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Tutorial 9

Problem 1: Pauli Commutation Relations

1a. Consider the two Pauli operators $P \in \mathcal{P}^{\otimes n}$ and $G \in \mathcal{P}^{\otimes n}$. These operators are said to intersect trivially at position i if $P_i = G_i$ or $P_i, G_i = I$. They intersect non-trivially if $P_i \neq G_i$ and $P_i, G_i \neq I$. Show that P and G will commute if they intersect non-trivially in an even number of locations and anti-commute if they intersect in an odd number of locations.

Solution:

1. Let N be the number of qubits where P_i and G_i both act non-trivially ($P_i, G_i \neq I$) and $P_i \neq G_i$.
2. At each such qubit, P_i and G_i are different Pauli matrices, so they anti-commute: $P_i G_i = -G_i P_i$.
3. The total commutation factor is $(-1)^N$: $PG = (-1)^N GP$.
4. If N is even, $(-1)^N = 1$, so P and G commute.
5. If N is odd, $(-1)^N = -1$, so P and G anti-commute.

1b. Do the Pauli operators $X_1 Z_2 Y_5$ and $X_2 Y_5 X_7$ commute or anti-commute?

Solution: They anti-commute as they intersect non-trivially only once on qubit 2.

1c. Do the Pauli operators $X_1 Z_2$ and $Z_1 X_2$ commute or anti-commute?

Solution: They commute as they intersect non-trivially an even number of times on qubits 1 and 2.

Problem 2: The two-qubit repetition code for phase flips

Figure 1 shows the two-qubit repetition code protocol for detecting phase-flip errors.

2a. What are the logical basis states of this code?

Solution: The logical basis states are $|0\rangle_L = |{+}{+}\rangle$ and $|1\rangle = |{-}{-}\rangle$.

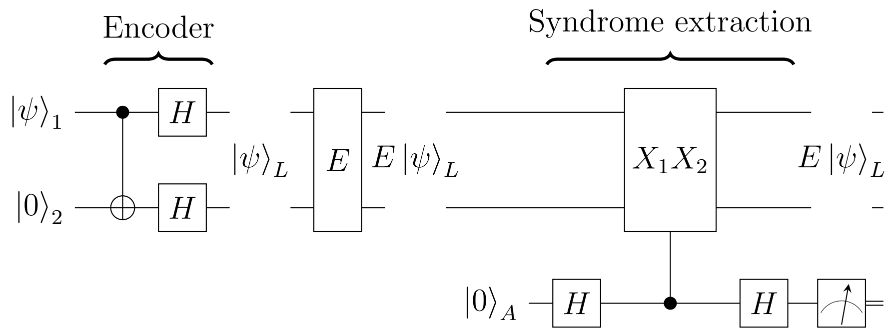

Figure 1: The two-qubit repetition code for phase flips

2b. Show that the stabiliser generator X_1X_2 acts as the identity on the basis states.

Solution: The Pauli-X operator acts as follows on the congugate basis states: $X|+\rangle = |+\rangle$ and $X|-\rangle = (-1)|-\rangle$. The X_1X_2 stabiliser therefore has the following action on the basis states:

$$X_1X_2|{+}{+}\rangle = |{+}{+}\rangle$$
$$X_1X_2|{-}{-}\rangle = (-1)(-1)|{-}{-}\rangle = |{-}{-}\rangle$$

2c. Show that immediately before the measurement of auxiliary qubit A the system is in the following state:

$$\frac{1}{2}(I + X_1X_2)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - X_1X_2)E|\psi\rangle_L|1\rangle_A$$

Solution:

We start with the state after the error has occurred:

$$E|\psi\rangle_L|0\rangle_A$$

where E is the error operator acting on the data qubits, $|\psi\rangle_L$ is the encoded logical state, and $|0\rangle_A$ is the initial state of the auxiliary qubit.

Step 1: Apply Hadamard Gate to the Auxiliary Qubit

The Hadamard gate transforms the auxiliary qubit as follows:

$$|0\rangle_A \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$$

The combined state becomes:

$$E\left|\psi\right\rangle_L\left(\frac{1}{\sqrt{2}}(\left|0\right\rangle_A+\left|1\right\rangle_A)\right)=\frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left|0\right\rangle_A+\frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left|1\right\rangle_A$$

Step 2: Apply Controlled-X_1X_2 Gate

The controlled-X_1X_2 gate applies the X_1X_2 operator to the data qubits when the auxiliary qubit is in state $\left|1\right\rangle_A$:

$$\frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left|0\right\rangle_A+\frac{1}{\sqrt{2}}X_1X_2E\left|\psi\right\rangle_L\left|1\right\rangle_A$$

Step 3: Apply Hadamard Gate to the Auxiliary Qubit Again

Applying the Hadamard gate to the auxiliary qubit transforms the states:

$$\left|0\right\rangle_A\xrightarrow{H}\frac{1}{\sqrt{2}}(\left|0\right\rangle_A+\left|1\right\rangle_A)$$
$$\left|1\right\rangle_A\xrightarrow{H}\frac{1}{\sqrt{2}}(\left|0\right\rangle_A-\left|1\right\rangle_A)$$

The total state becomes:

$$\frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left(\frac{1}{\sqrt{2}}(\left|0\right\rangle_A+\left|1\right\rangle_A)\right)+\frac{1}{\sqrt{2}}X_1X_2E\left|\psi\right\rangle_L\left(\frac{1}{\sqrt{2}}(\left|0\right\rangle_A-\left|1\right\rangle_A)\right)$$
$$=\frac{1}{2}E\left|\psi\right\rangle_L(\left|0\right\rangle_A+\left|1\right\rangle_A)+\frac{1}{2}X_1X_2E\left|\psi\right\rangle_L(\left|0\right\rangle_A-\left|1\right\rangle_A)$$

Step 4: Combine Like Terms

Grouping terms with $\left|0\right\rangle_A$ and $\left|1\right\rangle_A$:

$$\text{Coefficient of }\left|0\right\rangle_A:\frac{1}{2}E\left|\psi\right\rangle_L+\frac{1}{2}X_1X_2E\left|\psi\right\rangle_L=\frac{1}{2}(I+X_1X_2)E\left|\psi\right\rangle_L$$
$$\text{Coefficient of }\left|1\right\rangle_A:\frac{1}{2}E\left|\psi\right\rangle_L-\frac{1}{2}X_1X_2E\left|\psi\right\rangle_L=\frac{1}{2}(I-X_1X_2)E\left|\psi\right\rangle_L$$

Therefore, the state immediately before the measurement of the auxiliary qubit A is:

$$\frac{1}{2}(I+X_1X_2)E\left|\psi\right\rangle_L\left|0\right\rangle_A+\frac{1}{2}(I-X_1X_2)E\left|\psi\right\rangle_L\left|1\right\rangle_A$$

This is the desired expression, showing that the system is in the given state before measuring the auxiliary qubit.

2d. Show that the measurement of auxiliary qubit A_1 yields '0' if $[E, X_1X_2] = 0$ and '1' if $\{E, X_1X_2\} = 0$.

Solution:

From part 2c, immediately before the measurement of the auxiliary qubit A, the state of the system is:

$$\frac{1}{2}(I + X_1X_2)E\left|\psi\right\rangle_L\left|0\right\rangle_A + \frac{1}{2}(I - X_1X_2)E\left|\psi\right\rangle_L\left|1\right\rangle_A.$$

The probability of measuring A in state $\left|0\right\rangle_A$ is proportional to the squared norm of the coefficient of $\left|0\right\rangle_A$:

$$P(0) = \left\|\frac{1}{2}(I + X_1X_2)E\left|\psi\right\rangle_L\right\|^2.$$

Similarly, the probability of measuring A in state $\left|1\right\rangle_A$ is:

$$P(1) = \left\|\frac{1}{2}(I - X_1X_2)E\left|\psi\right\rangle_L\right\|^2.$$

Case 1: If E **commutes** with X_1X_2, i.e., $[E, X_1X_2] = 0$, then:

$$X_1X_2E\left|\psi\right\rangle_L = EX_1X_2\left|\psi\right\rangle_L = E\left|\psi\right\rangle_L,$$

since X_1X_2 stabilizes $\left|\psi\right\rangle_L$. Thus:

$$(I + X_1X_2)E\left|\psi\right\rangle_L = (I + I)E\left|\psi\right\rangle_L = 2E\left|\psi\right\rangle_L,$$

and

$$(I - X_1X_2)E\left|\psi\right\rangle_L = (I - I)E\left|\psi\right\rangle_L = 0.$$

Therefore, $P(0) = \|E\left|\psi\right\rangle_L\|^2$ and $P(1) = 0$. The measurement yields outcome 0.

Case 2: If E **anticommutes** with X_1X_2, i.e., $\{E, X_1X_2\} = 0$, then:

$$X_1X_2E\left|\psi\right\rangle_L = -EX_1X_2\left|\psi\right\rangle_L = -E\left|\psi\right\rangle_L.$$

Thus:

$$(I + X_1X_2)E\left|\psi\right\rangle_L = (I - I)E\left|\psi\right\rangle_L = 0,$$

and

$$(I - X_1X_2)E\left|\psi\right\rangle_L = (I + I)E\left|\psi\right\rangle_L = 2E\left|\psi\right\rangle_L.$$

Therefore, $P(0) = 0$ and $P(1) = \|E\left|\psi\right\rangle_L\|^2$. The measurement yields outcome 1.

Thus, the measurement of auxiliary qubit A yields 0 if $[E, X_1X_2] = 0$ and 1 if $\{E, X_1X_2\} = 0$.

2e. Complete syndrome table (Tab 1).

Error	$\mathbf{s}_1$
$I_1 \otimes I_2$	
$X_1 \otimes I_2$	
$I_1 \otimes X_2$	
$X_1 \otimes I_2$	
$X_1 \otimes X_2$	
$I_1 \otimes Z_2$	
$Z_1 \otimes I_I$	
$Z_1 \otimes Z_2$	

Table 1: Syndrome table for the 2-qubit repetition code for phase flips.

Solution:

We complete the syndrome table by determining whether each error operator E commutes or anticommutes with the stabilizer X_1X_2. The syndrome bit $\mathbf{s}_1$ is 0 if E commutes with X_1X_2 and 1 if it anticommutes. The completed syndrome table is shown in Tab 2.

Error	$\mathbf{s}_1$
$I_1 \otimes I_2$	0
$X_1 \otimes I_2$	0
$I_1 \otimes X_2$	0
$X_1 \otimes X_2$	0
$I_1 \otimes Z_2$	1
$Z_1 \otimes I_2$	1
$Z_1 \otimes Z_2$	0

Table 2: Solution: syndrome table

Explanation:

- For errors I_1I_2, X_1I_2, I_1X_2, and X_1X_2, they either act trivially or identically on qubits where the stabilizer acts, resulting in commutation ($\mathbf{s}_1 = 0$).
- Errors involving Z operators (I_1Z_2, Z_1I_2) anticommute with X on the same qubit. Since they differ on one qubit where both act non-trivially and differently, they anticommute ($\mathbf{s}_1 = 1$).
- For Z_1Z_2, the error anticommutes with X on both qubits (an even number), so the total effect is commutation ($\mathbf{s}_1 = 0$).

2f. Identify an X_L and Z_L logical operator for this code. Show that these operators have the correct action on the logical basis states.

Solution:

Identifying Logical Operators:

- **Logical X operator (X_L)**: We choose $X_L = Z_1 Z_2$.
- **Logical Z operator (Z_L)**: We choose $Z_L = X_1$ or $Z_L = X_2$.

Verification of Correct Action on Logical Basis States:

Action of X_L on Logical States:

$$X_L |0\rangle_L = Z_1 Z_2 |{+}{+}\rangle = (Z|+\rangle) \otimes (Z|+\rangle) = |-\rangle \otimes |-\rangle = |{-}{-}\rangle = |1\rangle_L ,$$
$$X_L |1\rangle_L = Z_1 Z_2 |{-}{-}\rangle = (Z|-\rangle) \otimes (Z|-\rangle) = (-|+\rangle) \otimes (-|+\rangle) = |{+}{+}\rangle = |0\rangle_L .$$

Thus, X_L flips the logical states, acting as a logical X operator.

Action of Z_L on Logical States:

$$Z_L |0\rangle_L = X_1 |{+}{+}\rangle = X|+\rangle \otimes |+\rangle = |+\rangle \otimes |+\rangle = |{+}{+}\rangle = |0\rangle_L ,$$
$$Z_L |1\rangle_L = X_1 |{-}{-}\rangle = X|-\rangle \otimes |-\rangle = (-|-\rangle) \otimes |-\rangle = -|{-}{-}\rangle = -|1\rangle_L .$$

Therefore, Z_L leaves $|0\rangle_L$ unchanged and introduces a phase -1 to $|1\rangle_L$, acting as a logical Z operator.

Commutation with Stabilizer:

Both $X_L = Z_1 Z_2$ and $Z_L = X_1$ commute with the stabilizer $X_1 X_2$:

- $[Z_1 Z_2, X_1 X_2] = 0$ because Z_i and X_i anticommute on each qubit, and the anticommutations on qubits 1 and 2 cancel out (even number).
- $[X_1, X_1 X_2] = X_1(X_1 X_2) - (X_1 X_2) X_1 = X_1^2 X_2 - X_1 X_2 X_1 = X_2 - X_2 = 0$.

This confirms that X_L and Z_L are valid logical operators for the code.

Anti-commutation with each other.

The two logical operators X_L and Z_L anti-commute with one another. We can verify this for both choices of the Z_L logical operator:

$$\{Z_1 Z_2, X_1\} = 0$$
$$\{Z_1 Z_2, X_2\} = 0$$

2g. What is the distance of this code?

Solution:

The **distance** of a quantum code is defined as the minimum weight (number of qubits acted upon) of a non-trivial logical operator or, equivalently, the minimum weight of an undetectable error that maps codewords to other codewords without being detected by the stabilizers.

In this code:

- The stabilizer generator is X_1X_2, which stabilizes the code space spanned by $|{+}{+}\rangle$ and $|{-}{-}\rangle$.
- The logical X operator is $X_L = Z_1Z_2$, acting on both qubits.
- The logical Z operator can be chosen as $Z_L = X_1$ or $Z_L = X_2$, each acting on a single qubit.

Minimum Weight of Logical Operators:

- $X_L = Z_1Z_2$ has a weight of 2 since it acts non-trivially on both qubits.
- $Z_L = X_1$ or $Z_L = X_2$ has a weight of 1 since it acts non-trivially on only one qubit.

The minimum weight of a non-trivial logical operator is therefore 1, corresponding to the logical Z operator $Z_L = X_1$ or $Z_L = X_2$.

Implications for Code Distance:

Since the minimum weight of a logical operator is 1, the **distance of the code is 1**.

Error Detection Capability:

- Single-qubit Z errors (phase flips) anticommute with the stabilizer X_1X_2 and are detectable.
- Single-qubit X errors (bit flips), such as X_1 or X_2, act as logical operators Z_L and are undetectable by the stabilizer.

Conclusion:

This code is specifically designed to detect phase-flip errors but not bit-flip errors. The distance being 1 means that the code cannot detect all single-qubit errors, as some single-qubit errors correspond to logical operations and cannot be detected by the stabilizer.

Therefore, while the code can detect single-qubit Z errors, it cannot detect single-qubit X errors. The **distance of the code is 1**.

Problem 3: The Five-Qubit Code

The five-qubit code is defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1 Z_2 Z_3 X_4 I_5 \\ I_1 X_2 Z_3 Z_4 X_5 \\ X_1 I_2 X_3 Z_4 Z_5 \\ Z_1 X_2 I_3 X_4 Z_5 \end{matrix} \right\rangle$$

3a. How many logical qubits are encoded by this code?

Solution: The number of logical qubits k encoded by a stabiliser code is given by:

$$k = n - \text{rank}(\mathcal{S}) = n - |S|$$

For this code, $n = 5$ and $|S| = 4$. The number of logical qubits is therefore

$$k = 5 - 4 = 1$$

3b. The logical basis states of the five-qubit code are given below.

$$|0_\text{L}\rangle = \frac{1}{4}(|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle$$
$$-|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle),$$

$$|1\rangle_L = \frac{1}{4}(|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle$$
$$-|00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle).$$

Show that both $X_L = X_1X_2X_3X_4X_5$ and $Z_L = Z_1Z_2Z_3Z_4Z_5$ are a valid choice of logical operators for the code.

Solution:
We can verify this by checking the action on the logical basis states:

1. $X_L|0\rangle_L = |1\rangle_L$:
   Applying X_L flips all qubits in each term of $|0\rangle_L$, transforming it into $|1\rangle_L$.

2. $Z_L|0\rangle_L = |0\rangle_L$:

   Applying Z_L assigns a phase of $+1$ to each $|0\rangle$ and -1 to each $|1\rangle$. Due to fact each ket of $|0\rangle_L$ has an even number of '1's, the overall state remains unchanged.

3. $Z_L|1\rangle_L = -|1\rangle_L$

   Similarly, applying Z_L to $|1\rangle_L$ introduces a global phase of -1 as each ket has an odd number of '1's, consistent with the logical Z operation.

3c. Complete the single-qubit syndrome table for this code:

Error	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$
$X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes X_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes X_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes X_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes X_5$				
$Z_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes Z_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes Z_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes Z_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Z_5$				

Table 3: Single-Qubit Syndrome Table (Tab 3) for the Five-Qubit Code.

Solution:

The completed syndrome table is shown in Tab 4.

To determine the syndrome bits $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ for each single-qubit error, we check the commutation relation between the error E and each stabilizer generator S_i. The syndrome bit $\mathbf{s}_i$ is set to:

$$\mathbf{s}_i = \begin{cases} 0 & \text{if } [E, S_i] = 0 \quad \text{(commute)} \\ 1 & \text{if } \{E, S_i\} = 0 \quad \text{(anticommute)} \end{cases}$$

Example:

Error $X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$: This error anti-commutes with only S_4. This results in the syndrome $(0, 0, 0, 1)$.

The completed table is in Tab 4.

Error	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$
$X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$	0	0	0	1
$I_1 \otimes X_2 \otimes I_3 \otimes I_4 \otimes I_5$	1	0	0	0
$I_1 \otimes I_2 \otimes X_3 \otimes I_4 \otimes I_5$	1	1	0	0
$I_1 \otimes I_2 \otimes I_3 \otimes X_4 \otimes I_5$	0	1	1	0
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes X_5$	0	0	1	1
$Z_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$	1	0	1	0
$I_1 \otimes Z_2 \otimes I_3 \otimes I_4 \otimes I_5$	0	1	0	1
$I_1 \otimes I_2 \otimes Z_3 \otimes I_4 \otimes I_5$	0	0	1	0
$I_1 \otimes I_2 \otimes I_3 \otimes Z_4 \otimes I_5$	1	0	0	1
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Z_5$	0	1	0	0

Table 4: Single-Qubit Syndrome Table for the Five-Qubit Code

3d. Explain why this is a correction code with distance $d \geq 3$.

Solution:

From the syndrome table, we see that each single-qubit error maps to a unique syndrome. The number of correctable errors t is given by $t = (d-1)/2$. Rearranging this, we find that $d = 3$.

3e. Find a pair of X_L and Z_L logical operators of weight 3.

Solution:

From 3b. we have two weight-five logical operators: $X_L = X_1X_2X_3X_4X_5$ and $Z_L = Z_1Z_2Z_3Z_4Z_5$. Any logical operator multipled by a stabiliser is also a logical operator. We can therefore reduce the weight of our logicals by multiplying by stabilisers. Recall that: $XZ = -\mathrm{i}Y$.

Multiplying $S_1 = X_1Z_2Z_3X_4I_5$ by X_L gives:

$$X'_L = (X_1X_2X_3X_4X_5)(X_1Z_2Z_3X_4I_5) = -(I_1Y_2Y_3I_4X_5)$$

Similarily, multiplying $S_1 = X_1Z_2Z_3X_4I_5$ by Z_L gives:

$$Z'_L = (Z_1Z_2Z_3Z_4Z_5)(X_1Z_2Z_3X_4I_5) = -(Y_1I_2I_3Y_4Z_5)$$

The above logical operators have weight $d = 3$. The distance of the code is therefore $d = 3$.

3f. What are the $[[n, k, d]]$ parameters of this code?

Solution: The number of physical qubits is $n = 5$. From 3a, $k = 1$. From 3e., $d = 3$. The code paramters are therefore $[[n = 5, k = 1, d = 3]]$.

Problem 4: The Surface Code

4a. Figure 2 shows the Tanner graph for a surface code defined over 5 qubits. List the four stabiliser generators that are measured by this code.

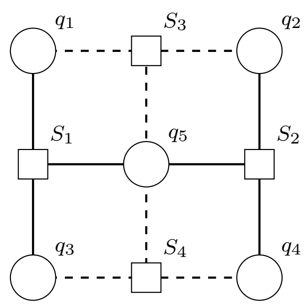

Figure 2: The five-qubit surface code. Dashed edges denote Z-type checks and solid edges X-type checks

Solution: The stabilisers of this code are:

$$S_1 = X_1 X_3 X_5$$
$$S_2 = X_2 X_4 X_5$$
$$S_3 = Z_1 Z_2 Z_5$$
$$S_4 = Z_3 Z_4 Z_5$$

4b. How many logical qubits does this code encode?

Solution: There are four stabiliser generators, $|S| = 4$. The logical qubit count is therefore $k = n - |S| = 5 - 4 = 1$.

4c. This code has distance $d = 2$. Find the logical operator pair Z_L, X_L.

Solution: In the surface code, logical operators span from edge-to-edge. The following is choice of logical operators:

$$X_L = X_1 X_2$$
$$Z_L = Z_1 Z_3$$

It is straightforward to verify that these logical operators commute with the stabilisers and anti-commute with another. An alternative choice of logical operators is:

$$X_L = X_3 X_4$$
$$Z_L = Z_2 Z_4$$

4d. Explain why this code is a detection code and not a correction code.

Solution: This code is a detection code as it has distance $d = 2$. The number of correctable errors t is given by the expression $t = (d-1)/2$. As such, any correction code must have $d \geq 3$.

4e. What are the $[[n, k, d]]$ parameters of this code?

Solution: The number of physical qubits $n = 5$, the logical qubit count is $k = 1$ and $d = 2$. The code has parameters $[[5, 1, 2]]$.

4f. Figure 3 shows the Tanner graph for a distance-4 surface code. Two X-errors have occurred on qubits q_{20} and q_6 activating a non-zero syndrome measurement for stabilisers S_{17} and S_{19}. Explain why $\mathcal{R} = X_6 X_{20}$ and $\mathcal{R}' = X_{10} X_{21}$ are both suitable recovery operations.

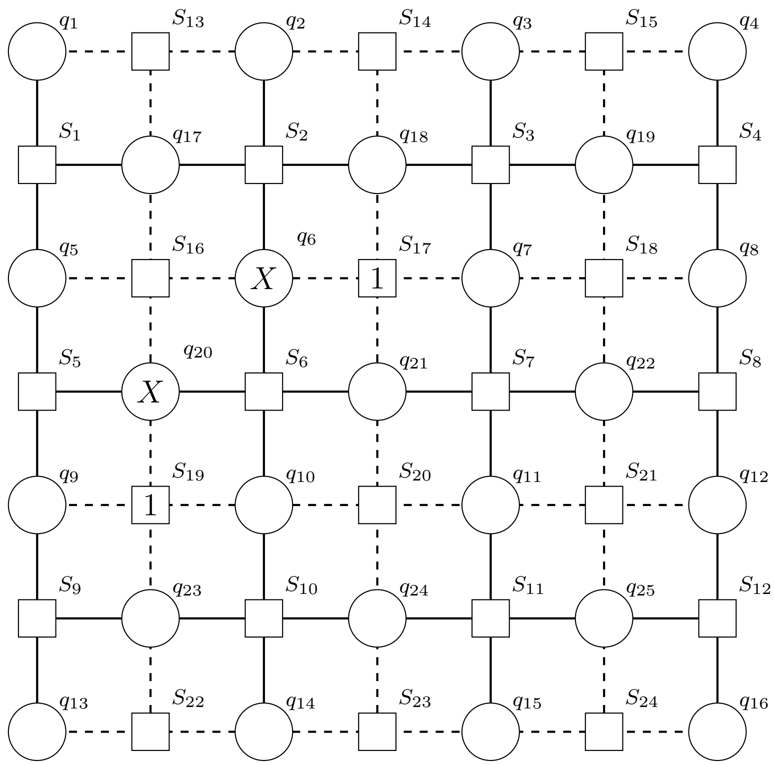

Figure 3: The distance-4 surface code. Dashed edges denote Z-type checks and solid edges X-type checks.

Solution: The original error is $E = X_6 X_{20}$. The determine whether or not our recovery operation is successful we first calculate the residual error. For the first recovery operation $\mathcal{R}$ is this given by:

$$R = \mathcal{R}E = I \in \mathcal{S}$$

which is in the stabiliser group. For the second recovery $\mathcal{R}'$ the residual error is:

$$R' = \mathcal{R}'E = X_6 X_{10} X_{20} X_{21} \in \mathcal{S}$$

This residual is equivalent to a stabiliser as it is equal to the operator measured by generator S_6.

4g. The recovery operator $\mathcal{R}'' = X_7 X_8 X_9$ would also reset the total syndrome of the surface code. Explain why this is not a suitable recovery operator.

Solution: The residual error for this recovery would be:

$$R'' = \mathcal{R}''E = X_6X_7X_8X_9X_{20} \in \mathcal{L}$$

This represents a chain of X-type Pauli operators spanning from the left edge of the surface code to the right edge. Error chains of this type are equivalent to X_L logical operators. The recovery operator $\mathcal{R}''$ would therefore change the logical information encoded by the code.