q_{31} S_8 q_{32} S_9 q_{33} S_{10}
q_{11} S_{29} q_{12} S_{30} q_{13} S_{31} q_{14} S_{32} q_{15}
S_{11} q_{34} S_{12} q_{35} S_{13} q_{36} S_{14} q_{37} S_{15}
q_{16} S_{33} q_{17} S_{34} q_{18} S_{35} q_{19} S_{36} q_{20}
S_{16} q_{38} S_{17} q_{39} S_{18} q_{40} S_{19} q_{41} S_{20}
1
q_{21} S_{37} q_{22} S_{38} q_{23} S_{39} q_{24} S_{40} q_{25}

Surface code: error chains

Error chains in the bulk have syndrome Hamming weight 2.



Surface code: error chains

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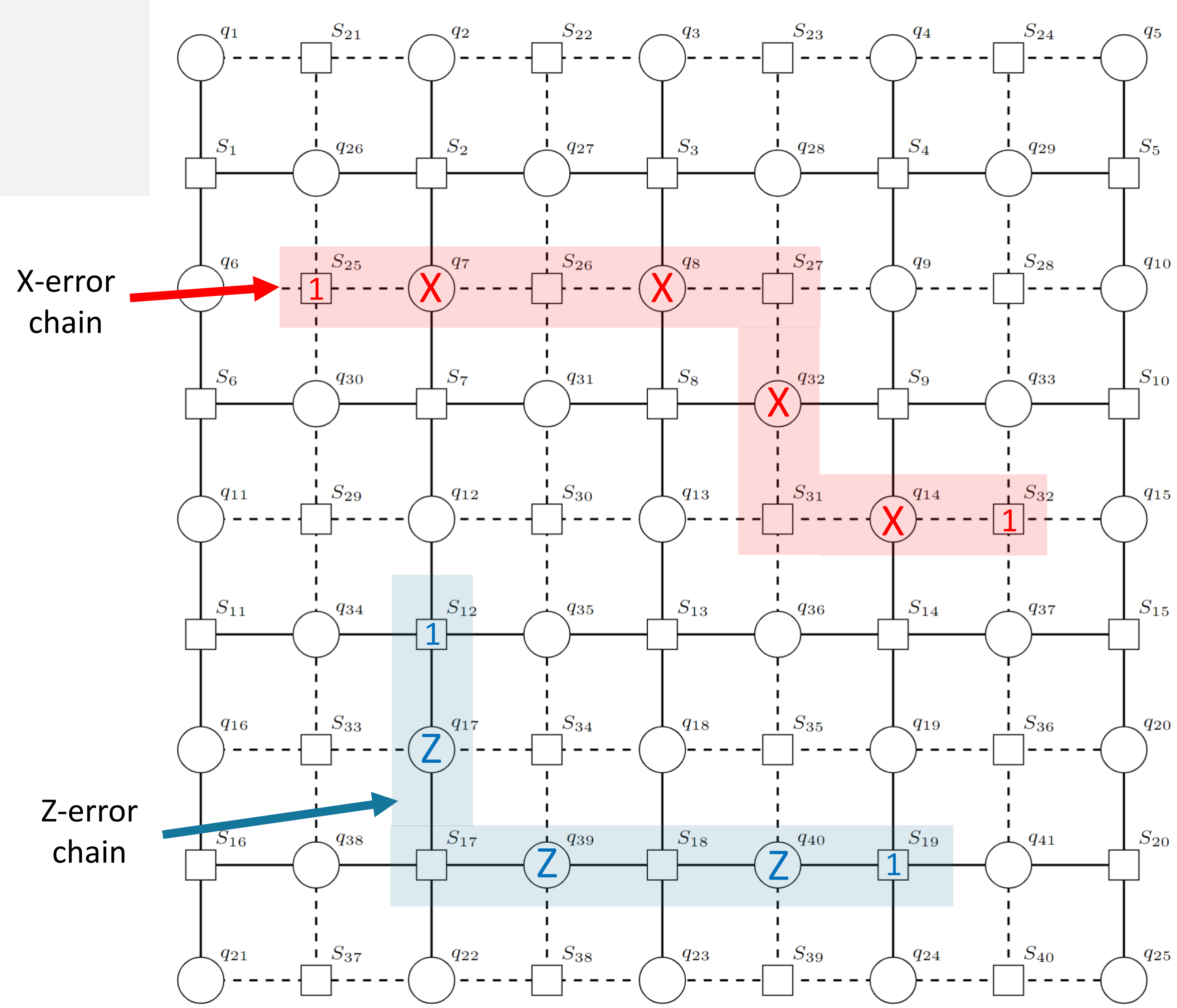
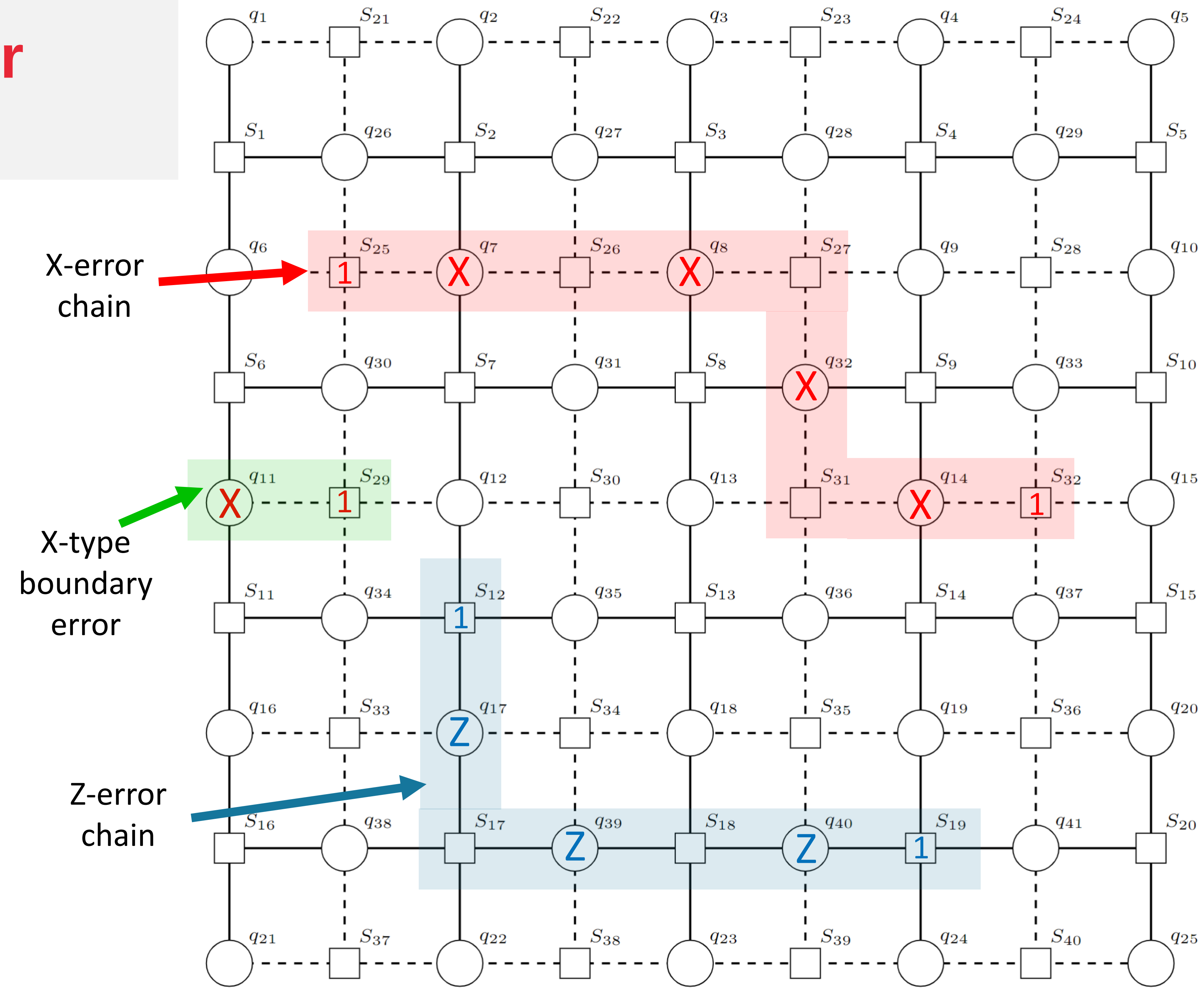

X-error chain
Z-error chain
q_1 S_{21} q_2 S_{22} q_3 S_{23} q_4 S_{24} q_5
S_1 q_{26} S_2 q_{27} S_3 q_{28} S_4 q_{29} S_5
q_6 S_{25} q_7 S_{26} q_8 S_{27} q_9 S_{28} q_{10}
S_6 q_{30} S_7 q_{31} S_8 q_{32} S_9 q_{33} S_{10}
q_{11} S_{29} q_{12} S_{30} q_{13} S_{31} q_{14} S_{32} q_{15}
S_{11} q_{34} S_{12} q_{35} S_{13} q_{36} S_{14} q_{37} S_{15}
q_{16} S_{33} q_{17} S_{34} q_{18} S_{35} q_{19} S_{36} q_{20}
S_{16} q_{38} S_{17} q_{39} S_{18} q_{40} S_{19} q_{41} S_{20}
q_{21} S_{37} q_{22} S_{38} q_{23} S_{39} q_{24} S_{40} q_{25}
1
X
X
X
X
1
1
Z
Z
Z
1

Surface code: error chains

Error chains in the bulk have syndrome Hamming weight 2.

Error chains starting at the boundary have syndrome Hamming weight 1.



Surface code: error chains

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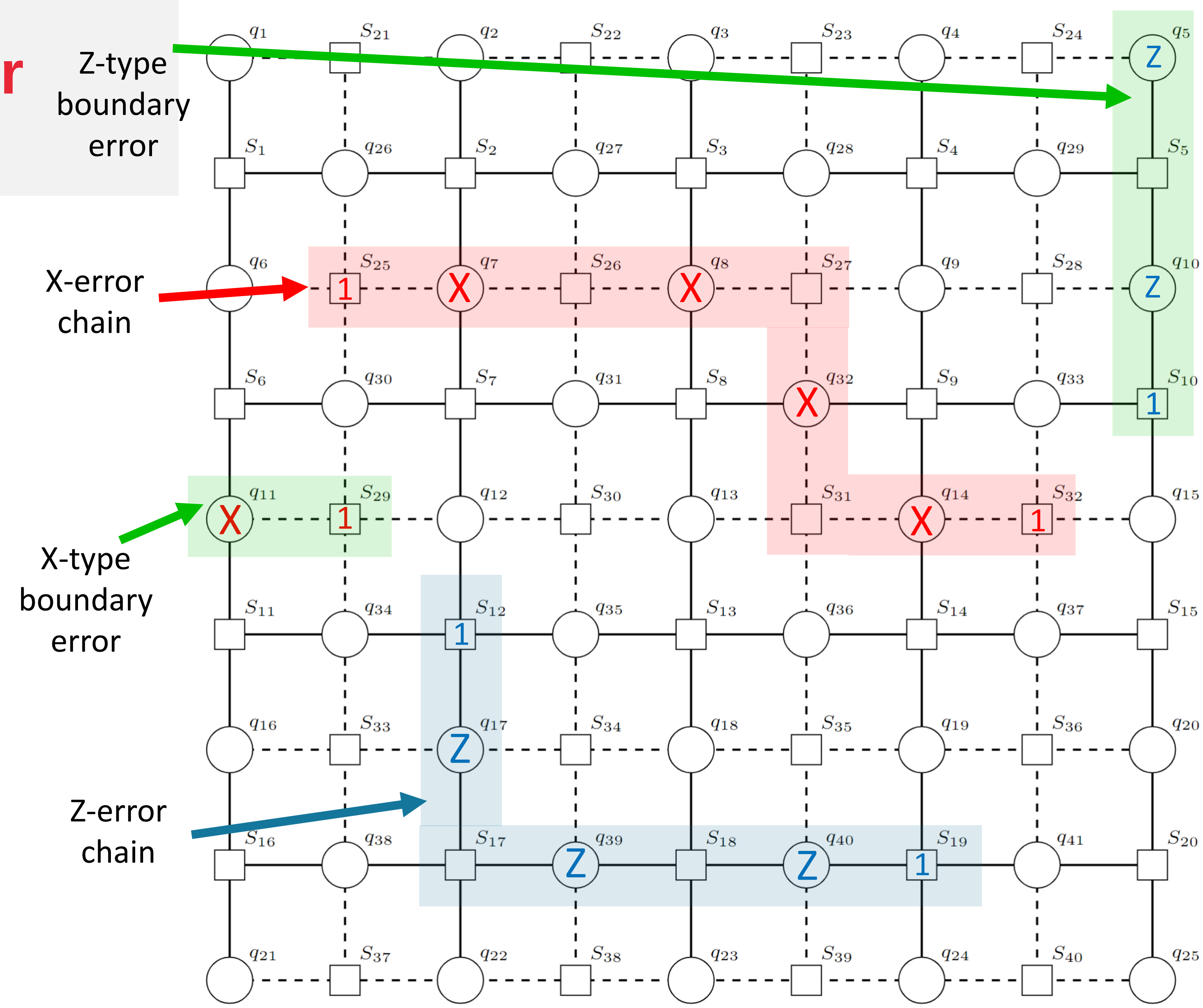
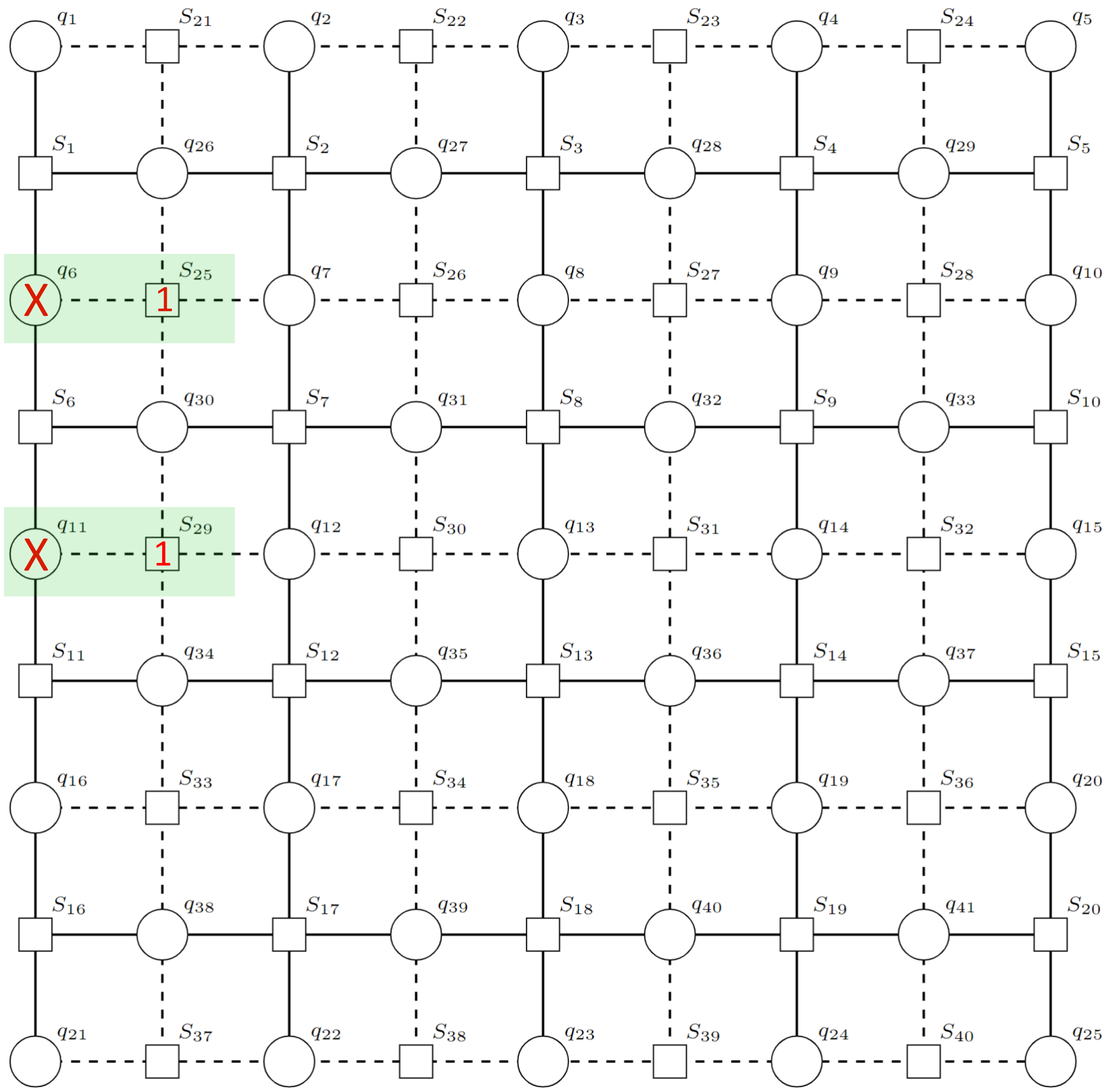
Error chains starting at the boundary have syndrome Hamming weight 1.


Z-type boundary error
X-error chain
X-type boundary error
Z-error chain
q_1 S_{21} q_2 S_{22} q_3 S_{23} q_4 S_{24} q_5
S_1 q_{26} S_2 q_{27} S_3 q_{28} S_4 q_{29} S_5
q_6 S_{25} q_7 S_{26} q_8 S_{27} q_9 S_{28} q_{10}
S_6 q_{30} S_7 q_{31} S_8 q_{32} S_9 q_{33} S_{10}
q_{11} S_{29} q_{12} S_{30} q_{13} S_{31} q_{14} S_{32} q_{15}
S_{11} q_{34} S_{12} q_{35} S_{13} q_{36} S_{14} q_{37} S_{15}
q_{16} S_{33} q_{17} S_{34} q_{18} S_{35} q_{19} S_{36} q_{20}
S_{16} q_{38} S_{17} q_{39} S_{18} q_{40} S_{19} q_{41} S_{20}
q_{21} S_{37} q_{22} S_{38} q_{23} S_{39} q_{24} S_{40} q_{25}
Z Z 1
1 X X X X 1
X 1
1 Z Z Z 1

Zero-syndrome error chains

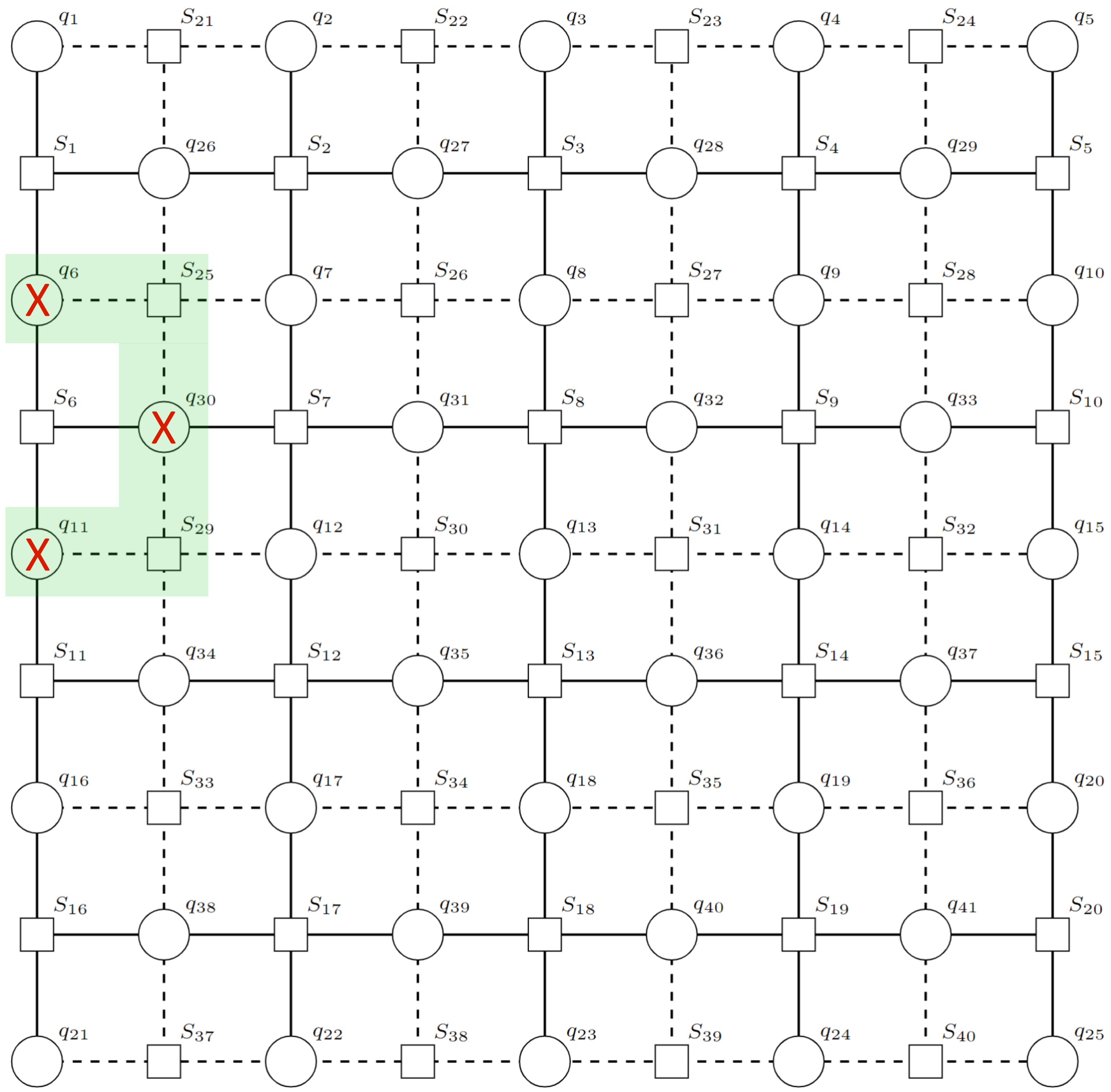
If two boundary error chains merge, we get an **undetectable** zero-weight syndrome.



Zero-syndrome error chains

If two boundary error chains merge, we get an **undetectable** zero-weight syndrome.

The highlighted error chain is equivalent to stabiliser S_6. => It has trivial action on the codespace.



q_1 S_{21} q_2 S_{22} q_3 S_{23} q_4 S_{24} q_5
S_1 q_{26} S_2 q_{27} S_3 q_{28} S_4 q_{29} S_5
q_6 S_{25} q_7 S_{26} q_8 S_{27} q_9 S_{28} q_{10}
S_6 q_{30} S_7 q_{31} S_8 q_{32} S_9 q_{33} S_{10}
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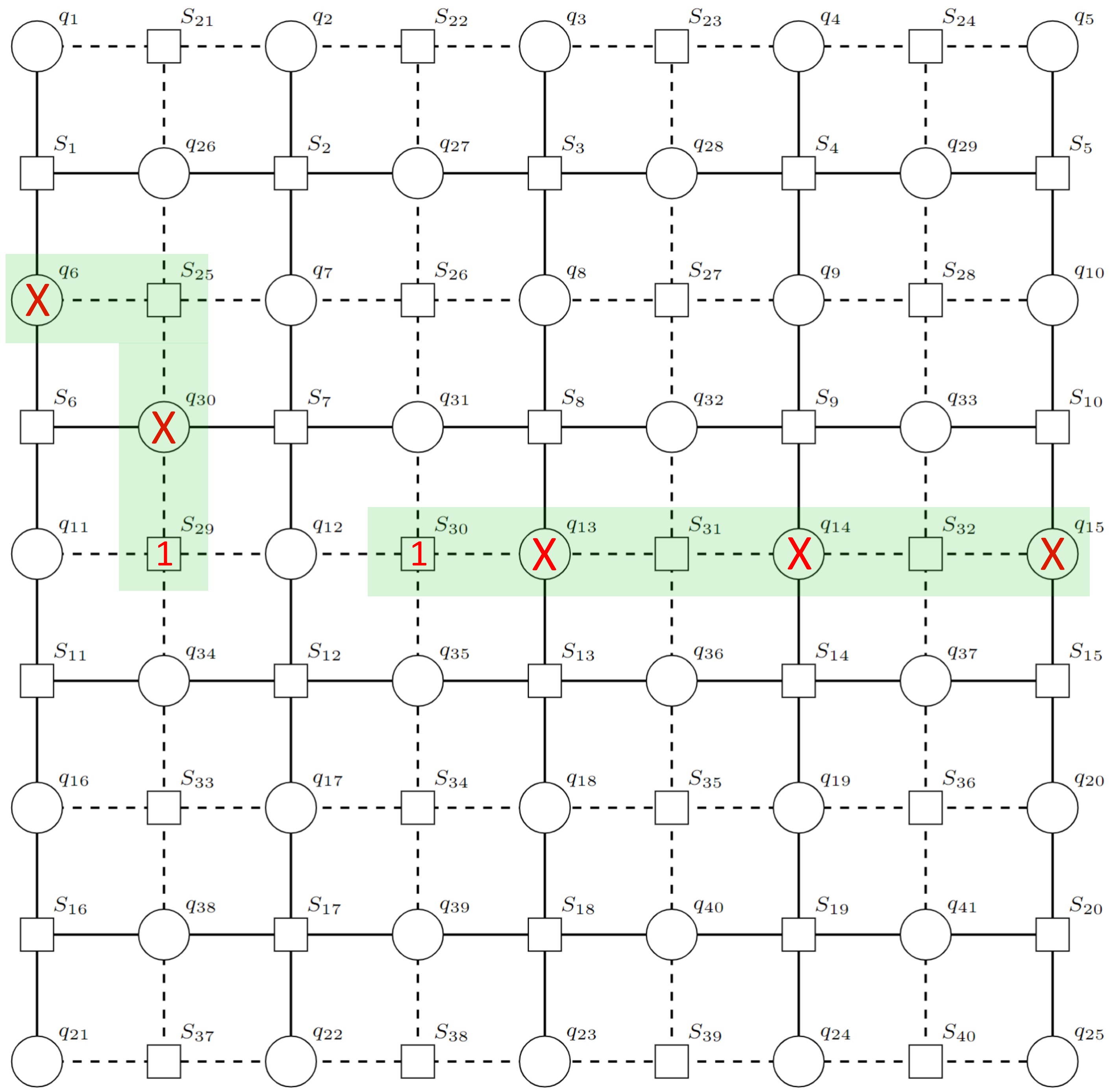
S_{11} q_{34} S_{12} q_{35} S_{13} q_{36} S_{14} q_{37} S_{15}
q_{16} S_{33} q_{17} S_{34} q_{18} S_{35} q_{19} S_{36} q_{20}
S_{16} q_{38} S_{17} q_{39} S_{18} q_{40} S_{19} q_{41} S_{20}
q_{21} S_{37} q_{22} S_{38} q_{23} S_{39} q_{24} S_{40} q_{25}
X
X
X

Logical operators

If two boundary error chains merge, we get an **undetectable** zero-weight syndrome.

If boundary operators from opposite ends of the lattice merge, we get a non-zero syndrome error chain that is not in the stabiliser group. This is a **logical operator**.



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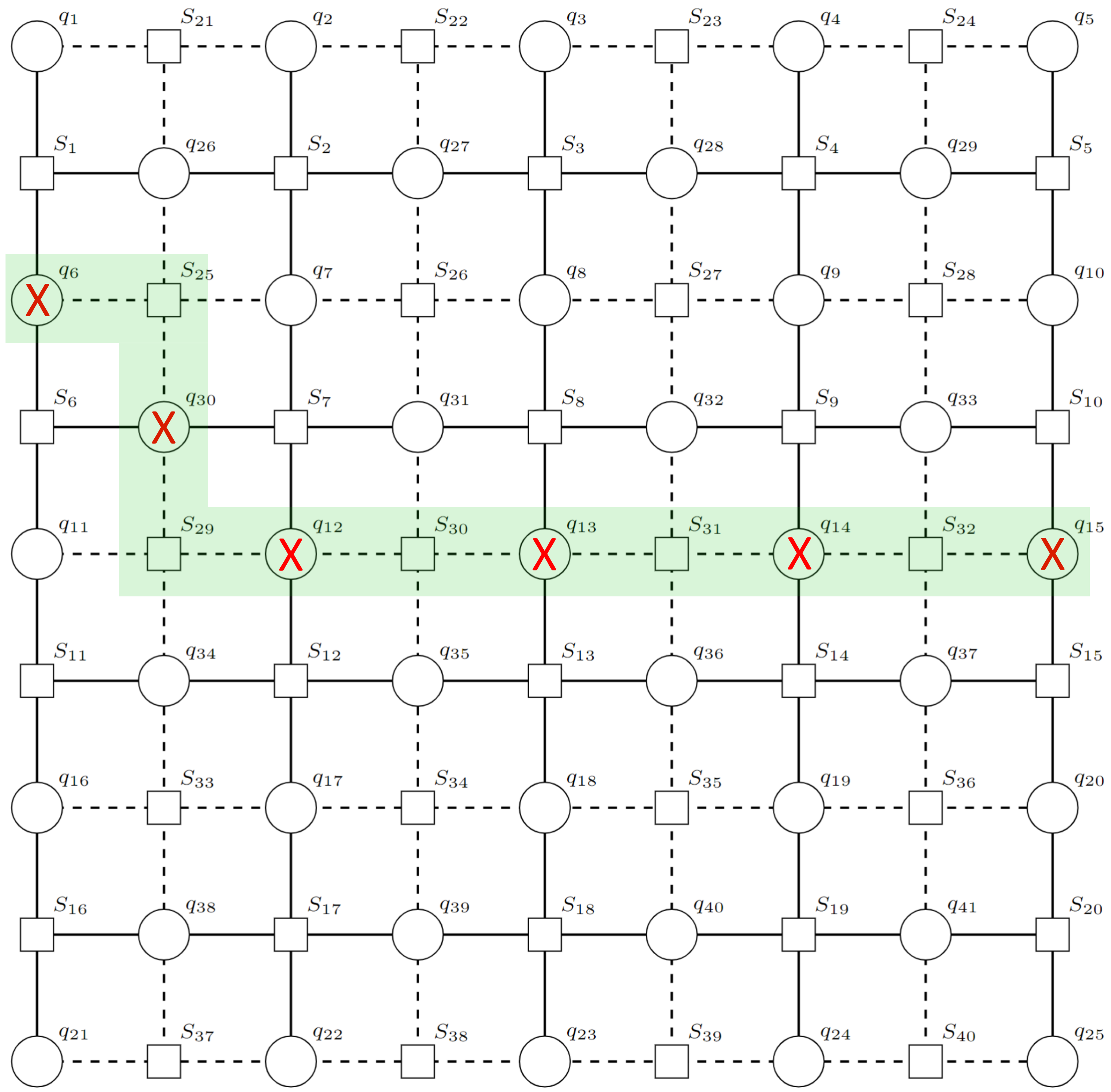

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q_{21} S_{37} q_{22} S_{38} q_{23} S_{39} q_{24} S_{40} q_{25}
X
X
X
X
X
X

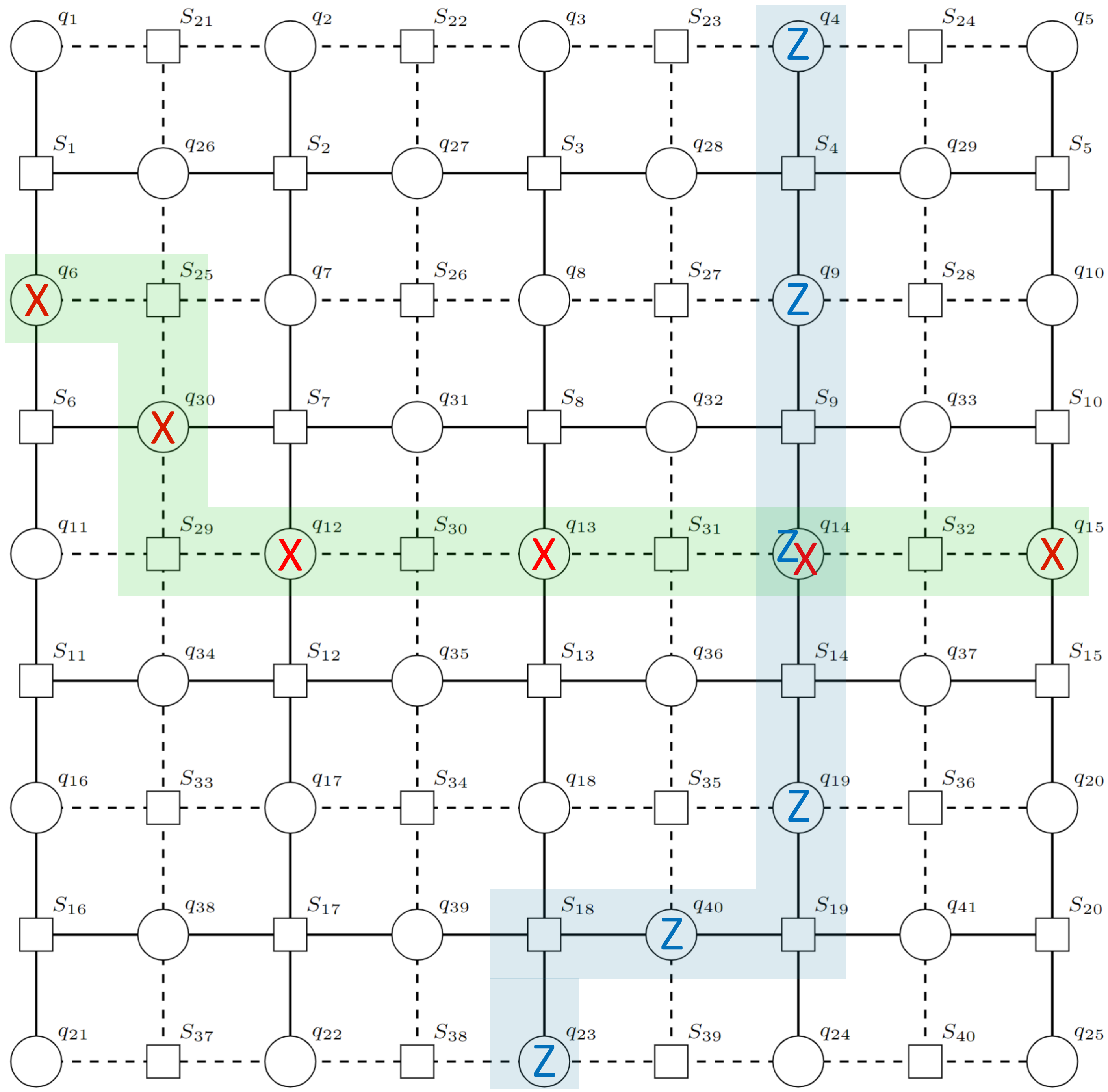
X- and Z-logical operators

X-Logical operators span from left-to-right boundary.

Z-Logical operators span from top-to-bottom boundary

We also see that the two logical operator anti-commute as they intersect on one qubit:

$$\{X_L, Z_L\} = 0$$



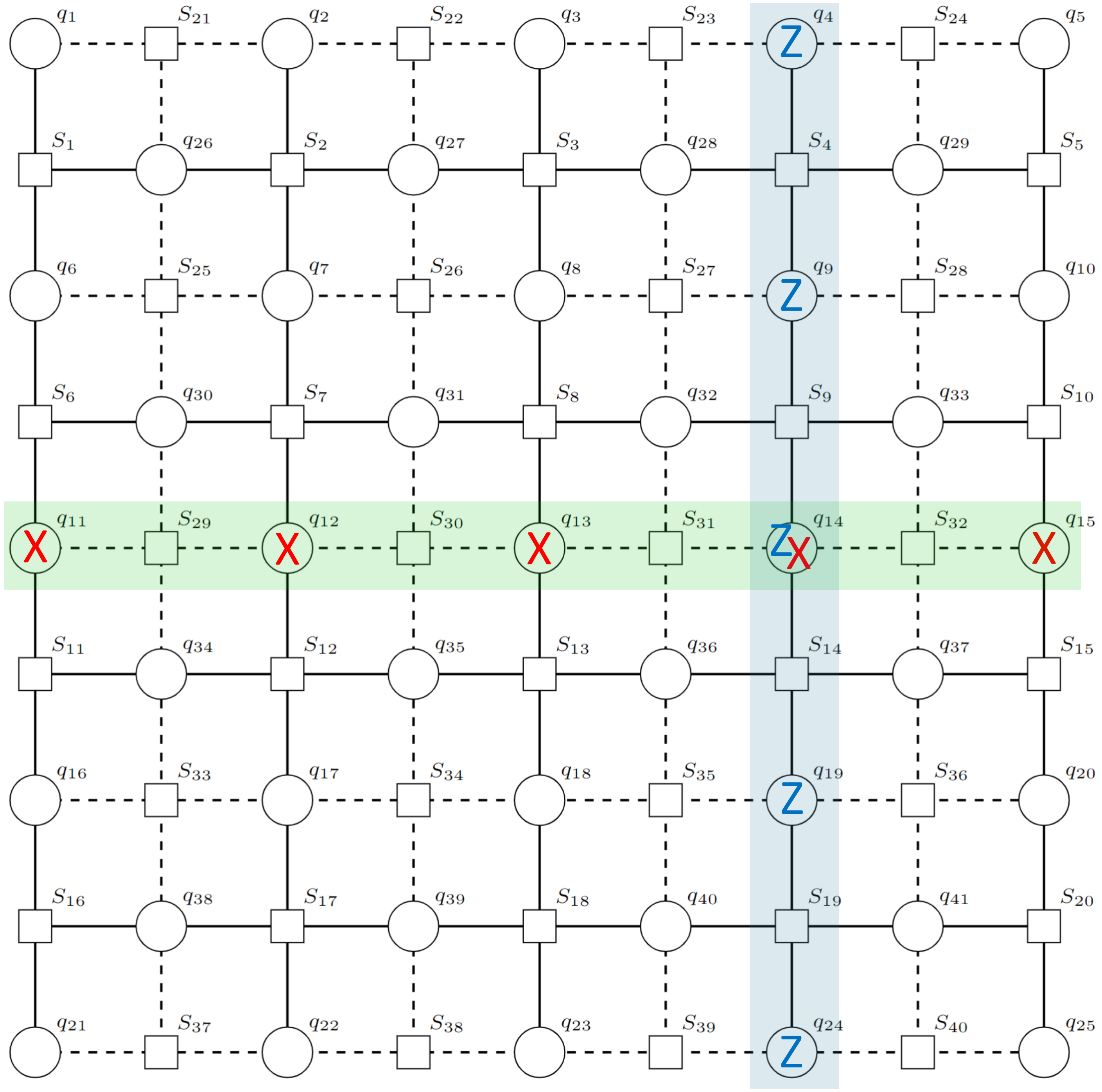
Surface code distance

The code distance corresponds to the minimum-weight logical operator.

In the surface code, these operators are error chains that span directly from one boundary to another.

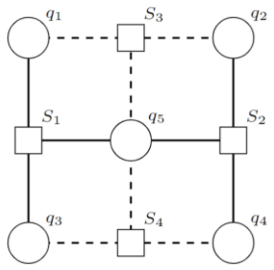
The distance of the surface code is given by the size of the lattice.

This surface code is defined on a 5x5 lattice. The distance $d = 5$.

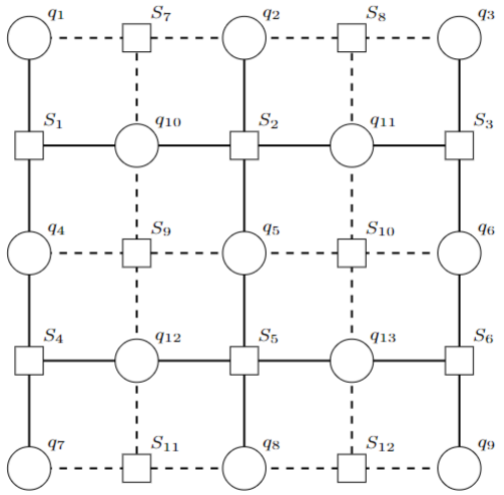


q_1 S_{21} q_2 S_{22} q_3 S_{23} q_4 S_{24} q_5
S_1 q_{26} S_2 q_{27} S_3 q_{28} S_4 q_{29} S_5
q_6 S_{25} q_7 S_{26} q_8 S_{27} q_9 S_{28} q_{10}
S_6 q_{30} S_7 q_{31} S_8 q_{32} S_9 q_{33} S_{10}
q_{11} S_{29} q_{12} S_{30} q_{13} S_{31} q_{14} S_{32} q_{15}
S_{11} q_{34} S_{12} q_{35} S_{13} q_{36} S_{14} q_{37} S_{15}
q_{16} S_{33} q_{17} S_{34} q_{18} S_{35} q_{19} S_{36} q_{20}
S_{16} q_{38} S_{17} q_{39} S_{18} q_{40} S_{19} q_{41} S_{20}
q_{21} S_{37} q_{22} S_{38} q_{23} S_{39} q_{24} S_{40} q_{25}
Z
Z
X X X ZX X
Z
Z

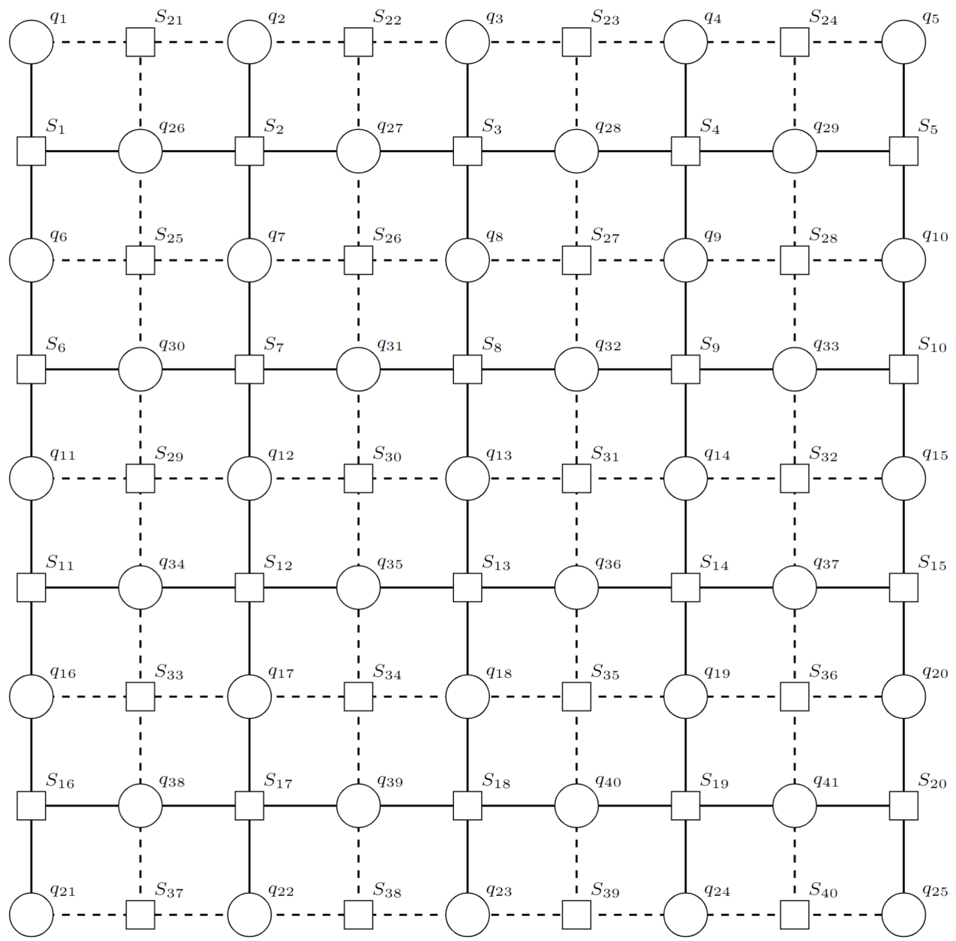
Scaling the Surface Code



$d = 2$ Surface code



$d = 3$ Surface code



$d = 5$ Surface code

The distance of surface code scales $d \sim O(\sqrt{n})$, where n in the number of physical qubits. By increasing the distance we can arbitrarily supress the logical error rate.

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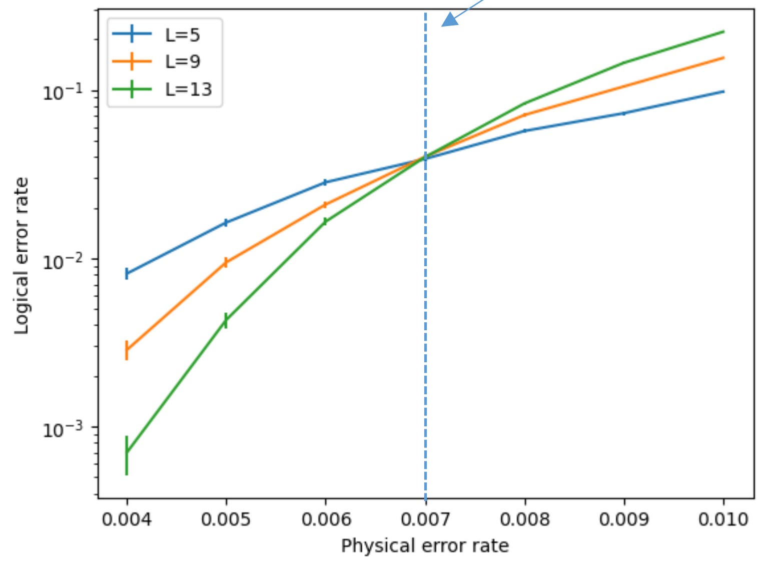
Surface Code Threshold

- p: physical error rate of each qubit
- p_L: logical error rate of logical qubit.

- The **threshold** p_{th} is the physical error rate below which the error correction code "works".

- Below threshold: $p_L(d+1) > p_L(d)$

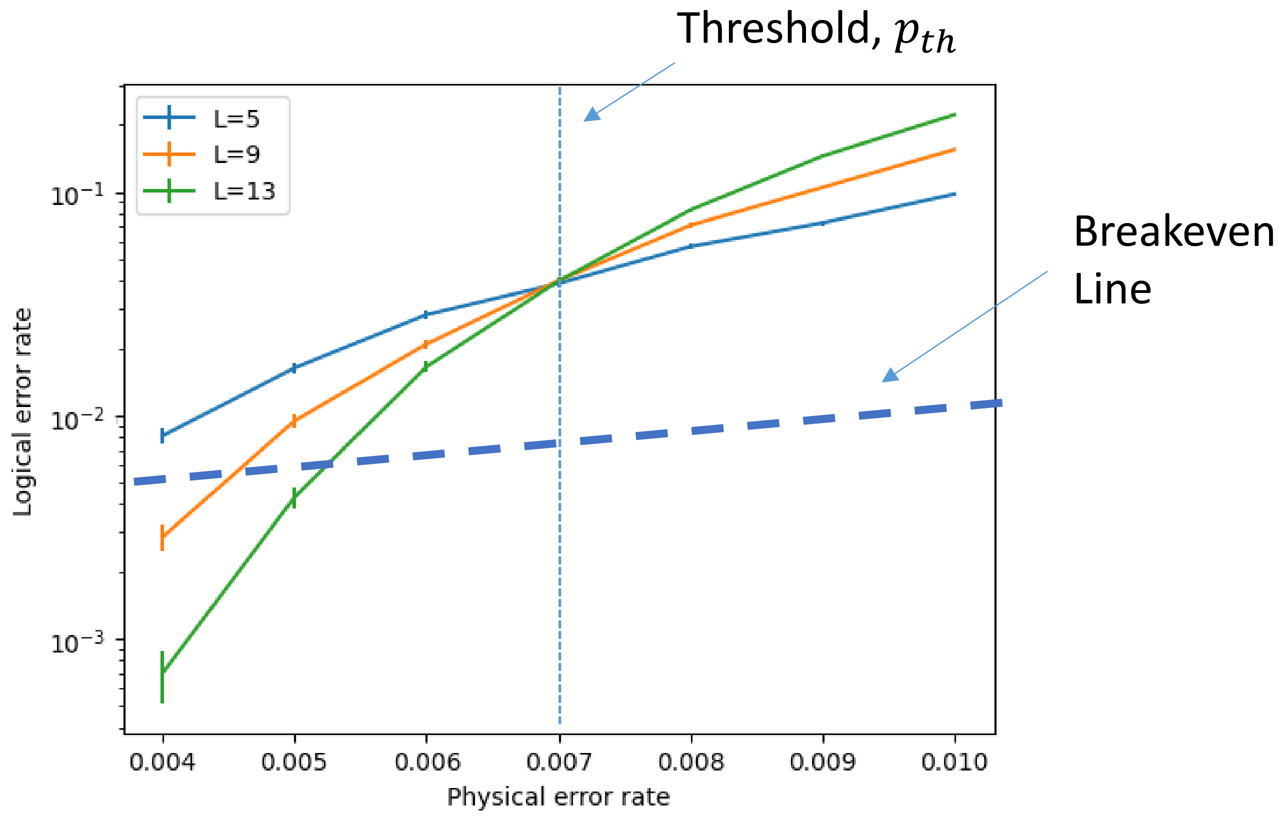
Threshold, p_{th}



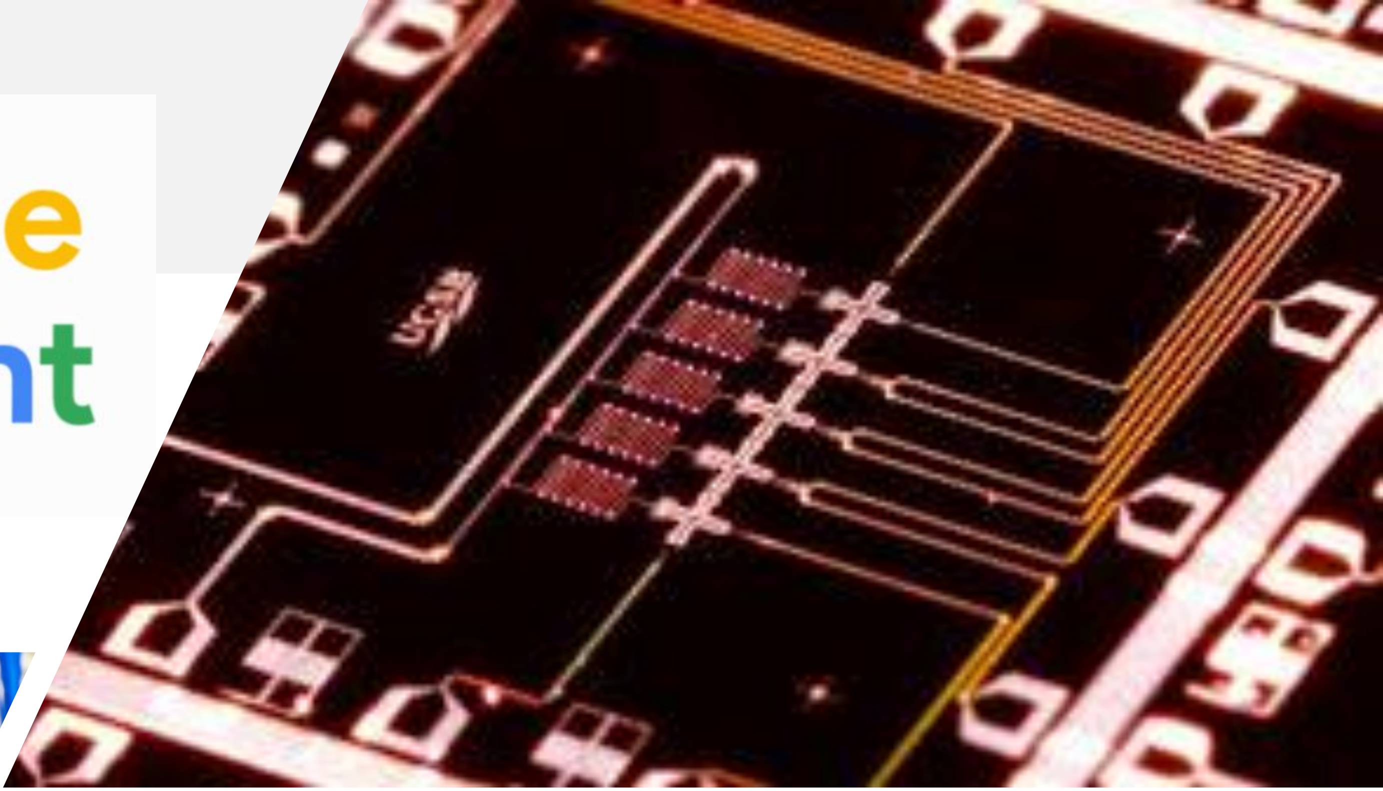
A surface code threshold plot simulated using the PyMatching package.

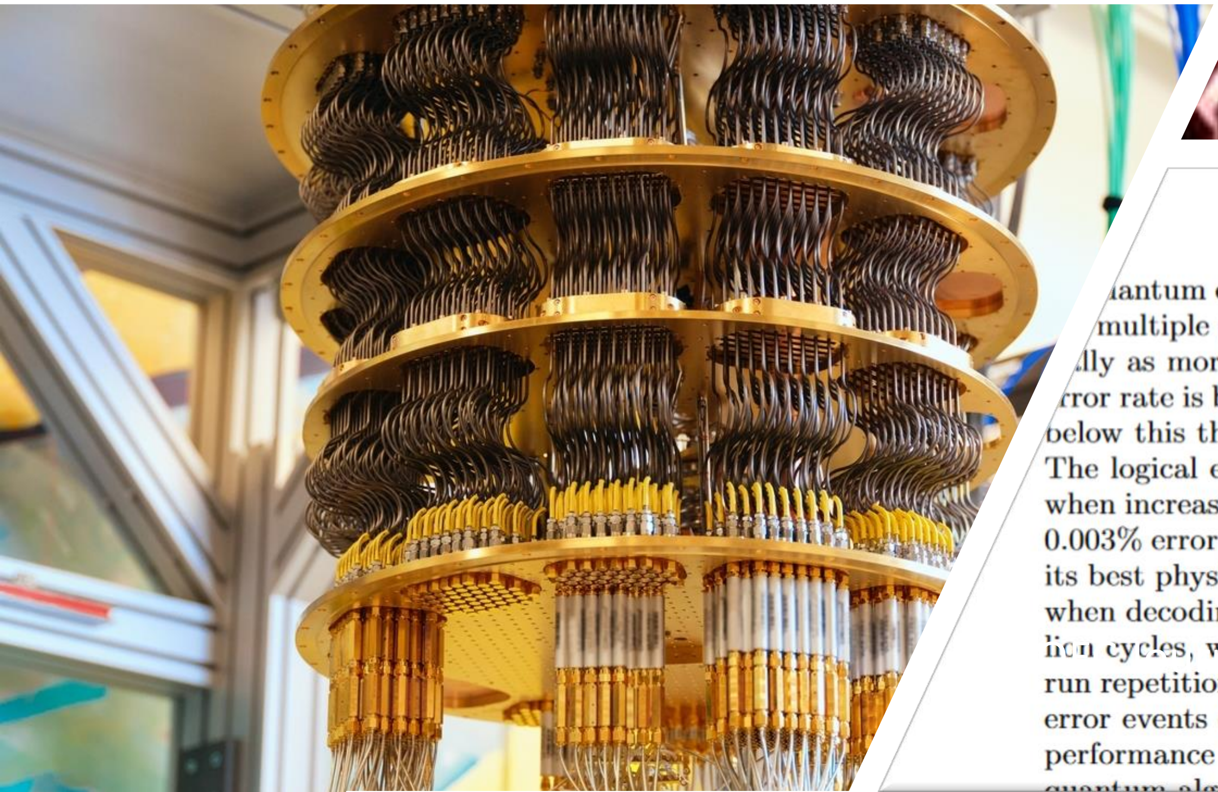
The Threshold and the Breakeven Point

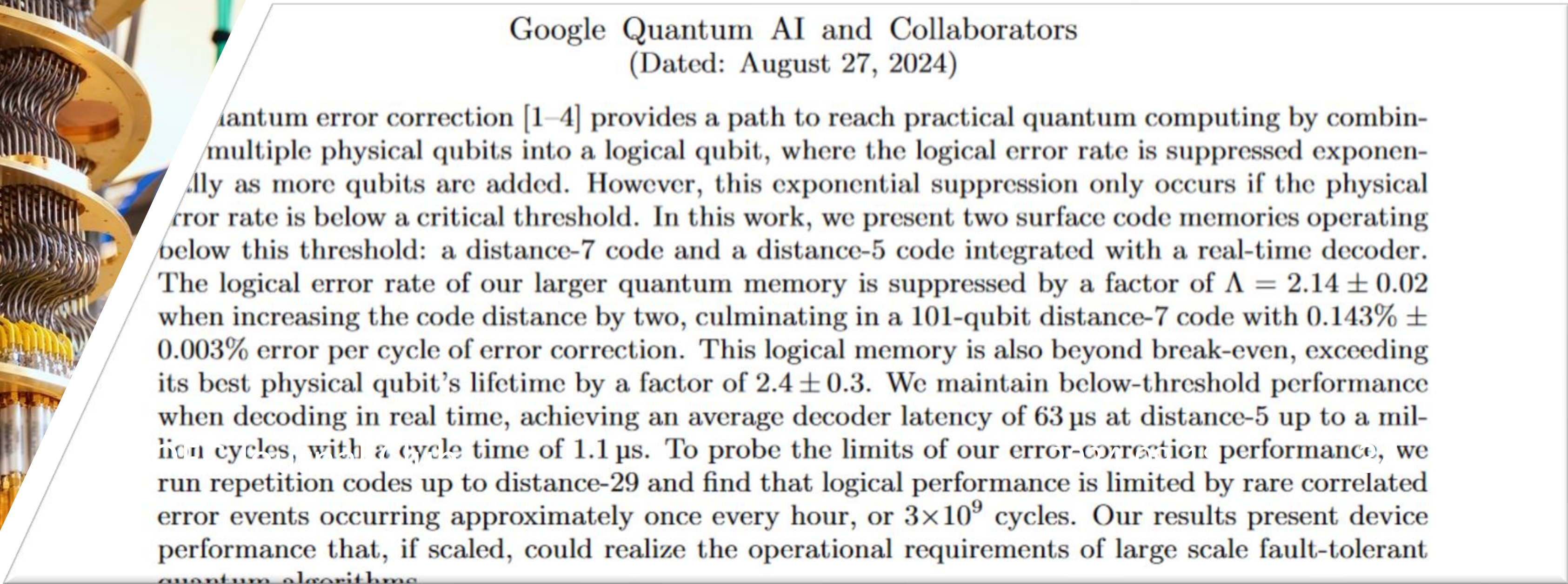
- An error correction code is below the breakeven point if $p_L < p$



Google's Surface Code Experiment







Google Quantum AI and Collaborators
(Dated: August 27, 2024)

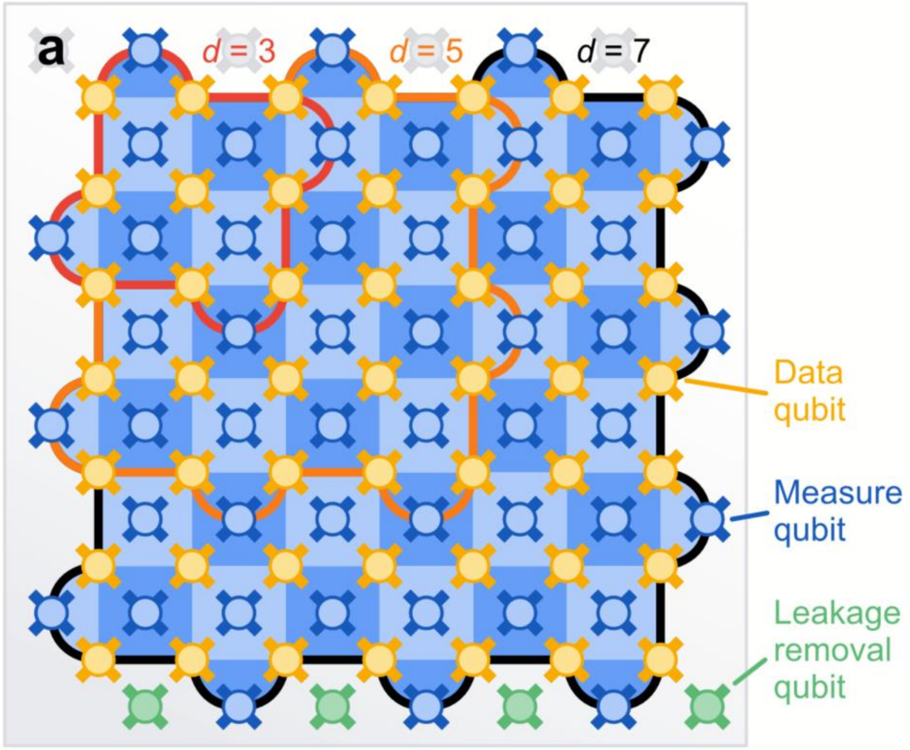
antum error correction [1–4] provides a path to reach practical quantum computing by combin-
multiple physical qubits into a logical qubit, where the logical error rate is suppressed exponen-
lly as more qubits are added. However, this exponential suppression only occurs if the physical
rror rate is below a critical threshold. In this work, we present two surface code memories operating
below this threshold: a distance-7 code and a distance-5 code integrated with a real-time decoder.
The logical error rate of our larger quantum memory is suppressed by a factor of $\Lambda = 2.14 \pm 0.02$ when increasing the code distance by two, culminating in a 101-qubit distance-7 code with $0.143\% \pm 0.003\%$ error per cycle of error correction. This logical memory is also beyond break-even, exceeding its best physical qubit's lifetime by a factor of 2.4 ± 0.3. We maintain below-threshold performance when decoding in real time, achieving an average decoder latency of 63 μs at distance-5 up to a million cycles, with a cycle time of 1.1 μs. To probe the limits of our error-correction performance, we run repetition codes up to distance-29 and find that logical performance is limited by rare correlated error events occurring approximately once every hour, or 3×10^9 cycles. Our results present device performance that, if scaled, could realize the operational requirements of large scale fault-tolerant quantum algorithms.

What have Google Achieved?

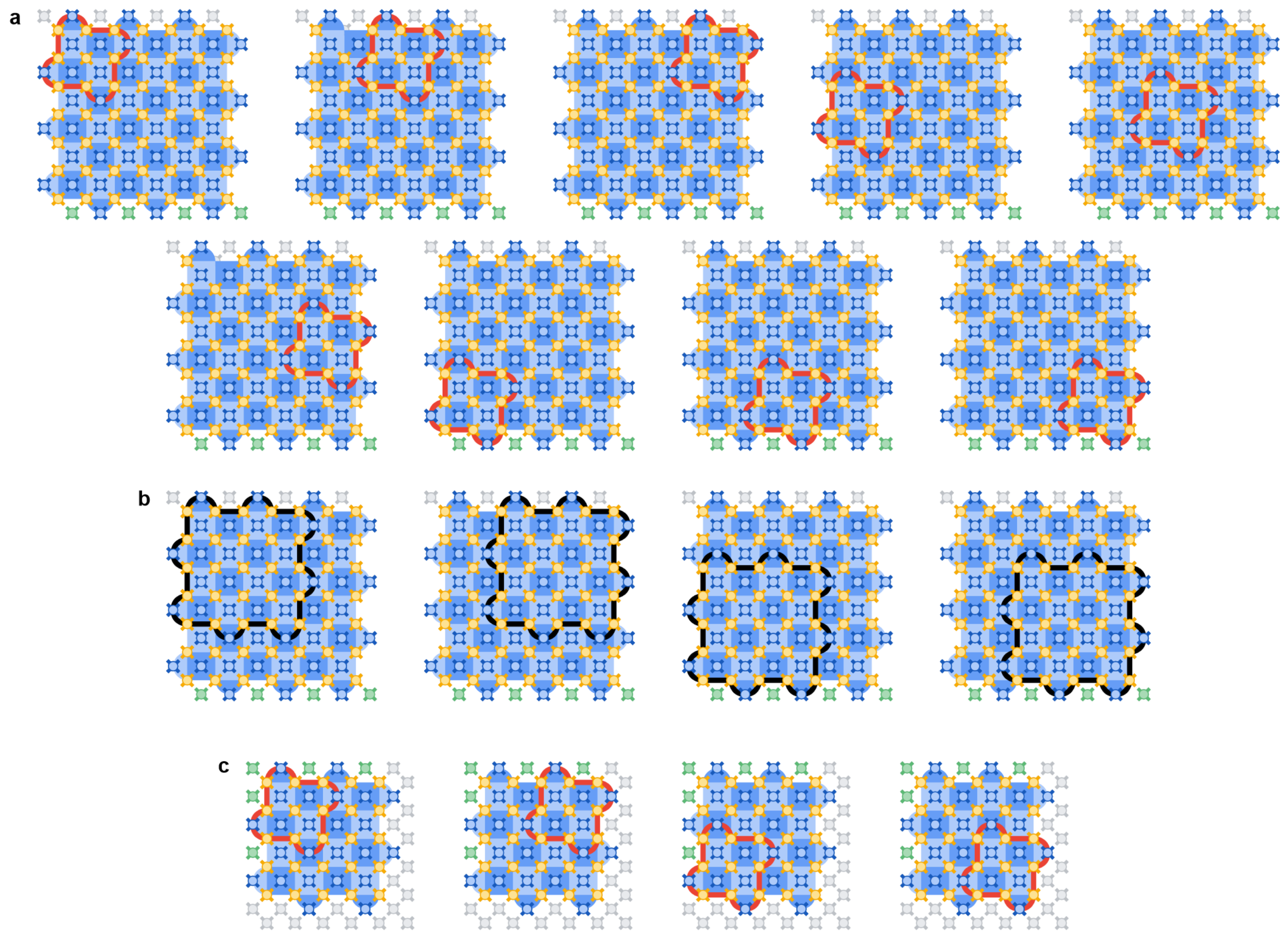
The first demonstration of a fault-tolerant logical qubit operating below the break-even point. **Quantum error correction works!**

Built a quantum processor with 107 superconductor-circuit qubits.

- Implemented distance $d = 3$, $d = 5$, and d=7 surface code.
- Demonstrated operation below the surface code threshold.
- Demonstrated operation below the break-even point for the $d = 5$ and $d = 7$ surface codes.



Google Quantum AI Team
arXiv:2408.13687



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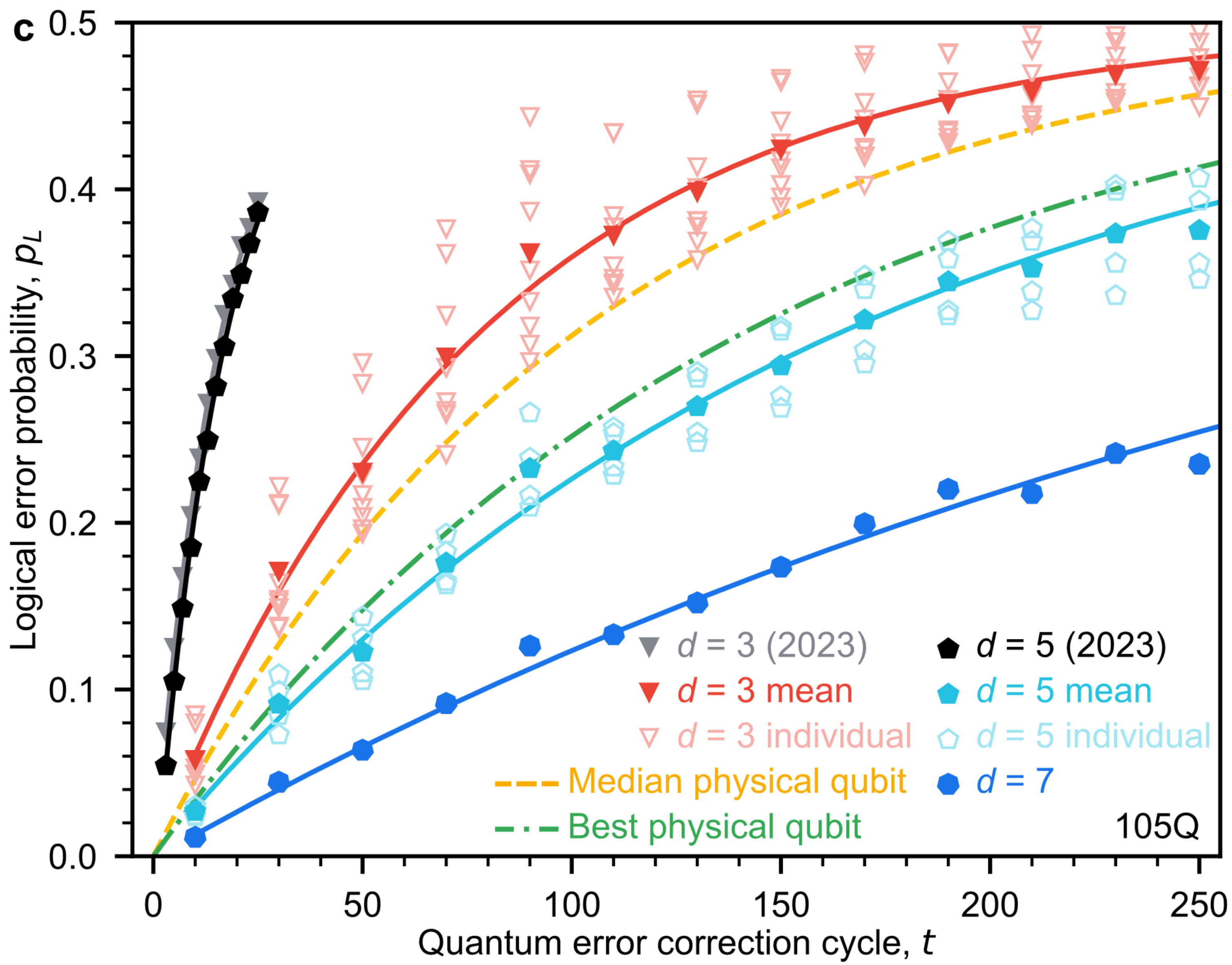
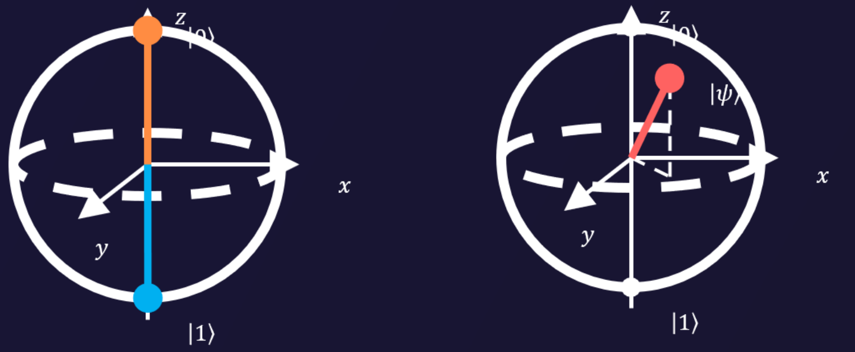
Google's Experimental Results

- Data from arXiv:2408.13687 showing surface code operation over 250 rounds.
- $d = 7$ surface code mean lifetime: 291 ± 6 µs
- Physical qubit mean lifetime: 85±7 µs



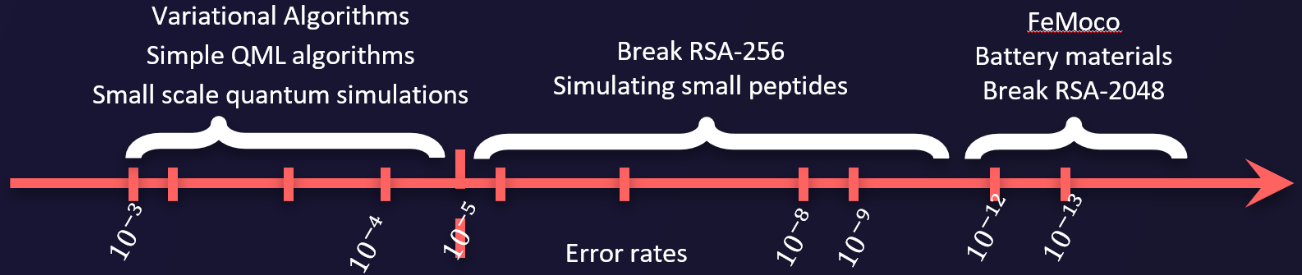
Reaching algorithm ally useful error regimes



Bit Qubit

Probability of Errors

Transistor Gates	Qubits
$p <\approx 1 \times 10^{-15}$	$p \approx 1 \times 10^{-3}$
0.0000000000000001	0.001



Extremely low logical error rates will be required for *operationally* useful quantum computing.

Figure credit: Tamas Noszko, Liam Veeder-Sweeney, Boren Gu, Adi Sireesh.

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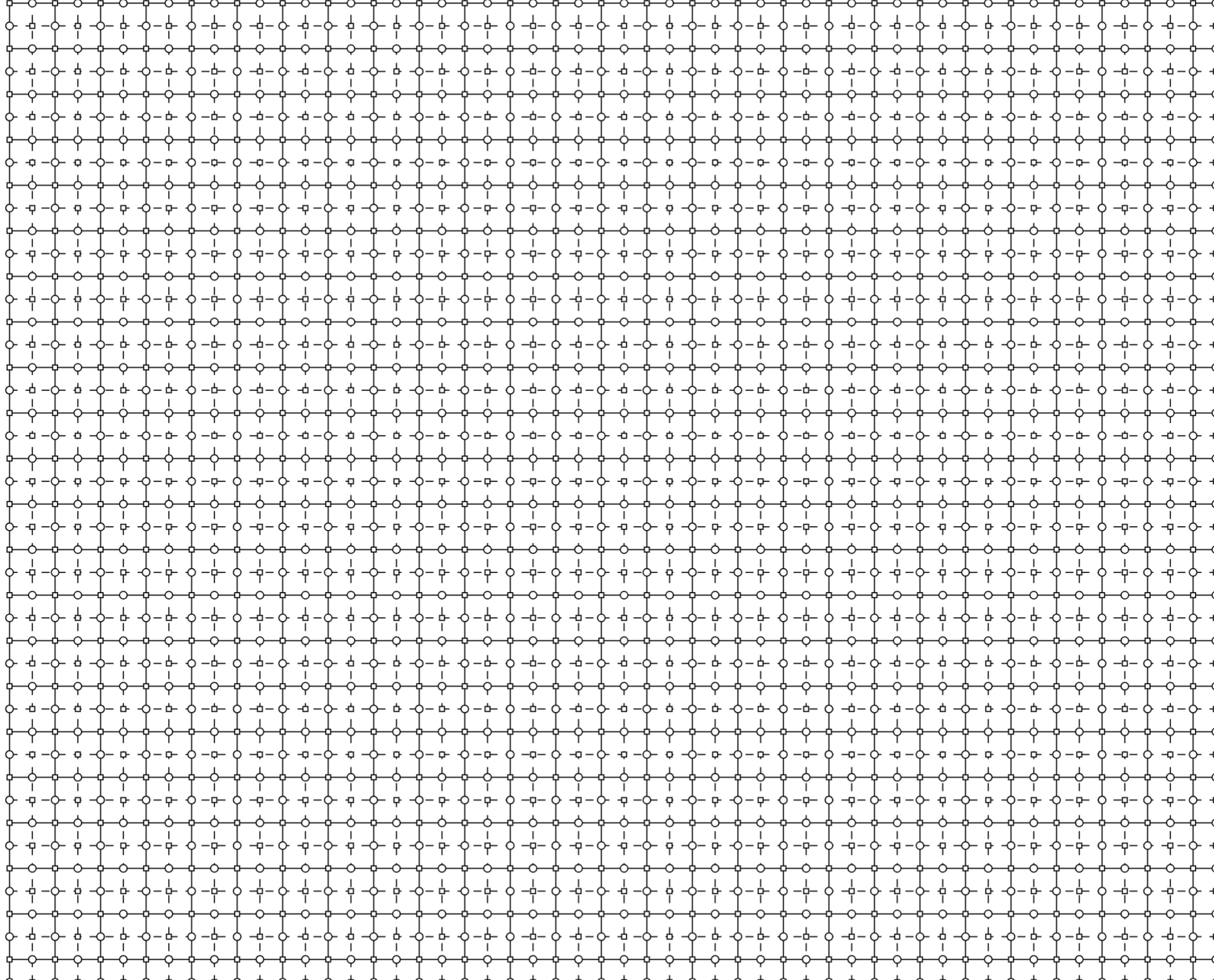
Q: What sort of distances will we need to supress the logical error rate to algorithmically useful levels $p_L \approx 10^{-12}$

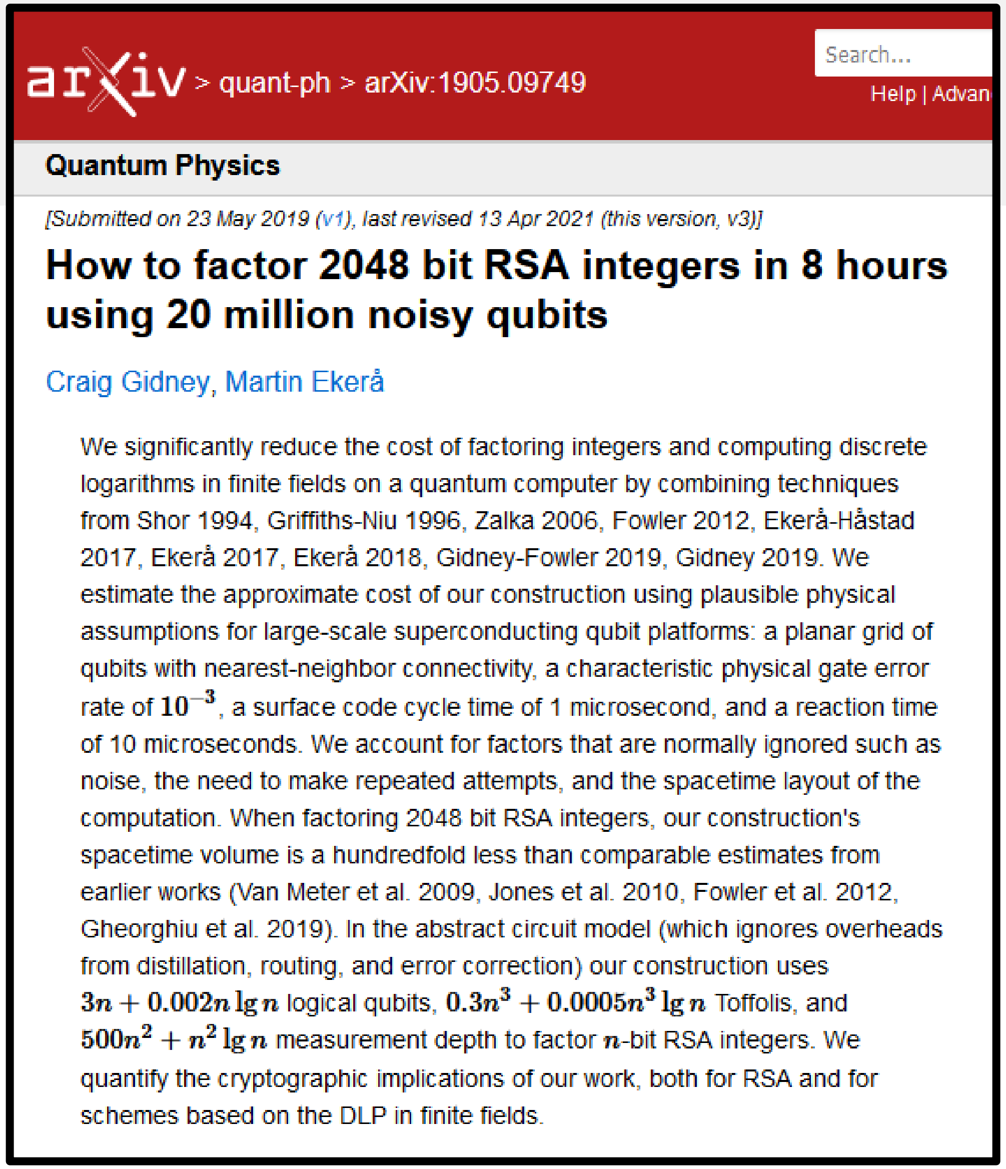
A: Estimates suggest $d \approx 27$ will be required. This corresponds to a surface code patch with > 1000 physical qubits.

Over 99.9% of the qubits in a quantum computer will be dedicated to error correction!



When will surface code quantum computers be useful?

- The Google Team have implemented surface codes on a 107 qubit chip.
- Estimates suggest operationally useful quantum computers will require surface codes of size $d \approx 27$ on chips >1000 qubits.



Quantum Physics

[Submitted on 23 May 2019 (v1), last revised 13 Apr 2021 (this version, v3)]

How to factor 2048 bit RSA integers in 8 hours using 20 million noisy qubits

Craig Gidney, Martin Ekerå

We significantly reduce the cost of factoring integers and computing discrete logarithms in finite fields on a quantum computer by combining techniques from Shor 1994, Griffiths-Niu 1996, Zalka 2006, Fowler 2012, Ekerå-Håstad 2017, Ekerå 2017, Ekerå 2018, Gidney-Fowler 2019, Gidney 2019. We estimate the approximate cost of our construction using plausible physical assumptions for large-scale superconducting qubit platforms: a planar grid of qubits with nearest-neighbor connectivity, a characteristic physical gate error rate of 10^{-3}, a surface code cycle time of 1 microsecond, and a reaction time of 10 microseconds. We account for factors that are normally ignored such as noise, the need to make repeated attempts, and the spacetime layout of the computation. When factoring 2048 bit RSA integers, our construction's spacetime volume is a hundredfold less than comparable estimates from earlier works (Van Meter et al. 2009, Jones et al. 2010, Fowler et al. 2012, Gheorghiu et al. 2019). In the abstract circuit model (which ignores overheads from distillation, routing, and error correction) our construction uses $3n + 0.002n \lg n$ logical qubits, $0.3n^3 + 0.0005n^3 \lg n$ Toffolis, and $500n^2 + n^2 \lg n$ measurement depth to factor n-bit RSA integers. We quantify the cryptographic implications of our work, both for RSA and for schemes based on the DLP in finite fields.

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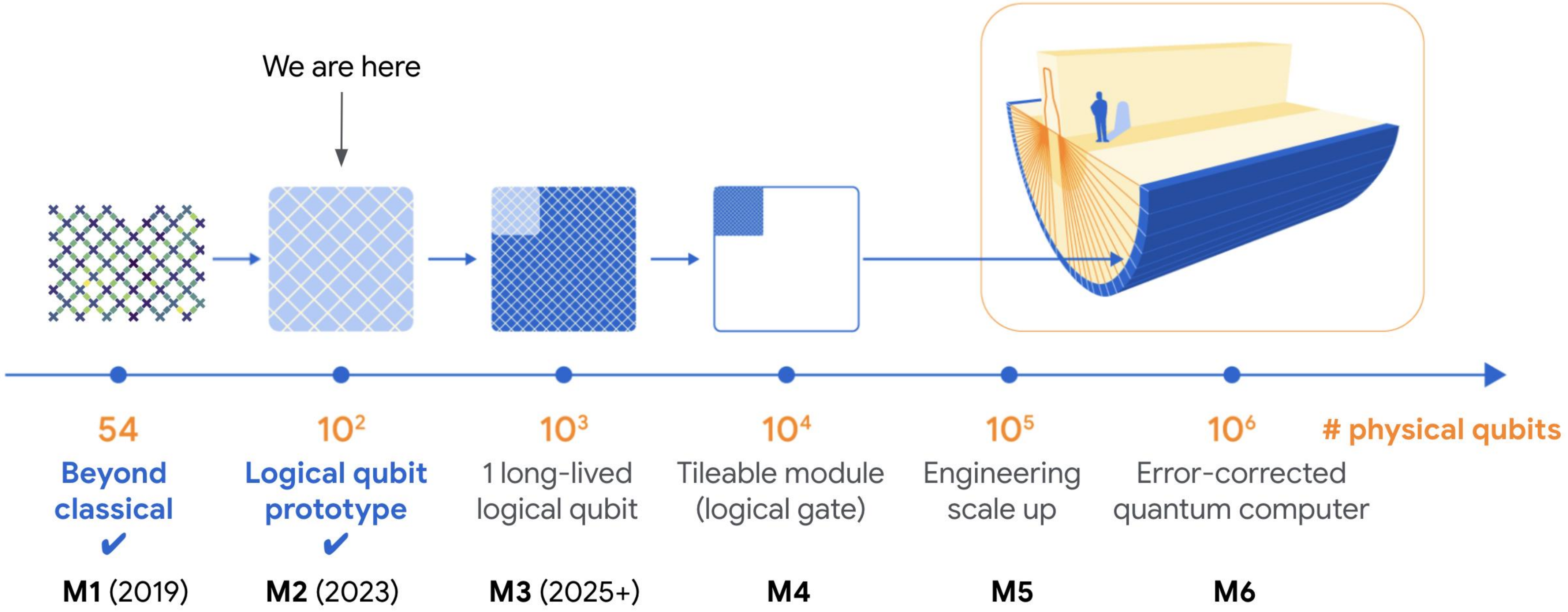
Google's Roadmap to Fault-Tolerance



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