Problem 1: Pauli Commutation Relations

1a. Consider the two Pauli operators $P \in \mathcal{P}^{\otimes n}$ and $G \in \mathcal{P}^{\otimes n}$. These operators are said to intersect trivially at position i if $P_i = G_i$ or $P_i, G_i = I$. They intersect non-trivially if $P_i \neq G_i$ and $P_i, G_i \neq I$. Show that P and G will commute if they intersect non-trivially in an even number of locations and anti-commute if they intersect in an odd number of locations.

1b. Do the Pauli operators $X_1Z_2Y_5$ and $X_2Y_5X_7$ commute or anti-commute?

1c. Do the Pauli operators X_1Z_2 and Z_1X_2 commute or anti-commute?

Problem 2: The two-qubit repetition code for phase flips

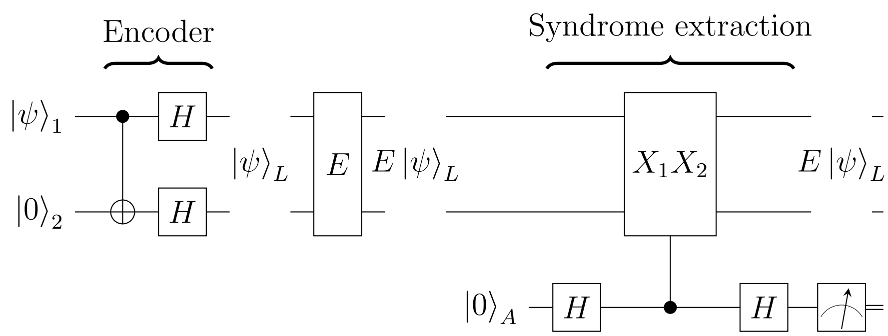

Figure 1: The two-qubit repetition code for phase flips

Figure 1 shows the two-qubit repetition code protocol for detecting phase-flip errors.

2a. What are the logical basis states of this code?

2b. Show that the stabiliser generator X_1X_2 acts as the identity on the basis states.

2c. Show that immediately before the measurement of auxiliary qubit A the system is in the following state:

$$\frac{1}{2}(I + X_1X_2)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - X_1X_2)E|\psi\rangle_L|1\rangle_A$$

2d. Show that the measurement of auxiliary qubit A_1 yields '0' if $[E, X_1X_2] = 0$ and '1' if $\{E, X_1X_2\} = 0$.

2e. Complete syndrome table (Tab 1).

2f. Identify an X_L and Z_L logical operator for this code. Show that these operators have the correct action on the logical basis states.

2g. What is the distance of this code?

Error	$\mathbf{s}_1$
$I_1 \otimes I_2$	
$X_1 \otimes I_2$	
$I_1 \otimes X_2$	
$X_1 \otimes I_2$	
$X_1 \otimes X_2$	
$I_1 \otimes Z_2$	
$Z_1 \otimes I_I$	
$Z_1 \otimes Z_2$	

Table 1: Syndrome table for the 2-qubit repetition code for phase flips.

Problem 3: The Five-Qubit Code

The five-qubit code is defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1 Z_2 Z_3 X_4 I_5 \\ I_1 X_2 Z_3 Z_4 X_5 \\ X_1 I_2 X_3 Z_4 Z_5 \\ Z_1 X_2 I_3 X_4 Z_5 \end{matrix} \right\rangle$$

3a. How many logical qubits are encoded by this code?

3b. The logical basis states of the five-qubit code are given below.

$$|0_{\mathrm{L}}\rangle = \frac{1}{4}(|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ -|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle),$$

$$|1\rangle_L = \frac{1}{4}(|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\ -|00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle).$$

Show that both $X_L = X_1 X_2 X_3 X_4 X_5$ and $Z_L = Z_1 Z_2 Z_3 Z_4 Z_5$ are a valid choice of logical operators for the code.

3c. Complete the single-qubit syndrome table for this code:

Error	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$
$X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes X_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes X_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes X_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes X_5$				
$Z_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes Z_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes Z_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes Z_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Z_5$				

Table 2: Single-Qubit Syndrome Table (Tab 2) for the Five-Qubit Code.

3d. Explain why this is a correction code with distance $d \geq 3$.

3e. Find a pair of X_L and Z_L logical operators of weight 3.

3f. What are the $[[n, k, d]]$ parameters of this code?

Problem 4: The Surface Code

4a. Figure 2 shows the Tanner graph for a surface code defined over 5 qubits. List the four stabiliser generators that are measured by this code.

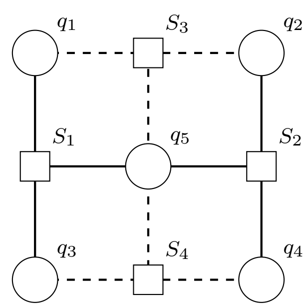

Figure 2: The five-qubit surface code. Dashed edges denote Z-type checks and solid edges X-type checks

4b. How many logical qubits does this code encode?

4c. This code has distance $d = 2$. Find the logical operator pair Z_L, X_L.

4d. Explain why this code is a detection code and not a correction code.

4e. What are the $[[n, k, d]]$ parameters of this code?

4f. Figure 3 shows the Tanner graph for a distance-4 surface code. Two X-errors have occurred on qubits q_{20} and q_6 activating a non-zero syndrome measurement for stabilisers S_{17} and S_{19}. Explain why $\mathcal{R} = X_6 X_{20}$ and $\mathcal{R}' = X_{10} X_{21}$ are both suitable recovery operations.

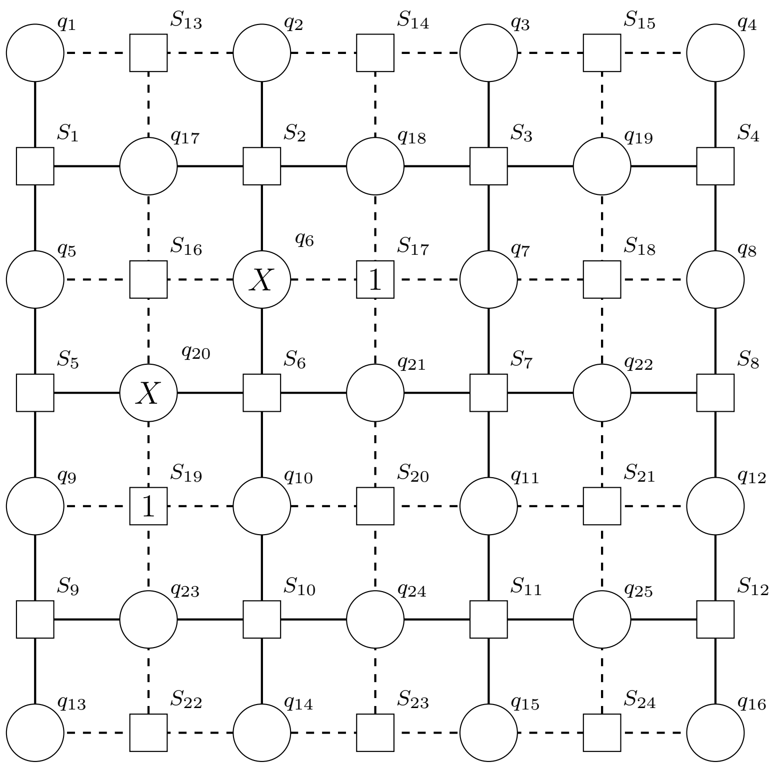

Figure 3: The distance-4 surface code. Dashed edges denote Z-type checks and solid edges X-type checks.

4g. The recovery operator $\mathcal{R}'' = X_7 X_8 X_9$ would also reset the total syndrome of the surface code. Explain why this is not a suitable recovery operator.