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Introduction to Quantum Computing

Lecture 10: Partial measurements

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Projectors

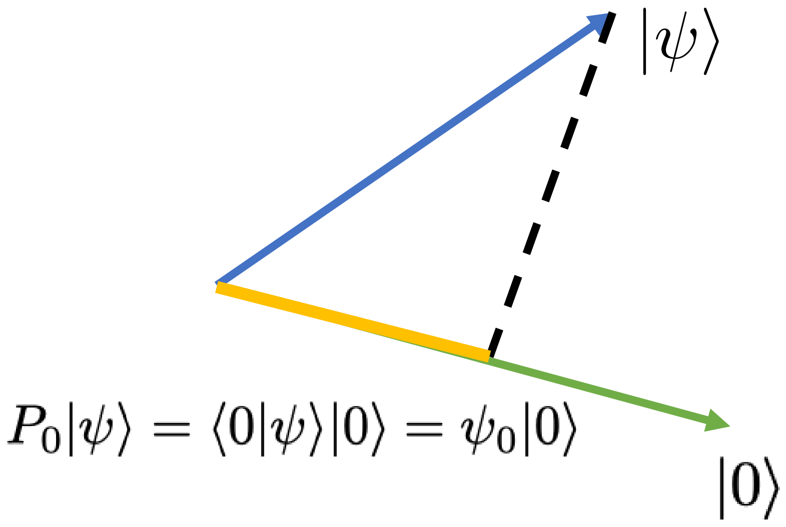
Projectors on computational basis

- Projector on computational basis state $|x\rangle$
  - $|0\rangle\langle 0| \equiv \begin{bmatrix}1\\0\end{bmatrix} \times \begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
  - $|1\rangle\langle 1| \equiv \begin{bmatrix}0\\1\end{bmatrix} \times \begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Definition of projectors

$$P_S^2 = P_S$$

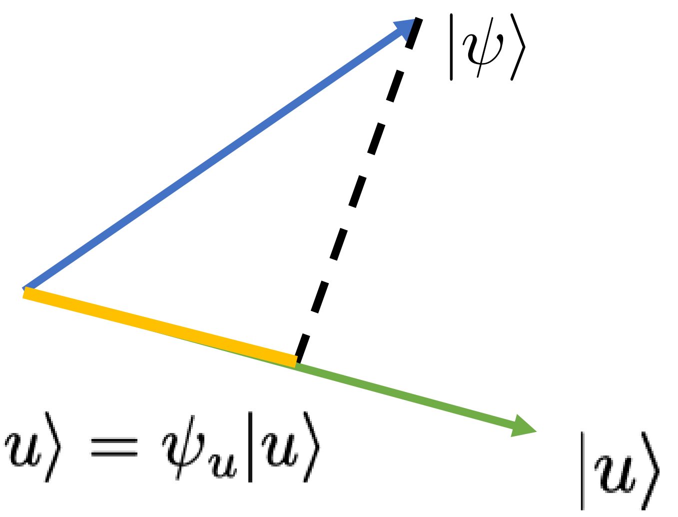
$$P_0|\psi\rangle = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\psi_0\\\psi_1\end{bmatrix} = \begin{bmatrix}\psi_0\\0\end{bmatrix} = \psi_0\begin{bmatrix}1\\0\end{bmatrix} = \psi_0|0\rangle$$



- Projector on $|-\rangle$
  - $P_{|-\rangle} = |-\rangle\langle -| \equiv \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} \times \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$
  - $P_{|-\rangle}|\psi\rangle = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}\psi_0\\\psi_1\end{bmatrix} = \frac{\psi_0 - \psi_1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = \frac{\psi_0 - \psi_1}{\sqrt{2}}|-\rangle$

Projectors on quantum states

- $$(|u\rangle\langle u|)|\psi\rangle = |u\rangle \underbrace{\langle u|\psi\rangle}_{\in\mathbb{C}} = \langle u|\psi\rangle|u\rangle = \psi_u|u\rangle$$



$$P_u|\psi\rangle = \langle u|\psi\rangle|u\rangle = \psi_u|u\rangle$$

$$(|u\rangle\langle u|)|v\rangle = \sum_j u_i u_j^* v_j = u_i(\sum_j u_j^* v_j) = u_i\langle u|v\rangle$$

Projectors properties

- $P_{\mathcal{S}}^2 = P_{\mathcal{S}}$

$$(|u\rangle\langle u|)|u\rangle\langle u| = |u\rangle\langle u|$$

- $P_{\mathcal{S}}^\dagger = P_{\mathcal{S}}$

$$(|u\rangle\langle u|)^\dagger = |u\rangle\langle u|$$

In this course

Self-adjoint Projectors

$$P_{\mathcal{S}}^2 = P_{\mathcal{S}} = P_{\mathcal{S}}^\dagger$$

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Revisiting basis measurements

Measurement of orthonormal basis

Any orthonormal basis $\{|v_i\rangle\}$ that span $\mathcal{H}$ has and associated measurement

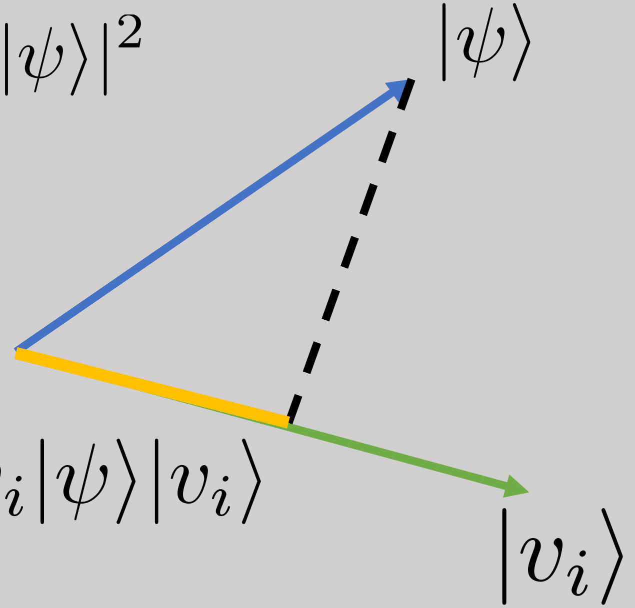
Satisfying a completeness relation: $\sum_{i=0}^{d-1} |v_i\rangle\langle v_i| = I_d$

$P_i = |v_i\rangle\langle v_i|$ are projectors on the states of the basis

Probability of outcome i reads: $P(i) = ||P_i|\psi\rangle||^2 = |\langle v_i|\psi\rangle|^2$

The quantum state is updated to

$$\frac{P_i|\psi\rangle}{||P_i|\psi\rangle||} = \frac{\langle v_i|\psi\rangle}{||P_i|\psi\rangle||}|v_i\rangle = e^{i\phi}|v_i\rangle$$



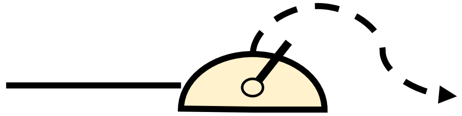
$$P_i|\psi\rangle = \langle v_i|\psi\rangle|v_i\rangle$$

Example:+/- basis

$$\mathcal{H}_{\mathcal{Q}} = \text{Span}\{|+\rangle, |-\rangle\}$$

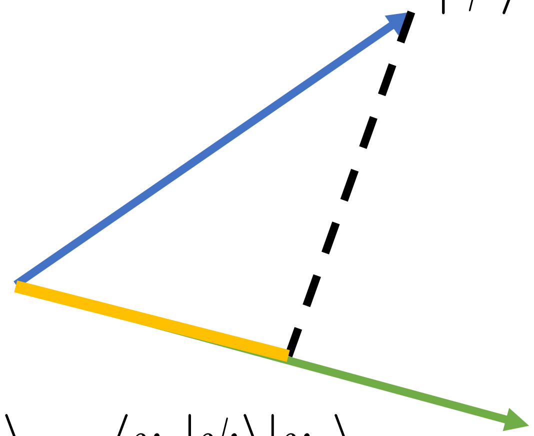
- Completness: $|+\rangle\langle+| + |-\rangle\langle-| \equiv \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \equiv I_2$

- $|\psi\rangle$



$|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$

$|\psi\rangle$



$P_i|\psi\rangle = \langle v_i|\psi\rangle|v_i\rangle$

$|v_i\rangle$

$$P(-) = ||P_{|-\rangle}|\psi\rangle||^2 \qquad \text{Updated state: } \frac{P_{|-\rangle}|\psi\rangle}{||P_{|-\rangle}|\psi\rangle||}$$

$$P_{|-\rangle} = |-\rangle\langle-| \equiv \begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix} \times \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

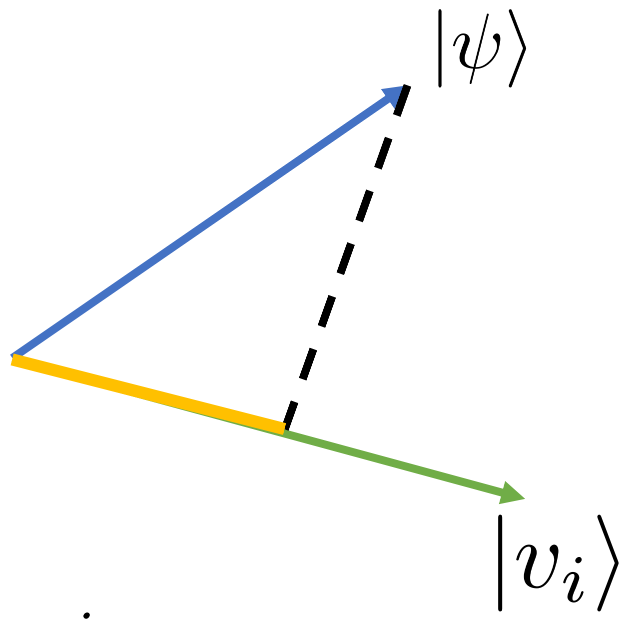
$$P_{|-\rangle}|\psi\rangle = \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\psi_0\\ \psi_1\end{bmatrix} = \frac{\psi_0 - \psi_1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix} = \frac{\psi_0 - \psi_1}{\sqrt{2}}|-\rangle$$

$$P(-) = |\psi_0 - \psi_1|^2/2 \qquad \text{Updated state: } |-\rangle$$

Global phase

The quantum state is updated to

$$\frac{P_i|\psi\rangle}{||P_i|\psi\rangle||} = \frac{\langle v_i|\psi\rangle}{||P_i|\psi\rangle||}|v_i\rangle = e^{i\phi}|v_i\rangle$$



- Define a state up to a global phase: $|\tilde{\psi}\rangle \equiv e^{i\varphi}|\psi\rangle$

Output probability: $P(i) = ||P_i|\tilde{\psi}\rangle||^2 = |e^{i\varphi}\langle v_i|\psi\rangle|^2 = |\langle v_i|\psi\rangle|^2$

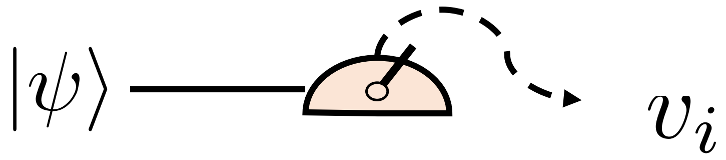
Projectors on basis states are rank-1 projectors

$$\forall \text{ basis } \{|v_i\rangle\}, \exists U_v \text{ s.t. } |v_i\rangle = U_v|i\rangle$$

$$P_i = |v_i\rangle\langle v_i| = U_v|i\rangle\langle i|U_v^\dagger = U_v \begin{bmatrix} \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix} U_v^\dagger$$

Rank 1 matrix

- Measurement basis $\{|v_i\rangle\}$:



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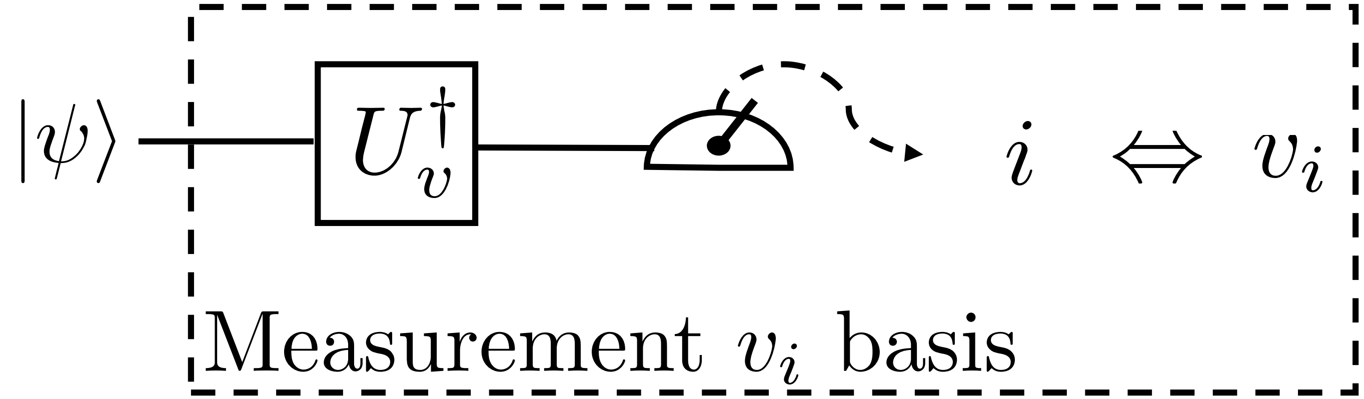
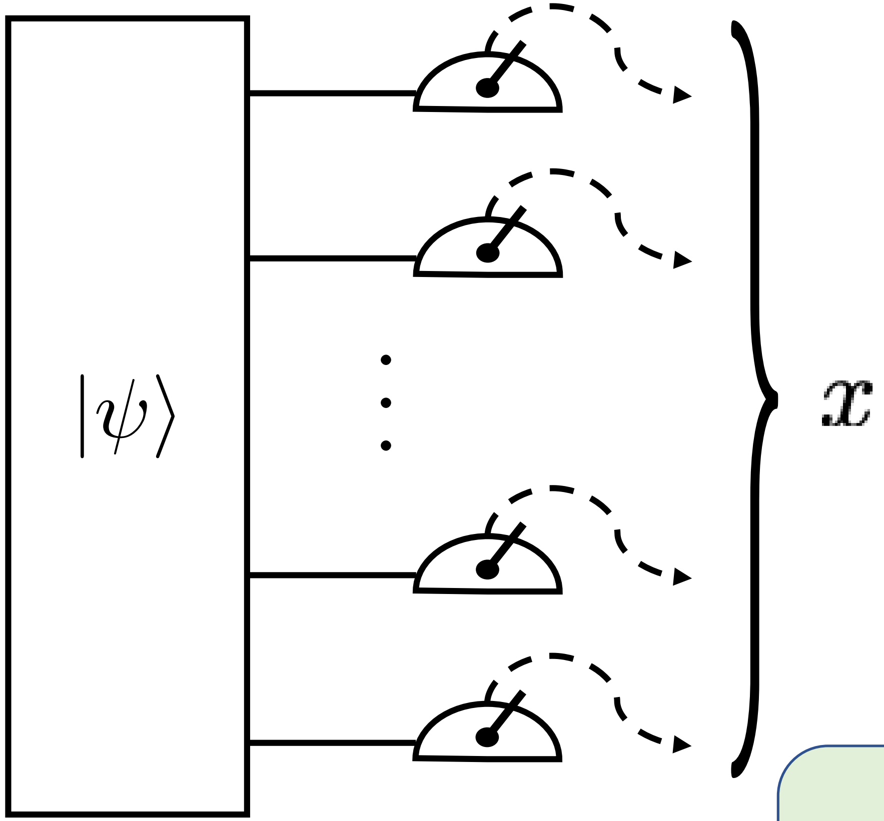


$$\langle v_i|\psi\rangle = \langle i|U_v^\dagger|\psi\rangle$$

Composition of measurement: computational basis

- Let's x encode the outcome of n bits



$$|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes .... \otimes |x_n\rangle$$

$$P_x = |x\rangle\langle x| = |x_1\rangle\langle x_1| \otimes .... \otimes |x_n\rangle\langle x_n|$$

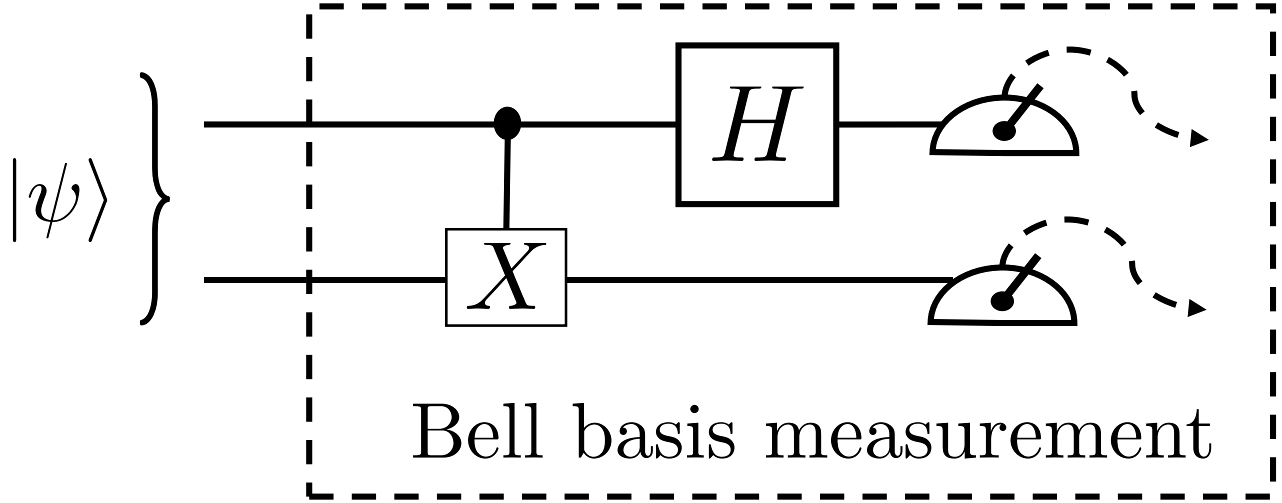
$$P(x) = ||P_x|\psi\rangle||^2 = |\langle x|\psi\rangle|^2$$

$$\text{Update: } \frac{P_x|\psi\rangle}{||P_x|\psi\rangle||} = \frac{\psi_x}{|\psi_x|}|x\rangle = e^{i\phi}|x\rangle$$

$$|\langle x|\psi\rangle|^2 = |\langle x| \sum_{y\in\{0,1\}^n} \psi_y|y\rangle|^2 = |\psi_x|^2$$

General multi-qubit basis measurement

- Bell basis measurement:



Bell basis

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

$$|\Phi^{\pm}\rangle\langle\Phi^{\pm}| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix}$$

$$|\Psi^{\pm}\rangle\langle\Psi^{\pm}| = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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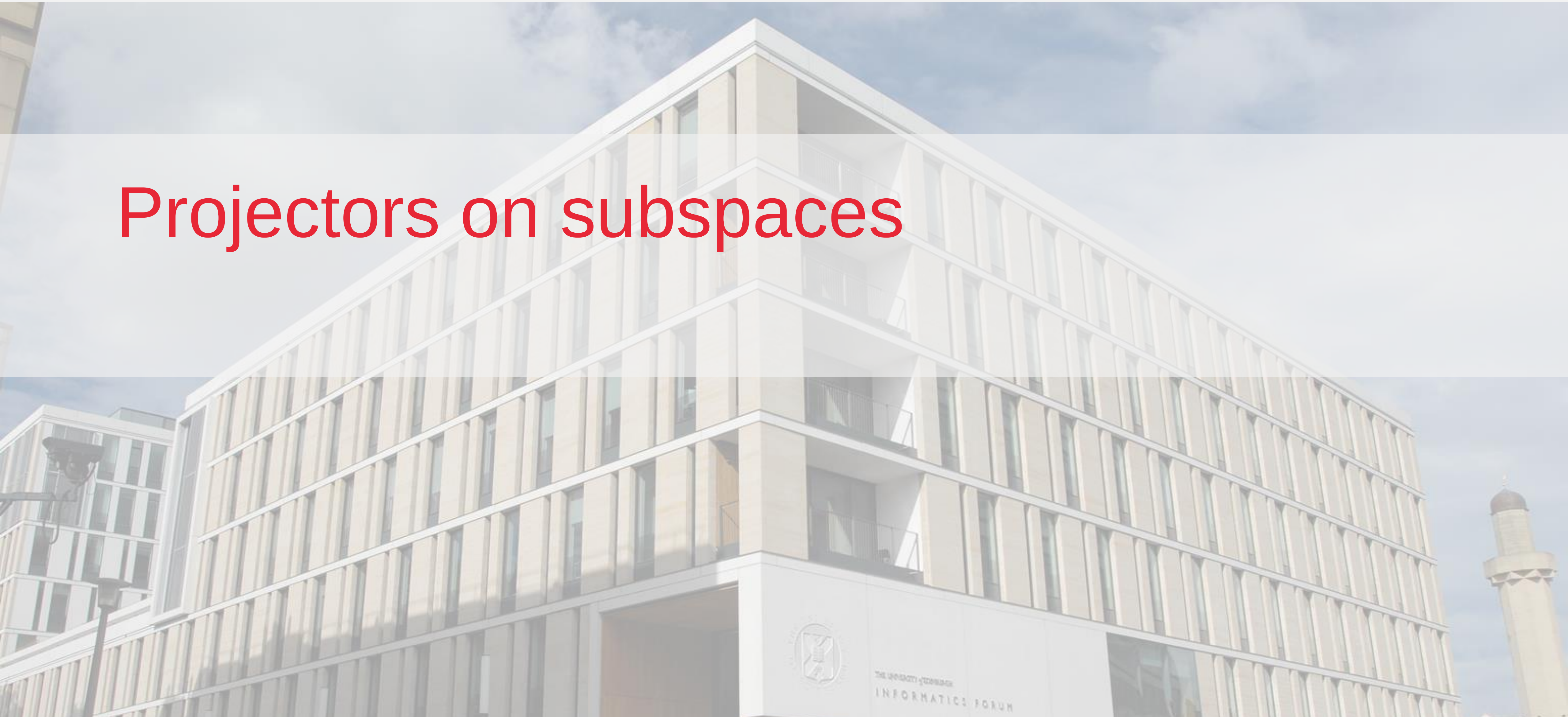
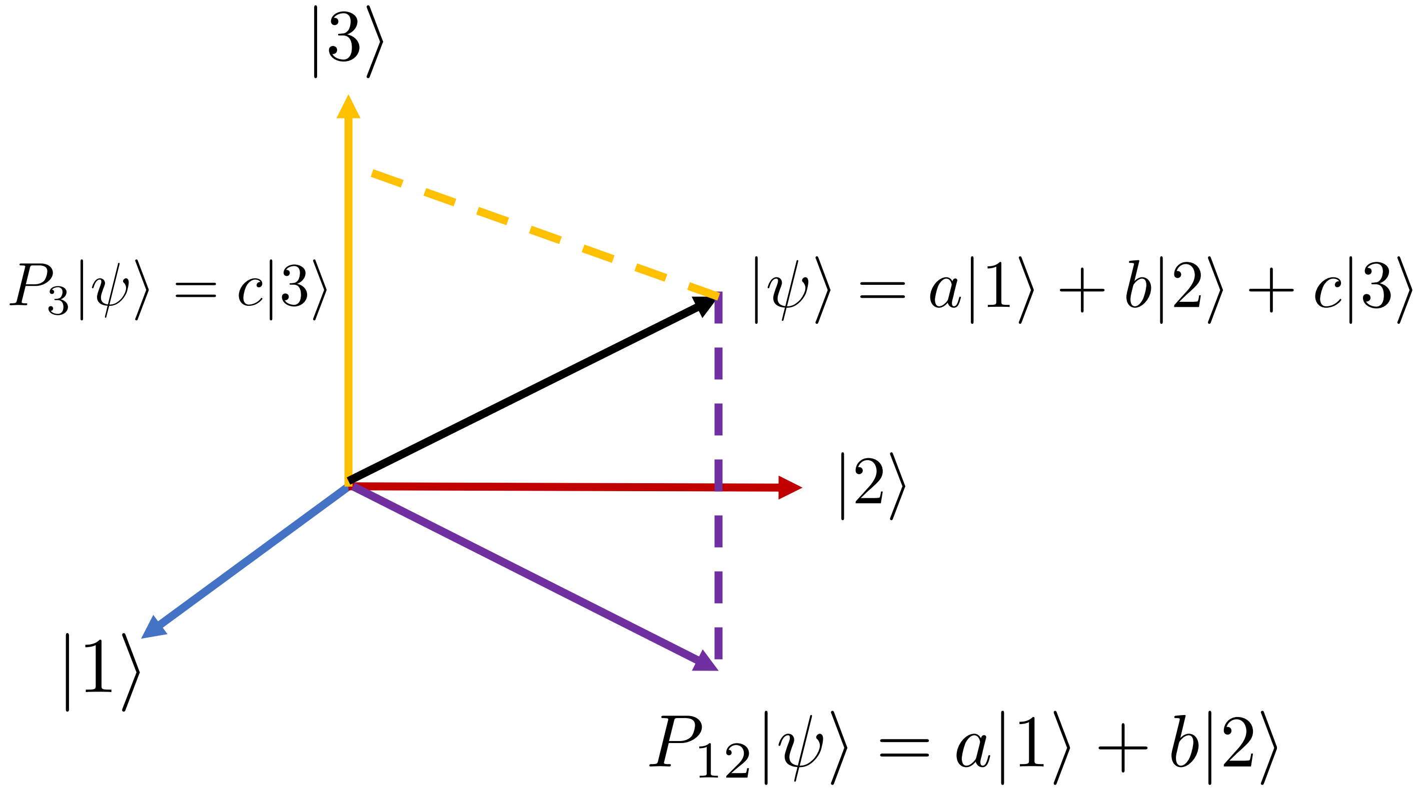
Projectors on subspaces



Projector on subspace

- Consider a 3-dimensional space

We want to project on subspace.
For example, the first two dimensions.



Rank 2 projector

- $$P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P_1 + P_2$$

Projectors on vector subspaces

- Projectors on a 1-dim vector subspace: $P_i = |v_i\rangle\langle v_i|$

- Projector on vector subspace $\mathcal{S}$ of dim $k\,(\mathcal{S} \subset \mathcal{H})$:

  - Being a vector space, $\mathcal{S}$ has an orthonormal basis $\{|u_i\}_{i=0}^{k-1}$

  - $P_{\mathcal{S}} = \sum_{i=0}^{k-1} |u_i\rangle\langle u_i|$

  - $P_{\mathcal{S}}^2 = P_{\mathcal{S}}$

  - $P_{\mathcal{S}}^\dagger = P_{\mathcal{S}}$

$$P_{\mathcal{S}}^2 = P_{\mathcal{S}} = P_{\mathcal{S}}^\dagger$$

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Projective measurement

Projective measurement

A projective measurement consist of a set of projectors P_i

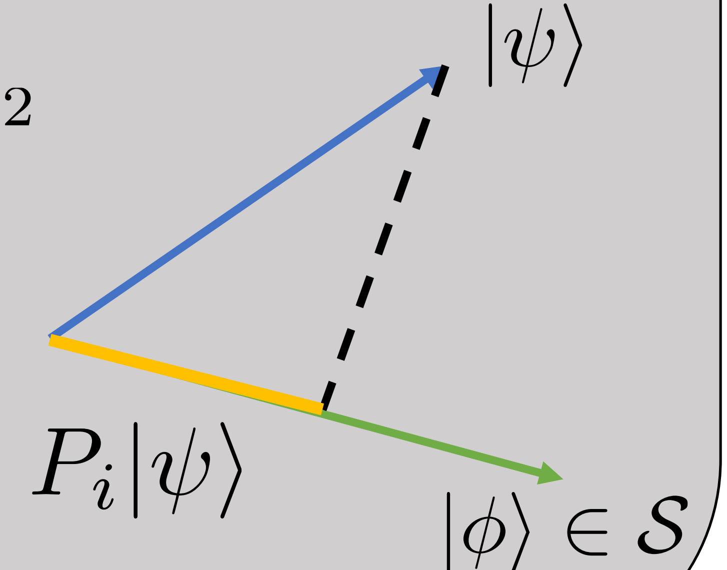
Satisfying a completeness relation: $\sum_{i=0}^{l} P_i = I_d$

Satisfy orthogonal relation: $P_m P_n = \delta_{n,m} P_m$

Probability of outcome i reads: $P(i) = ||P_i|\psi\rangle||^2$

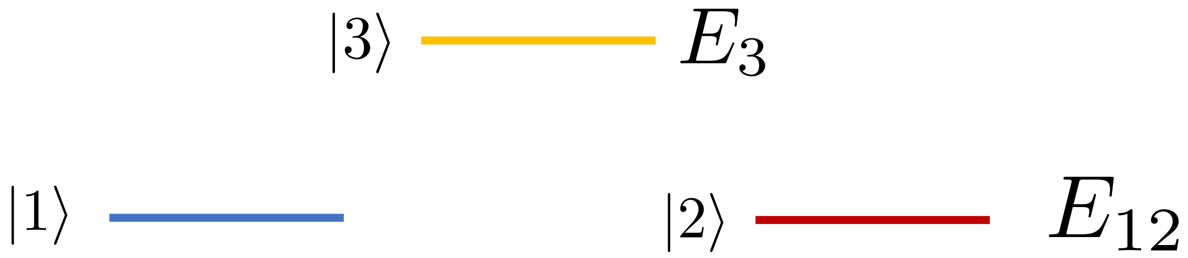
The quantum state is updated to

$$\frac{P_i|\psi\rangle}{||P_i|\psi\rangle||}$$



A degenerate 3 dimensional quantum system

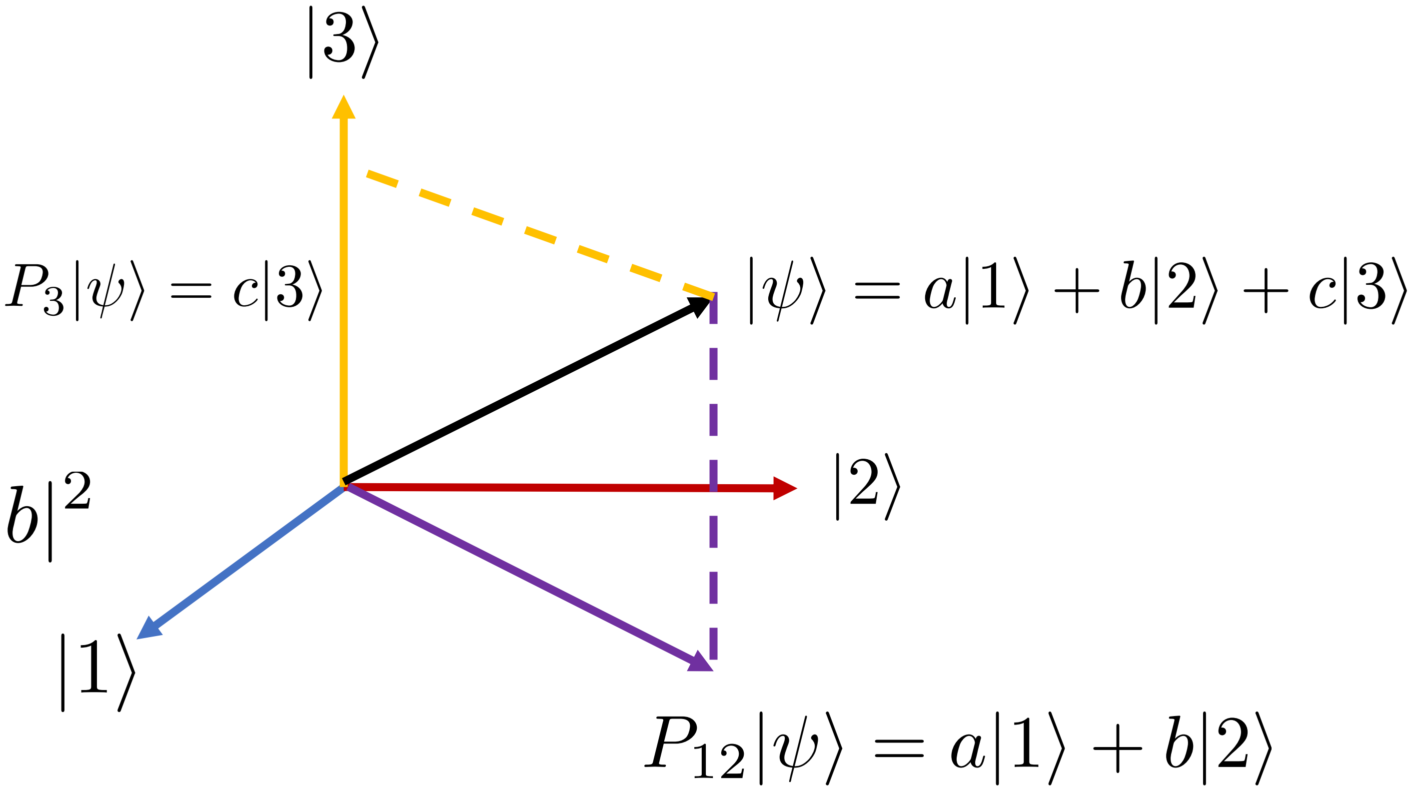
- Orthonormal basis:
- Completeness:
  - $P_3 + P_{12} = I$
  - $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Rank 2 projector

- $P(12) = ||P_{12}|\psi\rangle||^2 = |a|^2 + |b|^2$

Update: $|\psi'\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}}(a|1\rangle + b|2\rangle)$



Projective measurement

Completeness relation

$$\sum_{i=0}^{l} P_l = I_d$$

Completeness implies probabilities add to 1:

$$\sum_i P(i) = \sum_i ||P_i|\psi\rangle||^2$$
$$= \sum_i \langle\psi|P_i^\dagger P_i|\psi\rangle = \langle\psi|\sum_i P_i^\dagger P_i|\psi\rangle$$
$$= \langle\psi|\sum_i P_i|\psi\rangle = \langle\psi|\psi\rangle = 1$$

We use:

- $P(i) = ||P_i|\psi\rangle||^2$
- Linearity
- $P_{\mathcal{S}}^2 = P_{\mathcal{S}} = P_{\mathcal{S}}^\dagger$

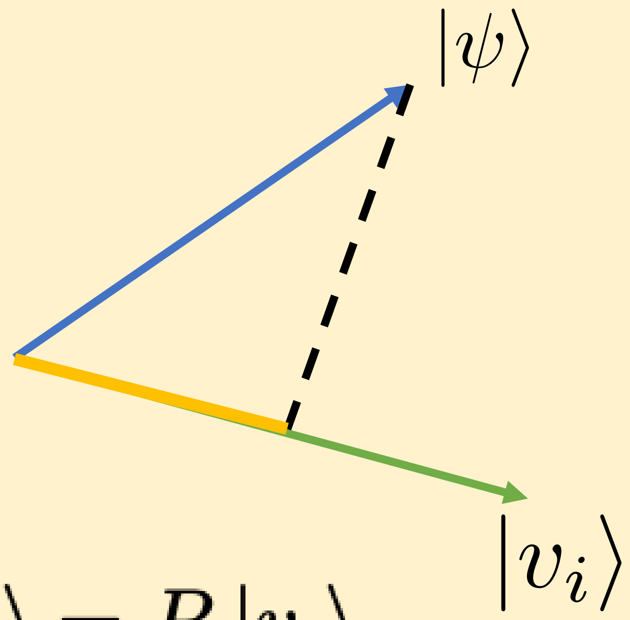
Reproducibility of measurement

- *Repeating the same measurement immediately after, gives the same answer.*

Results from: $P_i^2 = P_i$

$$P_i|v_i\rangle = |v_i\rangle\langle v_i|v_i\rangle = |v_i\rangle$$

$$P_j|v_i\rangle = |v_j\rangle\langle v_j|v_i\rangle = 0 \text{ if } i \neq j$$



$$P_i(P_i|\psi\rangle) = P_i(\psi_{v_i}|v_i\rangle) = \psi_{v_i}P_i|v_i\rangle = \psi_{v_i}|v_i\rangle = P_i|v_i\rangle$$

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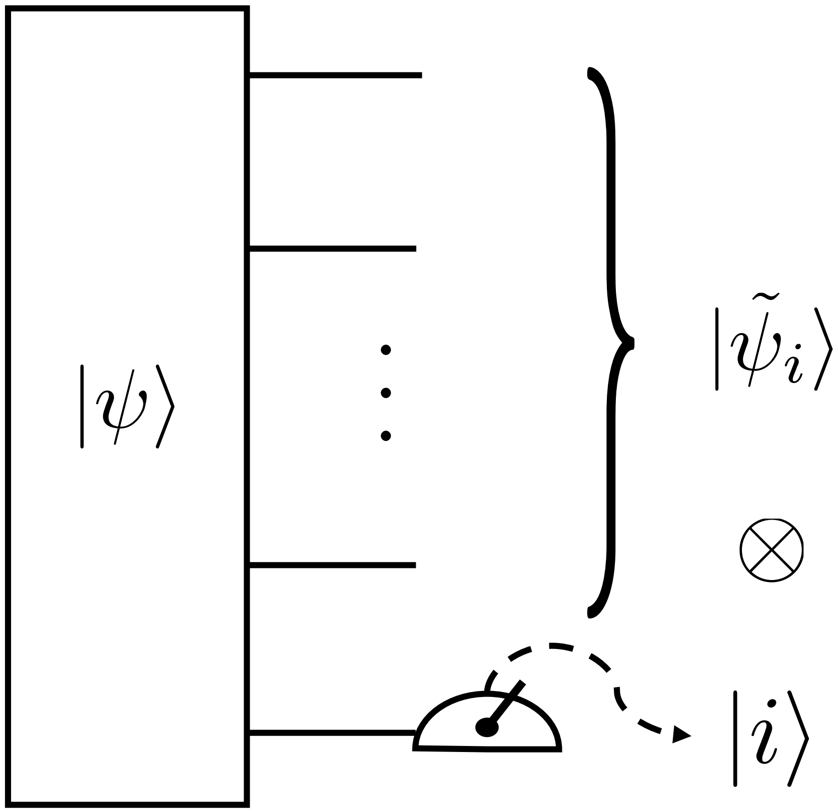
Partial measurement

Subsystem measurement

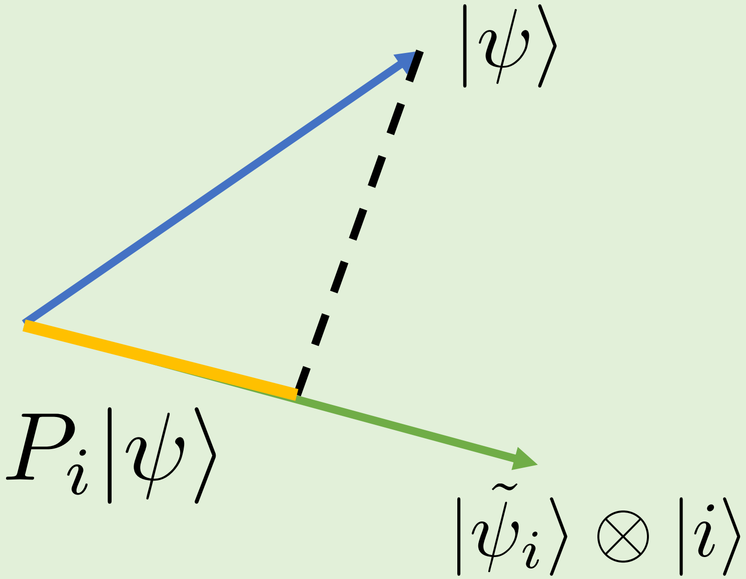


$$\tilde{P}_0 = I \otimes I \otimes \ldots \otimes |0\rangle\langle 0|$$

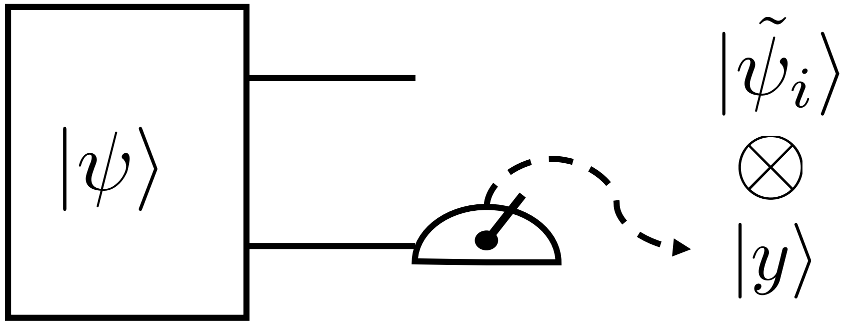
$$\tilde{P}_1 = I \otimes I \otimes \ldots \otimes |1\rangle\langle 1|$$

$$\tilde{P}_0 + \tilde{P}_1 = I_{\mathcal{H}^{\otimes n}}$$

$$\sum_i c_i (A_i \otimes C) = (\sum_i c_i A_i) \otimes C$$



Two qubit example



$$\tilde{P}_0 = I \otimes |0\rangle\langle 0|$$
$$\tilde{P}_1 = I \otimes |1\rangle\langle 1|$$
$$\tilde{P}_0 + \tilde{P}_1 = I$$

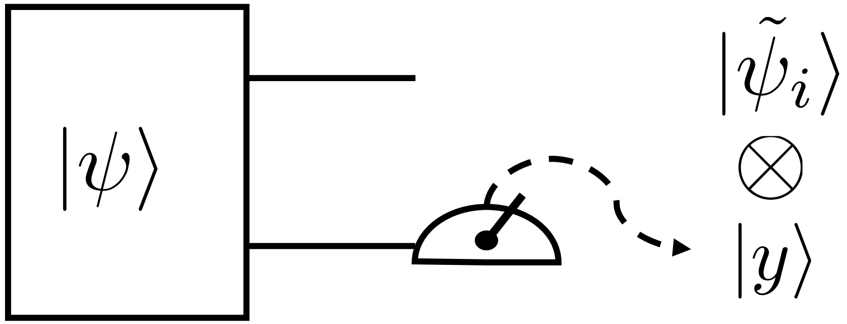
$$|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{01}|0\rangle \otimes |1\rangle + \psi_{10}|1\rangle \otimes |0\rangle + \psi_{11}|1\rangle \otimes |1\rangle$$

$$\tilde{P}_0|i\rangle \otimes |0\rangle = (I \otimes |0\rangle\langle 0|)(|i\rangle \otimes |0\rangle) = I|i\rangle \otimes |0\rangle \underbrace{\langle 0|0\rangle}_{=1} = |i\rangle \otimes |0\rangle$$

$$\tilde{P}_0|i\rangle \otimes |1\rangle = (I \otimes |0\rangle\langle 0|)(|i\rangle \otimes |1\rangle) = I|i\rangle \otimes |0\rangle \underbrace{\langle 0|1\rangle}_{=0} = 0$$

$$\tilde{P}_0|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{10}|1\rangle \otimes |0\rangle = (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle$$

Two qubit example



$$\tilde{P}_0 = I \otimes |0\rangle\langle 0|$$
$$\tilde{P}_1 = I \otimes |1\rangle\langle 1|$$
$$\tilde{P}_0 + \tilde{P}_1 = I$$

$$|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{01}|0\rangle \otimes |1\rangle + \psi_{10}|1\rangle \otimes |0\rangle + \psi_{11}|1\rangle \otimes |1\rangle$$

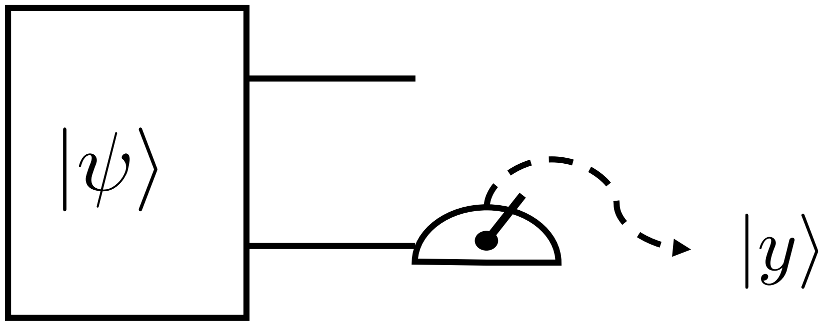
$$\tilde{P}_0|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{10}|1\rangle \otimes |0\rangle = (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle$$

$$||\tilde{P}_0|\psi\rangle||^2 = |\psi_{0,0}|^2 + |\psi_{1,0}|^2$$

$$\frac{\tilde{P}_0|\psi\rangle}{||\tilde{P}_0|\psi\rangle||} = \frac{1}{\sqrt{|\psi_{00}|^2 + |\psi_{01}|^2}}\,(\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle = |\tilde{\psi}_0\rangle \otimes |0\rangle$$

Two registers example

Two register of n and m qubits respectively :



$$\tilde{P}_y = I \otimes |y\rangle\langle y|$$

$$\sum_{y\in\{0,1\}^n} P_y = I_{\mathcal{H}^{\otimes n}}$$

$$|\psi\rangle = \sum_{x,w} \psi_{x,w} |x\rangle \otimes |w\rangle$$

$$\tilde{P}_y|\psi\rangle = (\sum_x \psi_{x,y}|x\rangle) \otimes |y\rangle \qquad ||\tilde{P}_y|\psi\rangle||^2 = \sum_x |\psi_{x,y}|^2$$

$$\frac{\tilde{P}_y|\psi\rangle}{||\tilde{P}_y|\psi\rangle||} = \frac{1}{\sqrt{\sum_x \psi_{x,y}}} \left(\sum_x \psi_{x,y}|x\rangle \right) \otimes |y\rangle = |\tilde{\psi}_y\rangle \otimes |y\rangle$$

References

Reading references

1. Adjoints and Hermitian operators NC 2.1.6
2. Projective measurement NC 2.2.5 (notation different from the course)
3. What is a phase? NC 2.2.7
4. Composite system and measurement NC 2.2.8

NC ≣ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information
Cambridge University Press (2010)