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Tutorial 2

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Problem 1: Tensor Product

a. Consider the quantum state:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{\sqrt{1}}{2}|1\rangle$$

1. Calculate $|\psi\rangle^{\otimes 2}$, where $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$.
2. Calculate $|+\rangle \otimes |-\rangle \otimes |+\rangle$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

b. Consider the four Pauli matrices:

$$I, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Calculate the tensor products:

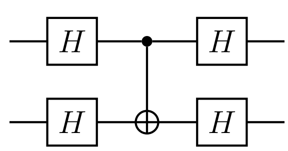
1. $Z \otimes X$
2. $X \otimes Y$
3. Using the two tensor products $Z \otimes X$ and $X \otimes Y$ above, compute their multiplication to obtain $(Z \otimes X)(X \otimes Y)$.
4. Compute the two 2×2 matrix products ZX and XY followed by the tensor products of their results to obtain $ZX \otimes XY$. Show that $(Z \otimes X)(X \otimes Y) = ZX \otimes XY$.

c. Consider two linear operators A, B.

1. Prove that if A, B are unitary operators, then $(A \otimes B)$ is also unitary.
2. Prove that if A, B are projector operators, then $(A \otimes B)$ is also a projector.

Problem 2: Concatenation and composition of gates

Consider the quantum circuit which consists of two Hadamard gates H, followed by a CNOT and finally with two more Hadamard gates:



a. We have seen in the lectures that a CNOT can be written in terms of a linear combination of tensor products of projectors into the computation basis, $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$, the identity matrix I and the Pauli matrix X. Using the rules of tensor product and linearity seen in the course, prove that this circuit is equivalent to reversed CNOT (control is on the lower qubit).

b. Calculate the output state $|\psi\rangle$ via application of the different quantum gates to the inputs:

1. $|\psi_1\rangle = a\,|0\rangle + b\,|1\rangle$ with $a, b \in \mathbb{C}$ is a general quantum state and $|\psi_2\rangle = |0\rangle$.
2. $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = a\,|0\rangle + b\,|1\rangle$ with $a, b \in \mathbb{C}$.

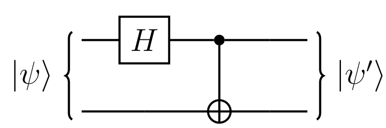
Problem 3: Measurement on Bell state basis

a. Consider the four Bell quantum states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

1. Verify that the Bell states form an orthonormal basis of the Hilbert system that describes the composite system.
2. Prove the *completeness relation* $\sum_{i=1}^{4} |u_i\rangle\langle u_i| = I_4$ where $|u_i\rangle$ are the set of four Bell states.

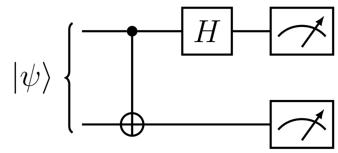
b. Consider the quantum circuit:



Calculate the output state when:

- $|\psi\rangle = |0\rangle\,|0\rangle$
- $|\psi\rangle = |0\rangle\,|1\rangle$
- $|\psi\rangle = |1\rangle\,|0\rangle$
- $|\psi\rangle = |1\rangle\,|1\rangle$

c. Consider the quantum circuit ending in a joint computational measurement of both qubits, leading to four possible outcomes $00, 01, 10$, and 11:



1. If we use the Bell state $|\Psi^+\rangle$ as input to the circuit, what is the probability of each of the 4 possible outcomes?

2. And what about when we use $|\Psi^-\rangle$ and $|\Phi^\pm\rangle$ as input?

3. What are the outcome probabilities resulting from the input state $|00\rangle$?

4. What will be the outcome probabilities when we input any state of the computational basis of the two qubits, i.e., $|x_1\rangle \otimes |x_2\rangle$ where $x_i \in \{0, 1\}$?