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Tutorial 2

Problem 1: Tensor Product

a. Consider the quantum state:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{\sqrt{1}}{2}|1\rangle$$

1. Calculate $|\psi\rangle^{\otimes 2}$, where $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$.
2. Calculate $|+\rangle \otimes |-\rangle \otimes |+\rangle$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

Solution: There are two ways to represent the tensor product $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$. One could use either the "bra-ket" notation or the matrix representation. Starting with the "bra-ket" notation, we have:

$$\begin{aligned} |\psi\rangle^{\otimes 2} = |\psi\rangle \otimes |\psi\rangle &= \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{\sqrt{1}}{2}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{\sqrt{1}}{2}|1\rangle\right) \\ &= \frac{3}{4}|0\rangle \otimes |0\rangle + \frac{\sqrt{3}}{4}|0\rangle \otimes |1\rangle + \frac{\sqrt{3}}{4}|1\rangle \otimes |0\rangle + \frac{1}{4}|1\rangle \otimes |1\rangle \\ &= \frac{3}{4}|00\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle, \end{aligned}$$

where we used the simplification of notation $|xy\rangle = |x\rangle \otimes |y\rangle$. However, if we use the matrix representation and recall that $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ we can write $|\psi\rangle$ in the matrix form:

$$|\psi\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$$

Thus, we can easily calculate $|\psi\rangle^{\otimes 2}$ as:

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3\\\sqrt{3}\\\sqrt{3}\\1\end{pmatrix}$$

The two representations are equivalent and can easily be verified if we calculate the matrix representations of the basis vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

For the second case, $|+\rangle|-\rangle|+\rangle$, working in the exact same way we can verify that:

$$|+\rangle|-\rangle|+\rangle = \frac{1}{2^{3/2}}\left(|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle\right),$$

where we used the simplification of notation $|xyz\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle$.

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b. Consider the four Pauli matrices:

$$I, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Calculate the tensor products:

1. $Z \otimes X$
2. $X \otimes Y$
3. Compute the product of the two 4×4 matrices $(Z \otimes X) \times (X \otimes Y)$.
4. Show that $(Z \otimes X) \times (X \otimes Y) = ZX \otimes XY$ to verify the identity $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

Solution: We will calculate step by step the tensor product $Z \otimes X$. We have:

$$Z \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Similarly, for $X \otimes Y$ we get:

$$X \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

The third question is just a matrix multiplication of two 4×4 matrices. Thus,

$$(Z \otimes X) \times (X \otimes Y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

Finally, for the last question, the identity states that we can first do the multiplication between the operators that act on the same subsystems and then take the tensor product. Recall the Pauli matrices properties:

$$ZX = iY = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, XY = iZ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

So,

$$(Z \otimes X) \times (X \otimes Y) = ZX \otimes XY =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

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c. Consider two linear operators A, B.

1. Prove that if A, B are unitary operators, then $(A \otimes B)$ is also unitary.
2. Prove that if A, B are projector operators, then $(A \otimes B)$ is also a projector.

Solution: Consider any two linear operators A and B. It is easy to prove that:

$$(A \otimes B)^* = A^* \otimes B^*$$
$$(A \otimes B)^T = A^T \otimes B^T$$

If we combine the above equations, we get:

$$(A \otimes B)^\dagger = (A^\dagger \otimes B^\dagger)$$

An operator U is called a unitary if it satisfies $UU^\dagger = U^\dagger U = I$. Consider the two unitary operators A, B which satisfy $AA^\dagger = A^\dagger A = I$ and $BB^\dagger = B^\dagger B = I$. Using the identity of Question (b.4) we get:

$$(A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B)$$
$$= (A^\dagger A \otimes B^\dagger B) = (I \otimes I) = I$$

It is easy to see that also:

$$(A \otimes B)(A \otimes B)^\dagger = I$$

and thus we proved that if A, B are unitary operators, then $(A \otimes B)$ is also unitary.

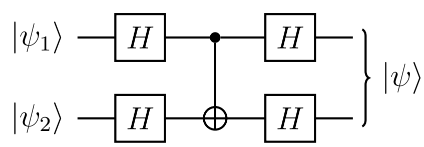
Now for the second question, an operator P is called a projector if it satisfies $P^2 = P$. Using again the identity of Question (b.4) we have

$$(A \otimes B)^2 = (A \otimes B)(A \otimes B) = (A^2 \otimes B^2) = (A \otimes B)$$

and so indeed if A, B are projector operators then $(A \otimes B)$ is also a projector.

Problem 2: Concatenation and composition of gates

Consider the quantum circuit which consists of two Hadamard gates H, followed by a CNOT and finally with two more Hadamard gates:



a. We have seen in the lectures that a CNOT can be written in terms of a linear combination of tensor products of projectors into the computation basis $P_{0/1}$, the identity matrix I and the Pauli matrix X. Using the rules of tensor product, linearity seen in the course, prove that this circuit is equivalent to a CNOT control is now the lower qubit.

Solution: First, recall that the $CNOT$ gate can be written as $CNOT = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = P_0 \otimes I + P_1 \otimes X$

The unitary operator corresponding to the action of the circuit can be written as $U = (H \otimes H)CNOT(H \otimes H)$. It's easy to see that the multiplication gives us:

$$(H \otimes H)CNOT(H \otimes H) = (H \otimes H)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)(H \otimes H)$$
$$\implies (H \otimes H)CNOT(H \otimes H) = P_+ \otimes I + P_- \otimes Z$$

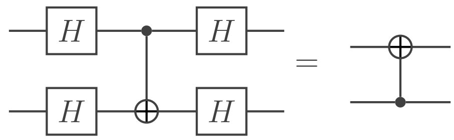
where $P_+ = |+\rangle\langle +|$ and $P_- = |-\rangle\langle -|$ are the projectors in the $\{|+\rangle, |-\rangle\}$ basis. Note that we used the identity0 $HXH = Z$. Now by expanding the unitary in its matrix representation, we get:

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\implies U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In order to understand if this is a well known 2-qubit gate, we can see how it acts on the computational basis states. We have:

$$U|00\rangle = |00\rangle, U|01\rangle = |11\rangle$$
$$U|10\rangle = |10\rangle, U|11\rangle = |01\rangle$$

It is clear then that the circuit U is actually a CNOT gate with the control and target qubit interchanged. Thus:



b. Calculate the output state $|\psi\rangle$ via application of the different quantum gates to the inputs:

1. $|\psi_1\rangle = a|0\rangle + b|1\rangle$ with $a, b \in \mathbb{C}$ is a general quantum state and $|\psi_2\rangle = |0\rangle$.

Solution: The circuit being equivalent to a CNOT controlled by the lower qubit, it is easy to see that the output will leave the state invariant, as $|\psi_2\rangle = |0\rangle$.

2. $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = a|0\rangle + b|1\rangle$ with $a, b \in \mathbb{C}$.

Solution: We start with the state:

$$|\psi\rangle = a|00\rangle + b|01\rangle$$

Then the action of the reversed CNOT is just:

$$U|\psi\rangle = a|00\rangle + b|11\rangle$$

Note that this state is an *entangled state* if both a and b are non-zero, which means that it cannot be written as a tensor product of the subsequent systems, i.e., $|\psi\rangle \neq |\psi_1\rangle \otimes |\phi_2\rangle$ for all $|\psi_1\rangle$ and $|\psi_2\rangle$.

Problem 3: Bell Measurement

a. Consider the four Bell quantum states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

1. Verify that the Bell states form an orthonormal basis of the Hilbert system that describes the composite system.

2. Prove the *completeness relation* $\sum_{i=1}^{4} |u_i\rangle \langle u_i| = I_4$ where $|u_i\rangle$ are the set of four Bell states.

Solution: First of all, recall that for computational basis vectors $|ij\rangle \equiv |i\rangle \otimes |j\rangle$:

$$\langle kl|ij\rangle = \delta_{ki}\delta_{lj}$$

We will prove that $|\Phi^+\rangle$ is orthogonal to $|\Phi^-\rangle$. You are advised to do the same for all other states. We have:

$$\langle \Phi^+|\Phi^-\rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle - |11\rangle) = \frac{1}{2}(\langle 00|00\rangle + \langle 11|00\rangle - \langle 00|11\rangle - \langle 11|11\rangle) = 0$$

so indeed $|\Phi^+\rangle$ and $|\Phi^-\rangle$ are orthogonal. The same applies for every other pair of Bell state. We can also prove easily that all Bell states are normalised to 1. We work again with $|\Phi^+\rangle$:

$$\sqrt{\langle \Phi^+|\Phi^+\rangle} = \sqrt{\frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle)} = \sqrt{\frac{1}{2}(\langle 00|00\rangle + \langle 11|00\rangle + \langle 00|11\rangle + \langle 11|11\rangle)} = 1$$

Thus, the four Bell quantum states do form an orthonormal basis for $H = \mathbb{C}^4$ (i.e. $H = \text{span}\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$) as:

- They are orthogonal
- They are normalised to one.
- The number of basis vectors are the same with the dimension of the Hilbert space.

For the second question, we must calculate all outer products $|u_i\rangle \langle u_i|$ for every bell state $|u_i\rangle$. We have:

$$|\Phi^+\rangle \langle \Phi^+| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, |\Phi^-\rangle \langle \Phi^-| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$|\Psi^+\rangle \langle \Psi^+| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, |\Psi^-\rangle \langle \Psi^-| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So clearly $\sum_{i=1}^{4} |u_i\rangle \langle u_i| = I_4$ where $|u_i\rangle$ are the set of four Bell states, and so we proved the *completeness relation*.

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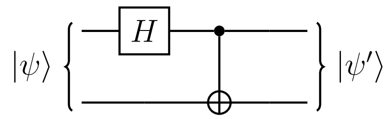
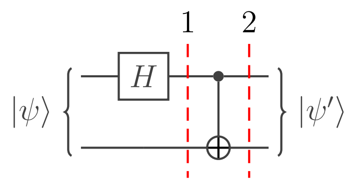
b. Consider the quantum circuit:



Calculate the output state when:

- $|\psi\rangle = |0\rangle |0\rangle$
- $|\psi\rangle = |0\rangle |1\rangle$
- $|\psi\rangle = |1\rangle |0\rangle$
- $|\psi\rangle = |1\rangle |1\rangle$

Solution: We will find the output state for the first case where $|\psi\rangle = |0\rangle |0\rangle$. We split again the circuit into two steps:



Step 1: On the first step, the operator $H \otimes I$ acts on the state $|\psi\rangle$ and gives us:

$$|\psi\rangle_1 = (H \otimes I)(|0\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Step 2: On the second step, we act with the CNOT and get:

$$|\psi'\rangle = CNOT\,|\psi\rangle_1 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

Working in the exact same way for the rest of the inputs you can verify that:

- If $|\psi\rangle = |0\rangle |1\rangle$ then $|\psi'\rangle = |\Psi^+\rangle$.
- If $|\psi\rangle = |1\rangle |0\rangle$ then $|\psi'\rangle = |\Phi^-\rangle$.
- If $|\psi\rangle = |1\rangle |1\rangle$ then $|\psi'\rangle = |\Psi^-\rangle$.

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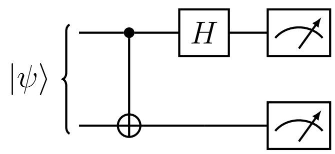
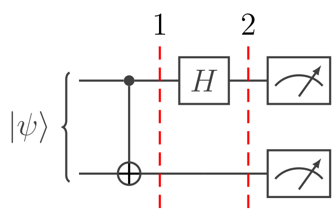
c. Consider the quantum circuit ending in a joint computational measurement of both qubits, leading to four possible outcomes $00, 01, 10$, and 11:



1. If we use the Bell state $|\Psi^+\rangle$ as input to the circuit, what is the probability of each of the 4 possible outcomes?
2. And what about when we use $|\Psi^-\rangle$ and $|\Phi^\pm\rangle$ as input?
3. What are the outcome probabilities resulting from the input state $|00\rangle$?
4. What will be the outcome probabilities when we input any state of the computational basis of the two qubits, i.e., $|x_1\rangle \otimes |x_2\rangle$ where $x_i \in \{0, 1\}$?

Solution: First of all, we consider as input the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Similarly to the previous exercises we break the quantum circuit into three parts with the first two parts referring to the action of the quantum gates and the third part being the measurement:



Step 1: We act with the CNOT gate and thus we get:

$$|\psi\rangle_1 = CNOT\,|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |1\rangle$$

Step 2: We act with the Hadamard operator on the first qubit $(H \otimes I)$:

$$|\psi\rangle_2 = (H \otimes I)\,|\psi\rangle_1 = (H \otimes I)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |1\rangle = |01\rangle$$

So we will measure with 100% probability the computational basis state $|01\rangle$. For the other three input states $|\Psi^-\rangle$, $|\Phi^+\rangle$ and $|\Phi^-\rangle$ we will measure with 100% probability the computational basis states $|11\rangle$, $|00\rangle$ and $|10\rangle$ respectively.

Now for the case where $|\psi\rangle = |00\rangle$ we have:

Step 1:

$$|\psi\rangle_1 = |00\rangle$$

Step 2:

$$|\psi\rangle_2 = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

In this case, we can measure $|00\rangle$ or $|10\rangle$ with an equal probability of 1/2. For question 4, we can work in the same manner to get:

- If the input state is $|\psi\rangle = |01\rangle$, then the output state is $|\psi'\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$. In this case, we can measure $|01\rangle$ or $|11\rangle$ with an equal probability of 1/2.
- If the input state is $|\psi\rangle = |10\rangle$, then the output state is $|\psi'\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$. In this case, we can measure $|01\rangle$ or $|11\rangle$ with an equal probability of 1/2.
- If the input state is $|\psi\rangle = |11\rangle$, then the output state is $|\psi'\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$. In this case, we can measure $|00\rangle$ or $|10\rangle$ with an equal probability of 1/2.
