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Tutorial 1

Problem 1: Quantum Operations

The *Hadamard* gate plays a very prominent role in quantum computation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

a. Prove that H is unitary, i.e. that it satisfies $HH^\dagger = H^\dagger H = I$.

Solution: In order to prove that a matrix U is unitary, it must satisfy $UU^\dagger = U^\dagger U = I$. First, we have to calculate the adjoint of the Hadamard operator. Recall that the matrix elements of the adjoint operator are related to that of the operator as $H^\dagger_{ij} = H^*_{ji}$. Thus:

$$H^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

So we can see that $H^\dagger = H$. In order for H to be unitary then the following must hold:

$$H^\dagger H = HH^\dagger = I$$

We have:

$$HH^\dagger = H^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

and thus $H^\dagger H = HH^\dagger = I$

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b. Prove that H is its own inverse by showing $H^2 = I$ where I is the identity operator.

Solution: This is a corollary of the previous result.

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c. Calculate the action of the operator on the vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution: We will show what is the action of the Hadamard on the computational basis vector $|0\rangle$ and on the vector $|+\rangle$.

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

We can see how H acts on the $|+\rangle$ by doing the matrix multiplication, but we can think of a more "clever" way. We proved on question b. that $H^2 = I$. So,

$$H|+\rangle = H(H|0\rangle) = H^2|0\rangle = |0\rangle$$

You can work in the same way with the other two examples and prove that $H|1\rangle = |-\rangle$ and that $H|-\rangle = |1\rangle$.

Extra information: $H^\dagger = H$ makes Hadamard a *Hermitian operator* and so $H^\dagger H = HH^\dagger$. The operators that satisfy $AA^\dagger = A^\dagger A$ are called *normal operators*.

Problem 2: Pauli matrices

Consider the four Pauli matrices:

$$I, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

a. Prove that for each Pauli matrix σ_i we have $\sigma_i^2 = I$ and $\sigma_i^\dagger = \sigma_i$.

Solution: We'll only make the proof for the Y Pauli matrix, but you should do the exact calculations on the rest. We have:

$$Y^2 = YY = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

We will also prove that Y satisfies $Y^\dagger = Y$ (i.e., is Hermitian):

$$Y^\dagger = \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^T \right]^* = \left[\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right]^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

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b. Show that the Pauli matrices are unitary matrices.

Solution: We proved that for all Pauli matrices $\sigma_i^\dagger = \sigma_i$ and that $\sigma_i^2 = I$. Clearly then, $\sigma_i^2 = \sigma_i \sigma_i = \sigma_i^\dagger \sigma_i = \sigma_i \sigma_i^\dagger = I$.

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c. Show that $Y = iXZ$.

Solution: We have:

$$iXZ = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

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d. Show that $HXH = Z$ and $HZH = X$.

Solution: First, we will prove that $HXH = Z$. We have:

$$
\begin{aligned}
HXH &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix} = Z
\end{aligned}
$$

We can work in the exact same way for $HZH = X$ or we can prove it differently. We proved that $HXH = Z$. We do a left and right multiplication with H and so we have $HHXHH = HZH$. But $H^2 = I$ and so $X = HZH$.

Problem 3: Measurement

Consider two quantum states $|L\rangle$ and $|R\rangle$ (eigenvalues of Pauli Y operator):

$$
|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)
$$
$$
|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)
$$

a. Consider the general quantum state:

$$
|\psi\rangle = \psi_0\,|0\rangle + \psi_1\,|1\rangle
$$

What are the probabilities of outcome $|R\rangle$ and $|L\rangle$ if we measure $|\psi\rangle$.

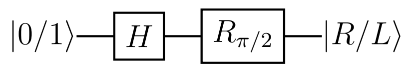
Solution: We start with the probability of measuring the outcome $|R\rangle$. If we measure the state $|\psi\rangle$, the probability of measuring $|R\rangle$ is given by:

$$
\begin{aligned}
Pr[R] = |\langle R|\psi\rangle|^2 &= \left|\frac{1}{\sqrt{2}}(\langle 0| - i\,\langle 1|)(\psi_0\,|0\rangle + \psi_1\,|1\rangle)\right|^2\\
&= \frac{1}{2}\,|\psi_0 - i\psi_1|^2
\end{aligned}
$$

where we used $\langle 0|1\rangle = 0$, since the vectors are orthogonal. Similarly, for the other probability we have:

$$
\begin{aligned}
Pr[L] = |\langle L|\psi\rangle|^2 &= \left|\frac{1}{\sqrt{2}}(\langle 0| + i\,\langle 1|)(\psi_0\,|0\rangle + \psi_1\,|1\rangle)\right|^2\\
&= \frac{1}{2}\,|\psi_0 + i\psi_1|^2
\end{aligned}
$$

b. Show that the states $|L\rangle$ and $|R\rangle$ can be generated from $|0\rangle$ and $|1\rangle$ using the following circuit:



where

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

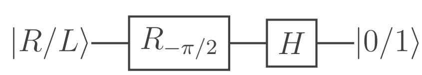
Solution:

$$R_{\pi/2}H|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = |R\rangle$$

$$R_{\pi/2}H|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = |L\rangle$$

c. What circuit will allow implementing a measurement on the $|L\rangle$ and $|R\rangle$ basis if our hardware only allows for measurement in the computational basis? Use H and R_θ gates.

Solution: In 1a) we saw that H is its own inverse, and it can be verified that $R_{-\pi/4}$ is inverse of $R_{\pi/4}$ gate, i.e. $R_{-\pi/4}R_{\pi/4} = R_{\pi/4}R_{-\pi/4} = I$. Hence, the circuit



can be used to map $|R\rangle$ and $|L\rangle$ into a computational basis where they can be distinguished by the measurement.