THE UNIVERSITY of EDINBURGH
informatics

Introduction to Quantum Computing

Lecture 13: Simon's Algorithm

Raul Garcia-Patron Sanchez

Find the period

$f:\{0,1\}^n \to \{0,1\}^n$

Promise: f is a 2-to-1 and periodic on $\{0,1\}^n$

$f(x) = f(x') \Leftrightarrow x = x'$ or $x = x' \oplus a$

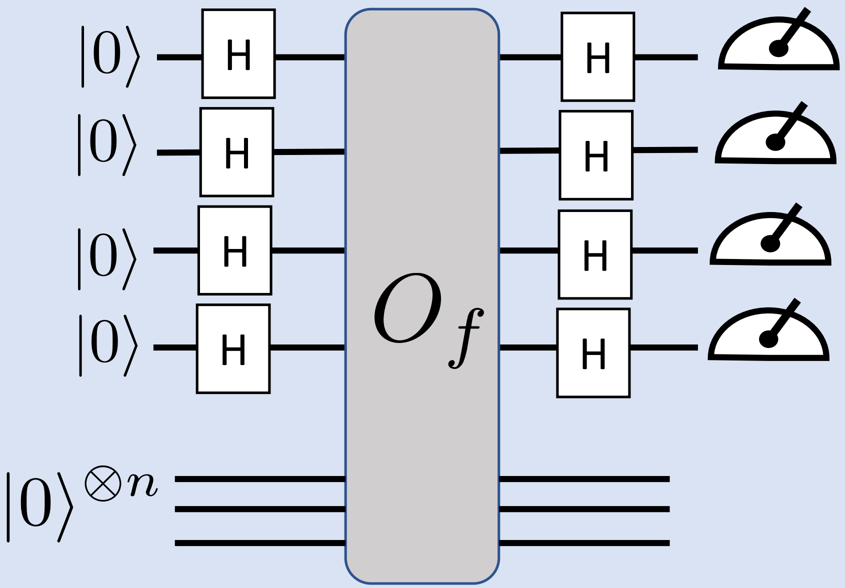
Problem: Find a

Exponential separation

- Inspired Shor Algorithm
- Can be used to break some cryptographic primitives (see references)

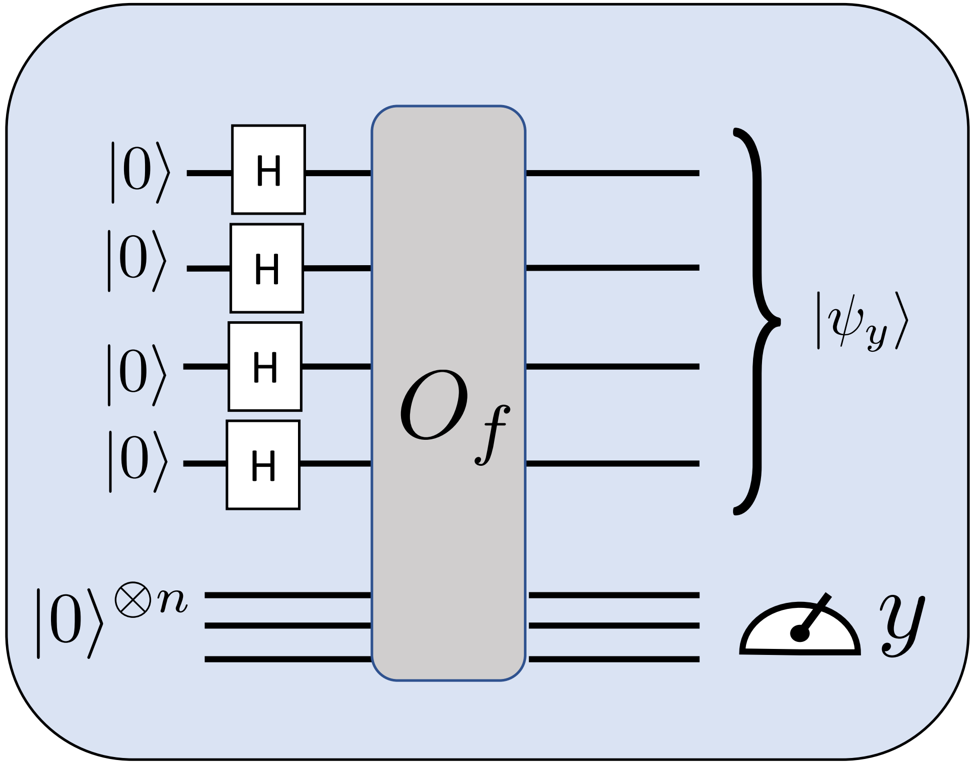
$$|x\rangle|0\rangle^n \xrightarrow{O_f} |x\rangle|f(x)\rangle$$

Quantum subroutine



Simon Algorithm

$$f:\{0,1\}^n \rightarrow \{0,1\}^n$$
$$f(x)=f(x') \Leftrightarrow x=x' \text{ or } x=x'\oplus a$$



- $$|0\rangle^n \otimes |0\rangle^n \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \otimes |0\rangle^{\otimes n}$$
$$\xrightarrow{O_f} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle|f(x)\rangle$$
$$= \frac{1}{\sqrt{2^n}} \sum_{\text{Image}} \left(\sum_{\text{Preimage of} f(x)} |x\rangle \right) |f(x)\rangle$$
- Measurement outcome y : $$|\psi_y\rangle = \frac{1}{\sqrt{2}}\left[|x_y\rangle + |x_y \oplus a\rangle\right]$$

Projective and partial measurements

Projective measurement

A projective measurement consist of a set of projectors P_i

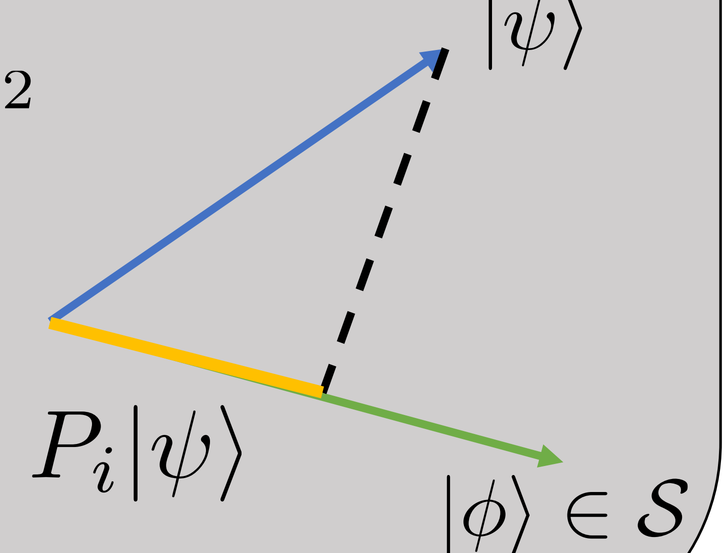
Satisfying a completeness relation: $\sum_{i=0}^{l} P_i = I_d$

Satisfy orthogonal relation: $P_m P_n = \delta_{n,m} P_m$

Probability of outcome i reads: $P(i) = ||P_i|\psi\rangle||^2$

The quantum state is updated to

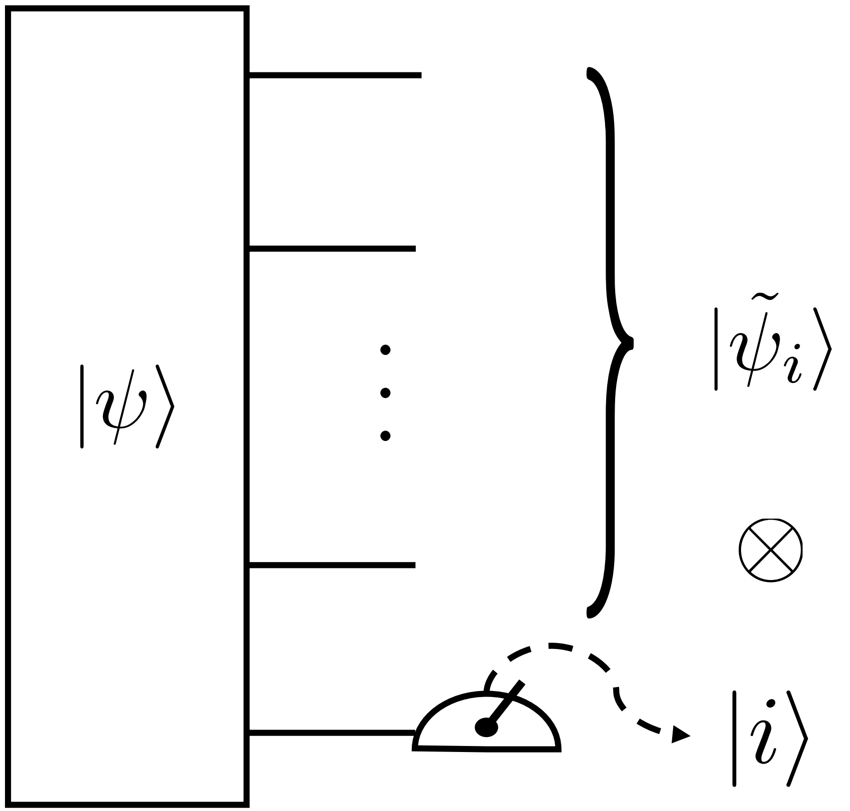
$$\frac{P_i|\psi\rangle}{||P_i|\psi\rangle||}$$



Projectors on vector subspaces

- Projectors on a 1-dim vector subspace: $P_i = |v_i\rangle\langle v_i|$
- Projector on vector subspace $\mathcal{S}$ of dim $k \, (\mathcal{S} \subset \mathcal{H})$:
  - Being a vector space, $\mathcal{S}$ has an orthonormal basis $\{|u_i\}_{i=0}^{k-1}$
  - $P_{\mathcal{S}} = \sum_{i=0}^{k-1} |u_i\rangle\langle u_i|$
  - $P_{\mathcal{S}}^2 = P_{\mathcal{S}}$
  - $P_{\mathcal{S}}^\dagger = P_{\mathcal{S}}$

Subsystem measurement

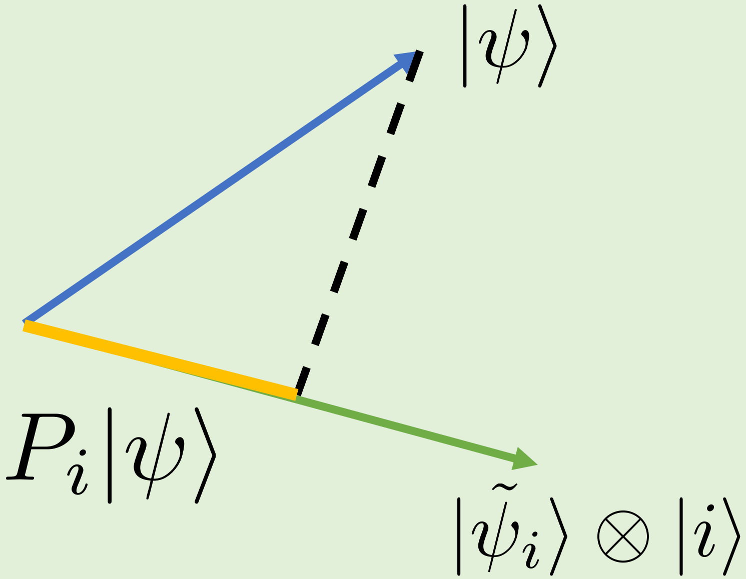


$$\tilde{P}_0 = I \otimes I \otimes \ldots \otimes |0\rangle\langle 0|$$

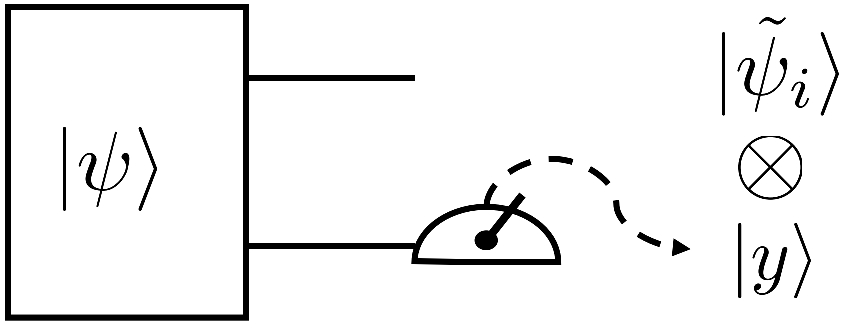
$$\tilde{P}_1 = I \otimes I \otimes \ldots \otimes |1\rangle\langle 1|$$

$$\tilde{P}_0 + \tilde{P}_1 = I_{\mathcal{H}^{\otimes n}}$$

$$\boxed{\sum_i c_i (A_i \otimes C) = (\sum_i c_i A_i) \otimes C}$$



Two qubit example



$$\tilde{P}_0 = I \otimes |0\rangle\langle 0|$$
$$\tilde{P}_1 = I \otimes |1\rangle\langle 1|$$
$$\tilde{P}_0 + \tilde{P}_1 = I$$

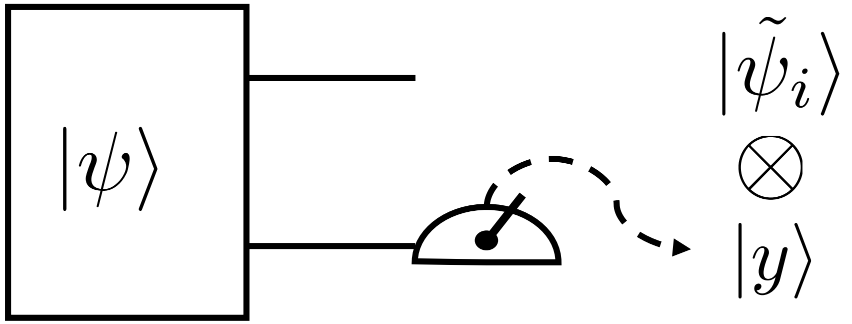
$$|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{01}|0\rangle \otimes |1\rangle + \psi_{10}|1\rangle \otimes |0\rangle + \psi_{11}|1\rangle \otimes |1\rangle$$

$$\tilde{P}_0|i\rangle \otimes |0\rangle = (I \otimes |0\rangle\langle 0|)(|i\rangle \otimes |0\rangle) = I|i\rangle \otimes |0\rangle \underbrace{\langle 0|0\rangle}_{=1} = |i\rangle \otimes |0\rangle$$

$$\tilde{P}_0|i\rangle \otimes |1\rangle = (I \otimes |0\rangle\langle 0|)(|i\rangle \otimes |1\rangle) = I|i\rangle \otimes |0\rangle \underbrace{\langle 0|1\rangle}_{=0} = 0$$

$$\tilde{P}_0|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{10}|1\rangle \otimes |0\rangle = (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle$$

Two qubit example



$$\tilde{P}_0 = I \otimes |0\rangle\langle 0|$$
$$\tilde{P}_1 = I \otimes |1\rangle\langle 1|$$
$$\tilde{P}_0 + \tilde{P}_1 = I$$

$$|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{01}|0\rangle \otimes |1\rangle + \psi_{10}|1\rangle \otimes |0\rangle + \psi_{11}|1\rangle \otimes |1\rangle$$

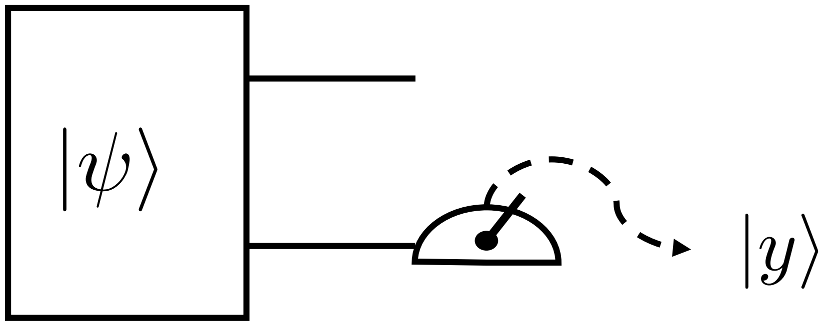
$$\tilde{P}_0|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{10}|1\rangle \otimes |0\rangle = (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle$$
$$||\tilde{P}_0|\psi\rangle||^2 = |\psi_{0,0}|^2 + |\psi_{1,0}|^2$$
$$\frac{\tilde{P}_0|\psi\rangle}{||\tilde{P}_0|\psi\rangle||} = \frac{1}{\sqrt{|\psi_{00}|^2 + |\psi_{01}|^2}} (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle = |\tilde{\psi}_0\rangle \otimes |0\rangle$$

Two registers example

Two register of n and m qubits respectively :

$$\tilde{P}_y = I \otimes |y\rangle\langle y|$$

$$\sum_{y\in\{0,1\}^n} P_y = I_{\mathcal{H}^{\otimes n}}$$



$$|\psi\rangle = \sum_{x,w} \psi_{x,w}|x\rangle \otimes |w\rangle$$

$$\tilde{P}_y|\psi\rangle = (\sum_x \psi_{x,y}|x\rangle) \otimes |y\rangle \qquad ||\tilde{P}_y|\psi\rangle||^2 = \sum_x |\psi_{x,y}|^2$$

$$\frac{\tilde{P}_y|\psi\rangle}{||\tilde{P}_y|\psi\rangle||} = \frac{1}{\sqrt{\sum_x \psi_{x,y}}}\left(\sum_x \psi_{x,y}|x\rangle\right) \otimes |y\rangle = |\tilde{\psi}_y\rangle \otimes |y\rangle$$

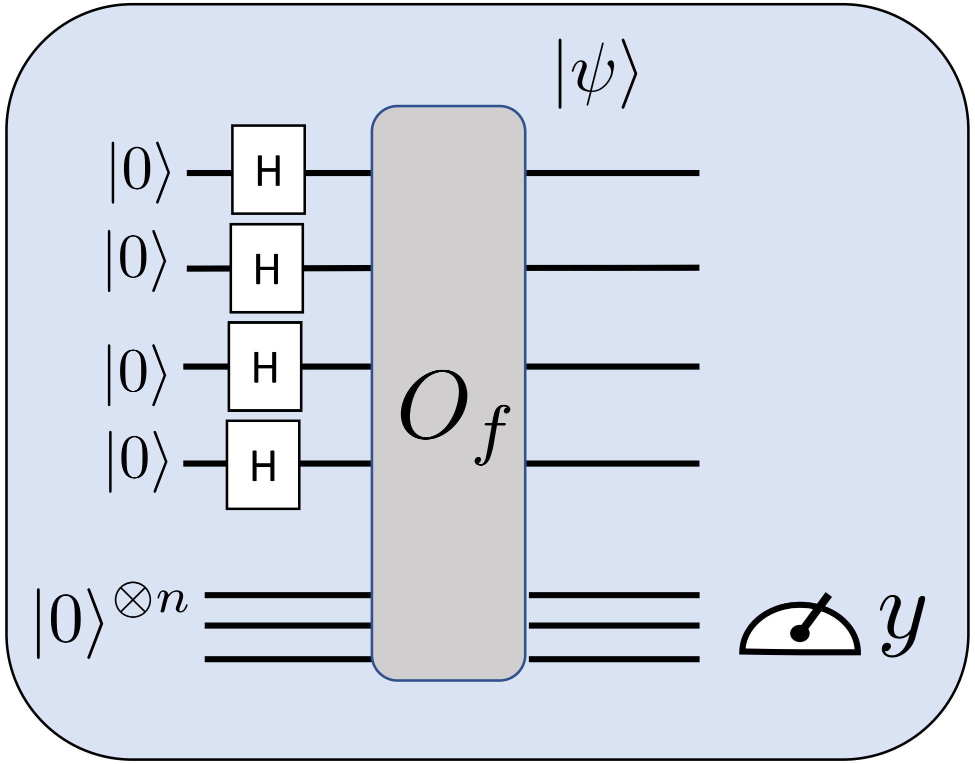
THE UNIVERSITY *of* EDINBURGH
informatics

Simon’s Algorithm

Raul Garcia-Patron Sanchez

Simon Algorithm

$$f : \{0,1\}^n \to \{0,1\}^n$$
$$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$$



- $$|0\rangle^n \otimes |0\rangle^n \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \otimes |0\rangle^{\otimes n}$$
$$\xrightarrow{O_f} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle |f(x)\rangle$$
$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{w\in\{0,1\}^n} \left(\sum_{x:x=f^{-1}(w)} |x\rangle \right) \otimes |w\rangle$$

Simon Algorithm

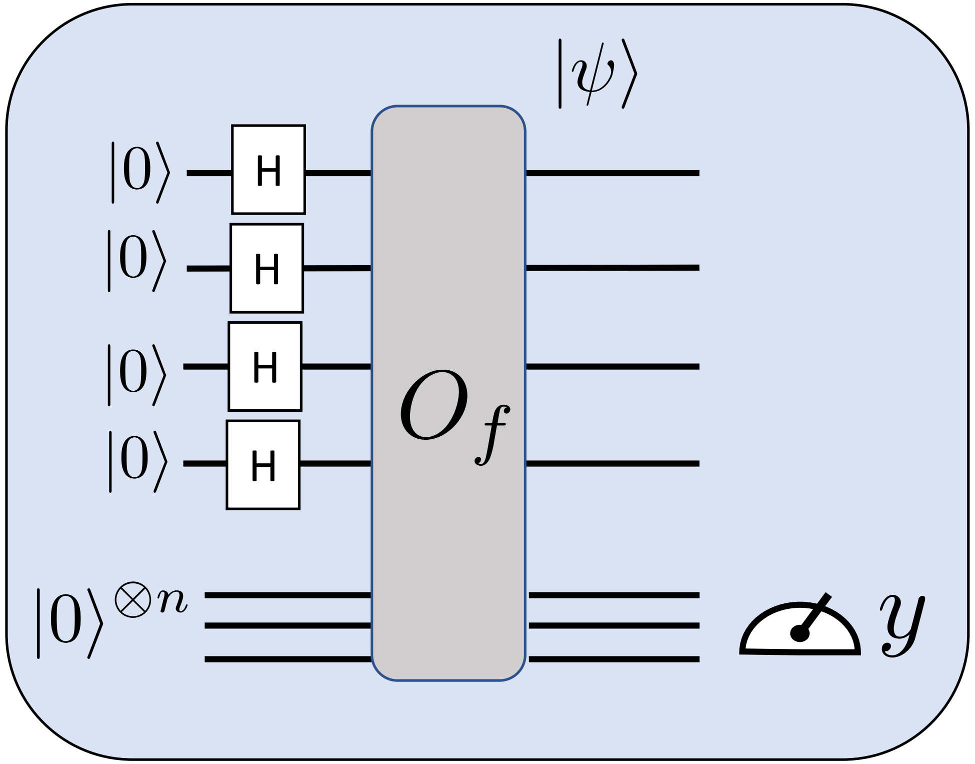
$$f : \{0,1\}^n \to \{0,1\}^n$$

$$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$$

- $$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{w \in \{0,1\}^n} \left(\sum_{x : x = f^{-1}(w)} |x\rangle \right) \otimes |w\rangle$$

$$P_y|\psi\rangle = \frac{1}{\sqrt{2^n}} \left(\sum_{x : x = f^{-1}(w)} |x\rangle \right) \otimes |y\rangle$$

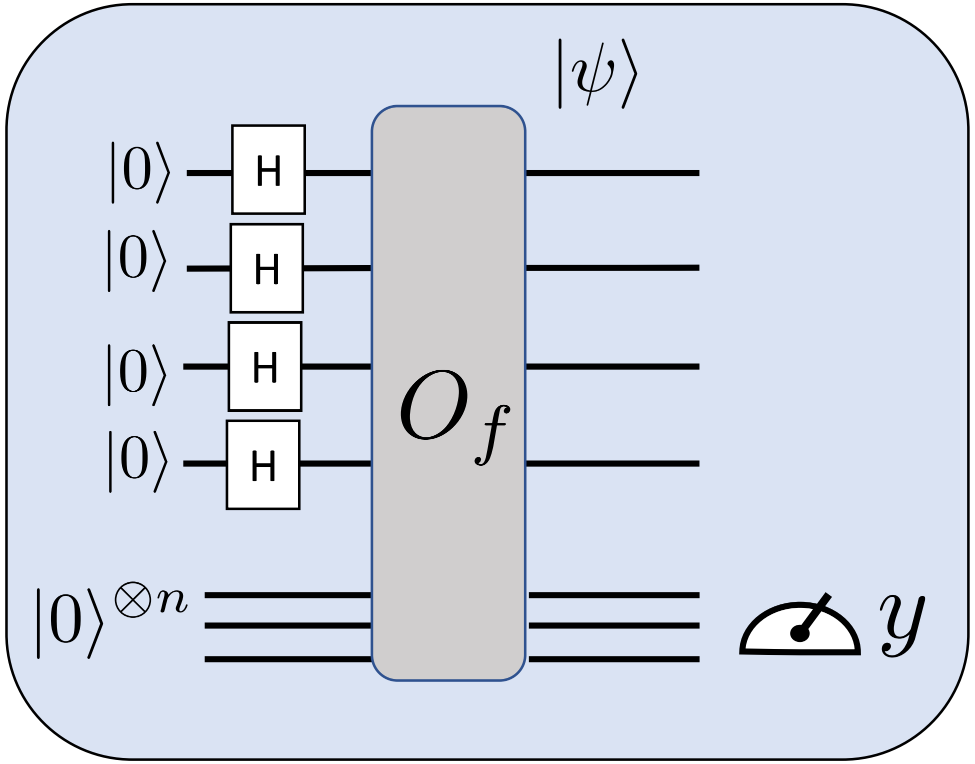
$$||P_y|\psi\rangle||^2 = \frac{|f^{-1}(y)|}{2^n}$$



$$P_y = I \otimes |y\rangle\langle y|$$

Simon Algorithm

$$f : \{0,1\}^n \to \{0,1\}^n$$
$$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$$



- $$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{w\in\{0,1\}^n} \left(\sum_{x:x=f^{-1}(w)} |x\rangle \right) \otimes |w\rangle$$

$$P_y|\psi\rangle = \frac{1}{\sqrt{2^n}} \left(\sum_{x:x=f^{-1}(y)} |x\rangle \right) \otimes |y\rangle$$

$$P_y = I \otimes |y\rangle\langle y|$$

$$||P_y|\psi\rangle||^2 = \frac{|f^{-1}(y)|}{2^n}$$

$$\frac{P_y|\psi\rangle}{||P_y|\psi\rangle||} = \frac{1}{\sqrt{|f^{-1}(y)|}} \sum_{x:x=f^{-1}(y)} |x\rangle \otimes |y\rangle$$

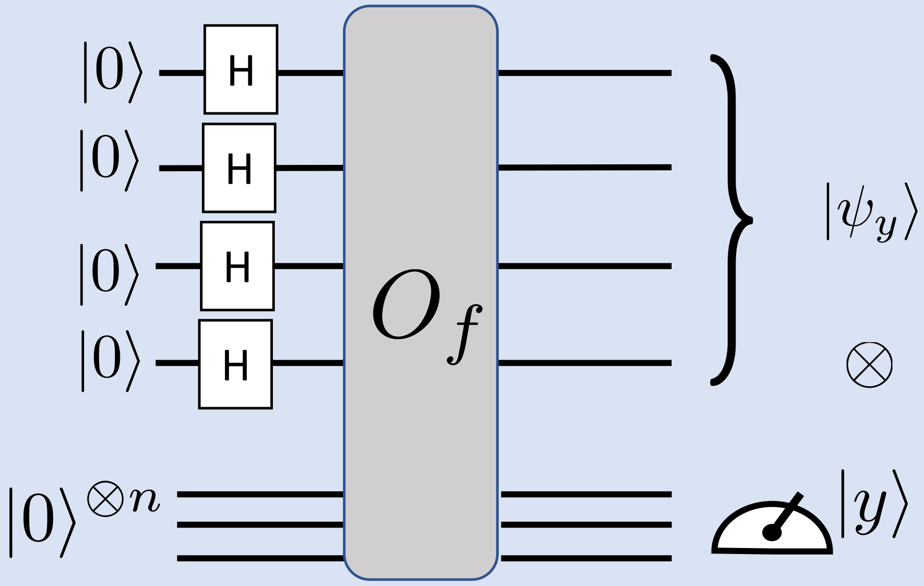
Simon Algorithm

$$f : \{0,1\}^n \to \{0,1\}^n$$

$$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$$

$$\frac{P_y|\psi\rangle}{||P_y|\psi\rangle||} = \frac{1}{\sqrt{|f^{-1}(y)|}} \sum_{x:x=f^{-1}(y)} |x\rangle \otimes |y\rangle$$

$$|\psi_y\rangle = |\phi_y\rangle \otimes |y\rangle = \left(\frac{1}{\sqrt{|f^{-1}(y)|}} \sum_{x:x=f^{-1}(y)} |x\rangle \right) \otimes |y\rangle$$



$$P_y = I \otimes |y\rangle\langle y|$$

- $f(x)$ being 2-1 function:

$$|\psi_y\rangle = \frac{1}{\sqrt{2}} \left[|x_y\rangle + |x_y \oplus a\rangle \right]$$

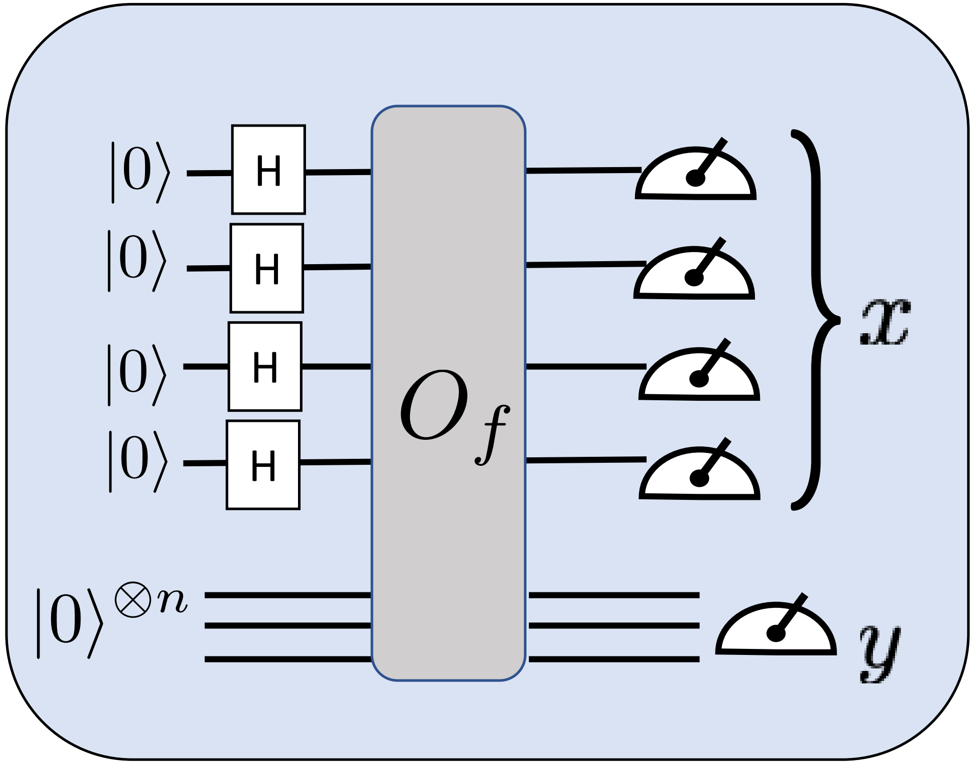
What if we measure after the oracle?

$$f : \{0,1\}^n \to \{0,1\}^n$$
$$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$$

- $f(x)$ being 2-1 function:

$$|\psi_y\rangle = \frac{1}{\sqrt{2}}\left[|x_y\rangle + |x_y \oplus a\rangle\right]$$

?

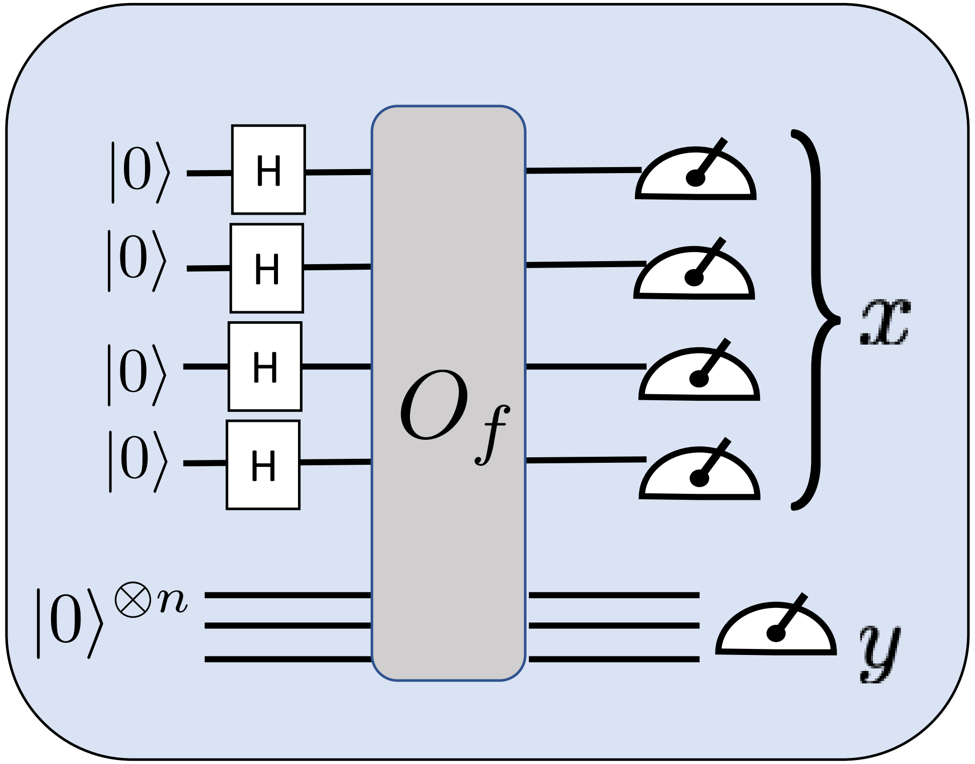


$$P_y = I \otimes |y\rangle\langle y|$$

What if we measure after the oracle?

$$f : \{0,1\}^n \to \{0,1\}^n$$

$$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$$



- $f(x)$ being 2-1 function:

$$|\psi_y\rangle = \frac{1}{\sqrt{2}}\left[|x_y\rangle + |x_y \oplus a\rangle\right]$$

$$P_y = I \otimes |y\rangle\langle y|$$

We get an input "x" associated to output "y" as y=f(x). But to guess a we need the other one and the probability to measure the same y again is going to be exponentially small.

Simon Algorithm $f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$

- Measurement outcome y :

$$|\psi_y\rangle = \frac{1}{\sqrt{2}}\left[|x_y\rangle + |x_y \oplus a\rangle\right]$$

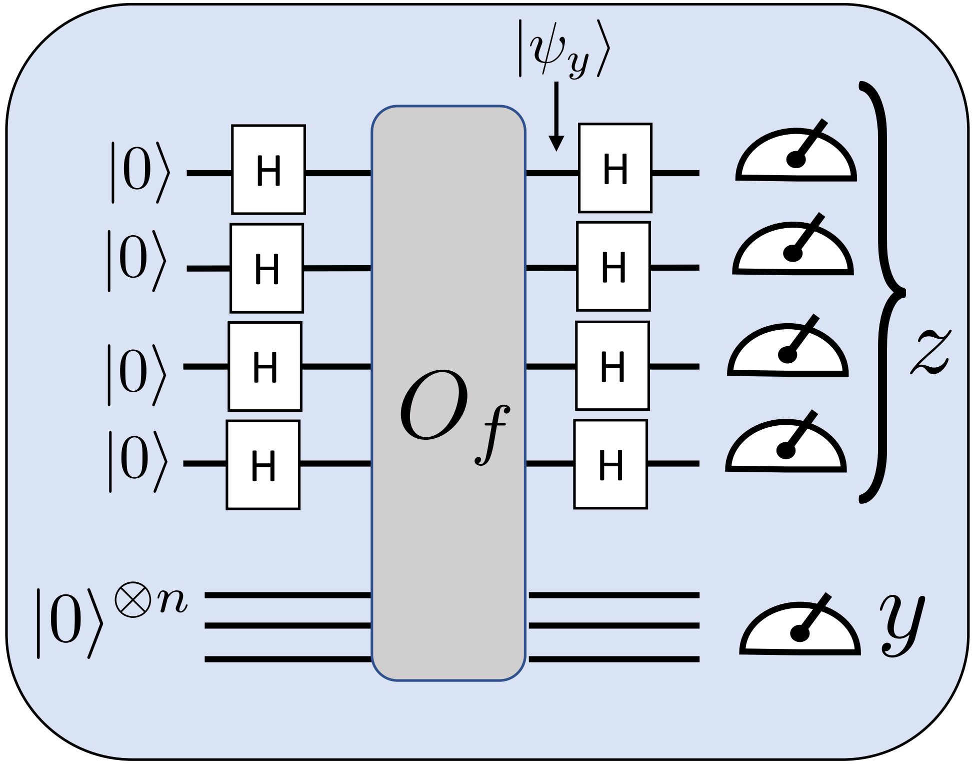
- Walsch-Hadamard:

$$|\psi_y\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^{n+1}}} \sum_{z\in\{0,1\}^n} \left[(-1)^{x_y\cdot z} + (-1)^{(x_y\oplus a)\cdot z}\right]|z\rangle$$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{z\in\{0,1\}^n} (-1)^{x_y\cdot z}\left[1 + (-1)^{a\cdot z}\right]|z\rangle$$

$$= \frac{1}{\sqrt{2^{n-1}}} \sum_{z:a\cdot z=0} (-1)^{x_y\cdot z}|z\rangle$$

$$\boxed{\{z \in \{0,1\}^n : a\cdot z = 0\}}$$



Walsh-Hadamard transform

$$|x\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle$$

We generate samples z of n bits that satisfy: $az = 0$.

Simon Algorithm

Quantum: $O(n)$
Classical: ?

$f:\{0,1\}^n \to \{0,1\}^n$

$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$

- Classical post-processing: solve system of linear eq. $\Rightarrow$ a

$$\begin{cases} a_1 z_1^{(1)} + a_2 z_2^{(1)} + \ldots + a_n z_n^{(1)} = 0 \bmod 2 \\ a_1 z_1^{(2)} + a_2 z_2^{(2)} + \ldots + a_n z_n^{(2)} = 0 \bmod 2 \\ \vdots \\ a_1 z_1^{(n-1)} + a_2 z_2^{(n-1)} + \ldots + a_n z_n^{(n-1)} = 0 \bmod 2 \end{cases}$$

1. Each outcome generates a samples z of n bits that satisfy: $$az = 0.$$
2. We generate n samples.
3. If n equations are independent we solve system of equations.

- We always have $a = 0$ as solution $\Rightarrow$ We need only $n-1$ equations.
- Probability of linear independent set $P > \frac{1}{4}$

Simon Algorithm

- $\{z \in \{0,1\}^n : a \cdot z = 0\}$ is of size $2^n/2 = 2^{n-1}$
- k lin. indep. z^i has size 2^k

- Probability of linear independent set

$$P = \prod_{k=0}^{n-2} \left(\frac{2^{n-1} - 2^k}{2^{n-1}} \right) = \prod_{j=1}^{n-1} \left(1 - \frac{1}{2^j} \right)$$

Relabeling: $j = n - k - 1$

$$= \prod_{j=1}^{n-1} \left(1 - \frac{1}{2^j} \right) \geq \prod_{j=1}^{\infty} \left(1 - \frac{1}{2^j} \right)$$

Extend the product with terms < 1

Use: $(1-a)(1-b) > 1 - (a+b)$

$$= \frac{1}{2} \prod_{j=2}^{\infty} \left(1 - \frac{1}{2^j} \right) > \frac{1}{2} \left[1 - \sum_{j=2}^{\infty} \frac{1}{2^j} \right]$$

$$> \frac{1}{2} \left[1 - \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{2^j} \right] > \frac{1}{4}$$

Geometric series

Classical queries

$$f : \{0,1\}^n \to \{0,1\}^n$$

$$f(x) = f(x') \Leftrightarrow x = x' \text{ or } x = x' \oplus a$$

- We need to find a pair x and x' such that $f(x) = f(x')$

  Solution $a = x \oplus x'$

- Success probability: $\Pr[x = x' \oplus a] = \dfrac{1}{2^n - 1}$

- For T queries we have at most T^2 different pairs:
  - $\Pr[\text{ find } a] \leq T^2 2^{-n}$
  - $T \geq \sqrt{P} 2^{n/2}$

Quantum: $O(n)$

Classical: $\Omega(2^{n/2})$

We do not need to measure lower register

- Measurement outcome y :

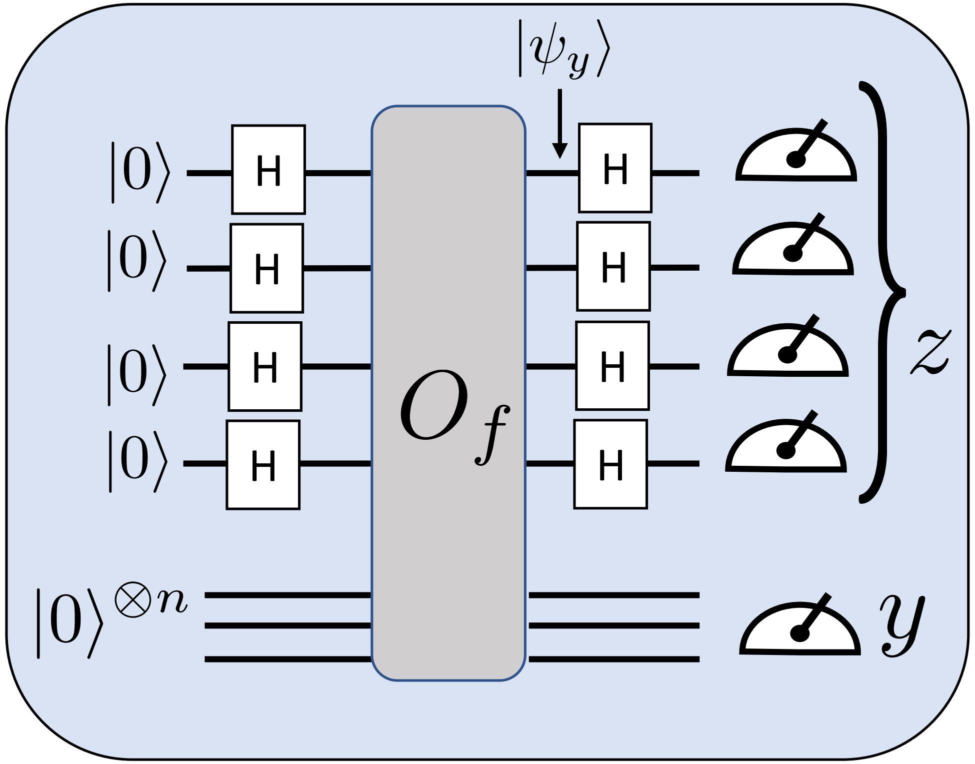
$$|\psi_y\rangle = \frac{1}{\sqrt{2}}\left[|x_y\rangle + |x_y \oplus a\rangle\right]$$

- Walsch-Hadamard:

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_{z:a\cdot z=0} (-1)^{x_y\cdot z}|z\rangle$$

$$\{z \in \{0,1\}^n : a\cdot z = 0\}$$

Neither “x_y” nor “y” play play a role in the post-processing. We could forget the value of “y” and nothing would change!



We do not need to measure lower register

- Measurement outcome y :

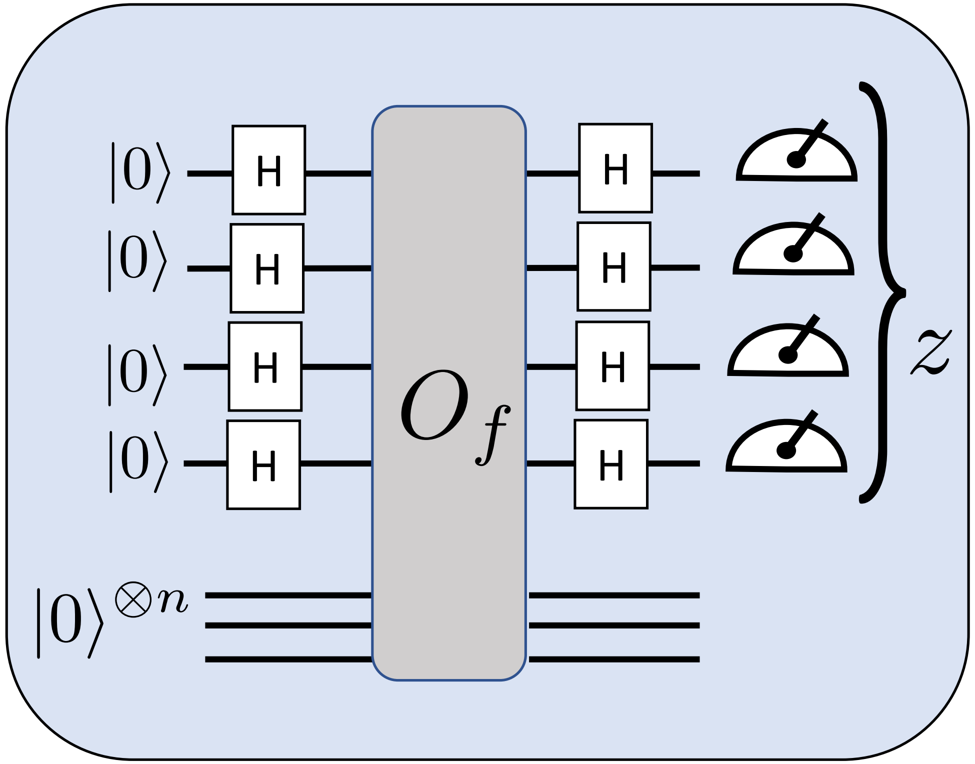
$$|\psi_y\rangle = \frac{1}{\sqrt{2}}\left[|x_y\rangle + |x_y \oplus a\rangle\right]$$

- Walsch-Hadamard:

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_{z:a\cdot z=0} (-1)^{x_y\cdot z}|z\rangle$$

$$\boxed{\{z \in \{0,1\}^n : a\cdot z = 0\}}$$

Neither "x_y" nor "y" play play a role in the post-processing. We could forget the value of "y" and nothing would change!



References

Reading references

1. Simon: NC 1.4.3 RdW 3 and G 8

NC ≣ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information Cambridge University Press (2010)
RdW ≣ Quantum Computing Lecture Notes, Ronald de Wolf, https://arxiv.org/abs/1907.09415
G ≣ Introduction to Quantum Computation, Sevag Gharibian, Lectures notes

Simon Algorithm breaks cryptographic primitives

1. T. Santoli and C. Schaffner, *Using Simon's algorithm to attack symmetric-key cryptographic primitives*, Quantum Information & Computation 17, 65 (2017).
2. H. Kuwakado and M. Morii, *Quantum distinguisher between the 3-round feistel cipher and the random permutation,* In 2010 IEEE International Symposium on Information Theory, pages 2682-2685, June (2010).
3. M. Kaplan, Gaetan L., A. Leverrier, and M. Naya-Plascencia, *Breaking symmetric cryptosystems using quantum period finding,* In Advances in Cryptology - CRYPTO 2016., volume 9815 of Lecture Notes in Computer Science, 2016.