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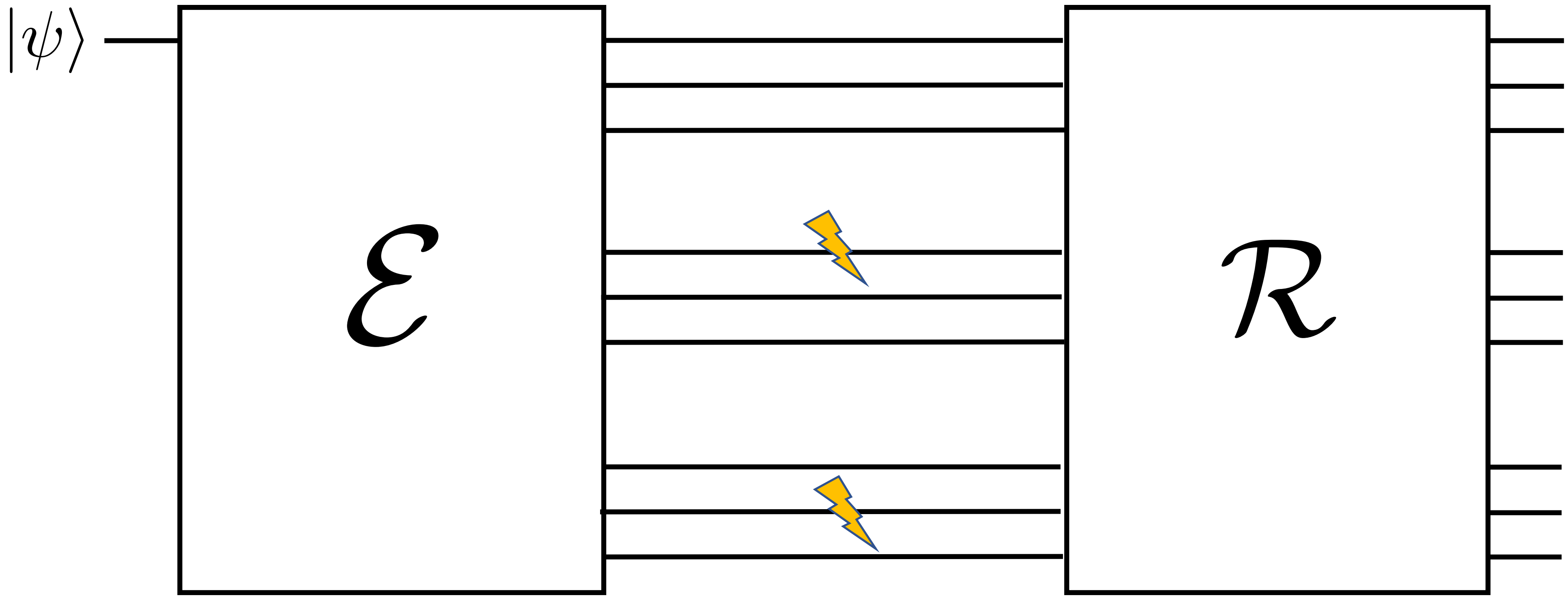
Introduction to Quantum Computing

Lecture 15: Parity Check

Raul Garcia-Patron Sanchez

Quantum Error Correction in one slide

- Noise is local: independent for every qubit
- Our operations are perfect: error due to interaction with local environment.



- In reality we have also correlated error
- Our operations have errors $\Rightarrow$ Theory of fault-tolerance

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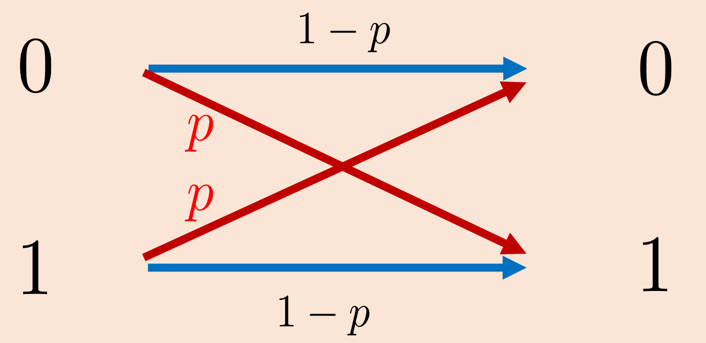
Classical repetition code and parity check

The repetition code

Encoding $0_L \to 000$ Codeword
$1_L \to 111$

Binary symmetric channel



$000 \to 010$

Error detection: majority vote

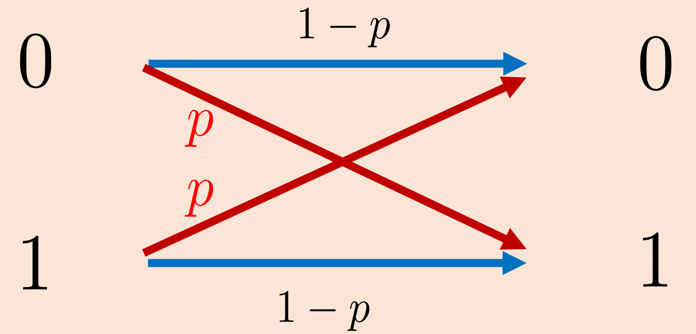
Error correction
$\{000, 100, 010, 001\} \to 0_L$
$\{111, 110, 011, 101\} \to 1_L$

The repetition code

Encoding $0_L \to 000$ Codeword

$1_L \to 111$

Binary symmetric channel



$000 \to 010$

Error detection: majority vote

Error correction

$\{000, 100, 010, 001\} \to 0_L$

$\{111, 110, 011, 101\} \to 1_L$

$[n, k, d]$ code

n : size of codeword

k : # logical bits

d : Hamming distance

e : correctable error

$$d = 2e + 1$$

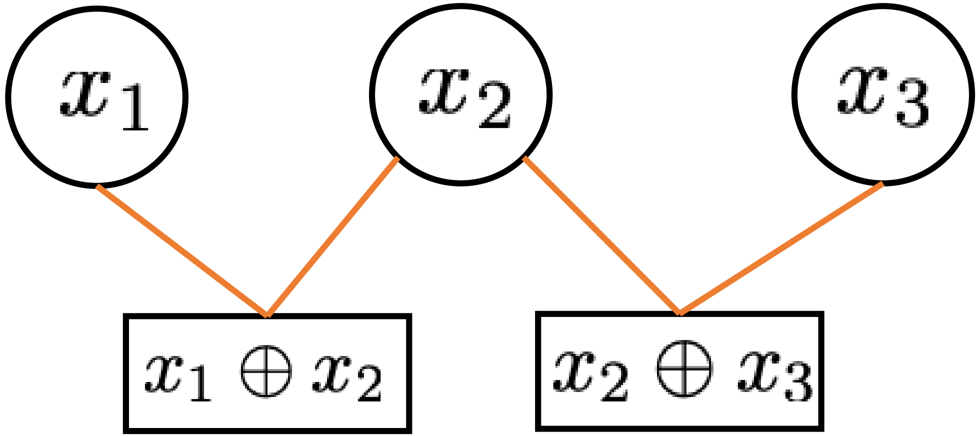
Repetition code

$[3, 1, 3]$ code

Parity checks

Encoding $0_L \rightarrow 000$ Codeword

$1_L \rightarrow 111$



Message	Parity Check 1	Parity Check 2	Error location	Output
000	0	0	no	000
001	0	1	3	000
010	1	1	2	000
100	1	0	1	000
011	1	0	1	111
101	1	1	2	111
110	0	1	3	111
111	0	0	no	111

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Quantum parity check

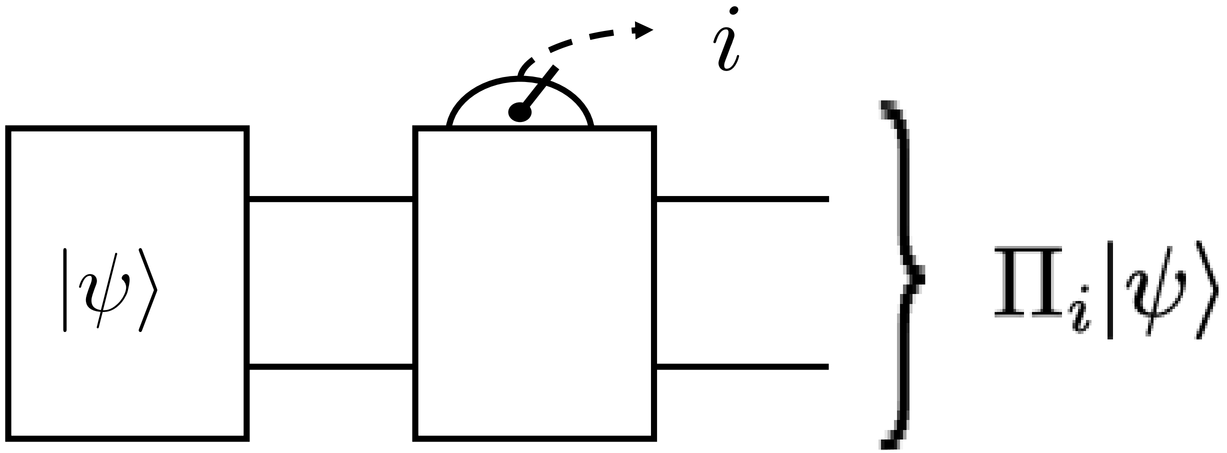
Example 2: Parity of 2 qubits

- $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o = \text{span}\{|00\rangle, |11\rangle\} \oplus \text{span}\{|01\rangle, |10\rangle\}$

$\oplus \equiv$ Direct sum

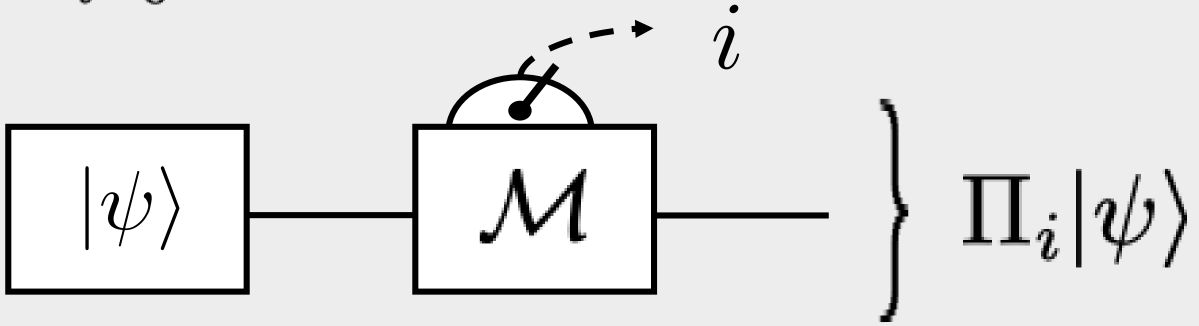
- $\Pi_e = |00\rangle\langle 00| + |11\rangle\langle 11|$
  $\Pi_o = |01\rangle\langle 01| + |10\rangle\langle 10|$

$$\left.\begin{aligned} \Pi_e &= |00\rangle\langle 00| + |11\rangle\langle 11| \\ \Pi_o &= |01\rangle\langle 01| + |10\rangle\langle 10| \end{aligned}\right\} \; \Pi_e + \Pi_o = I_4$$

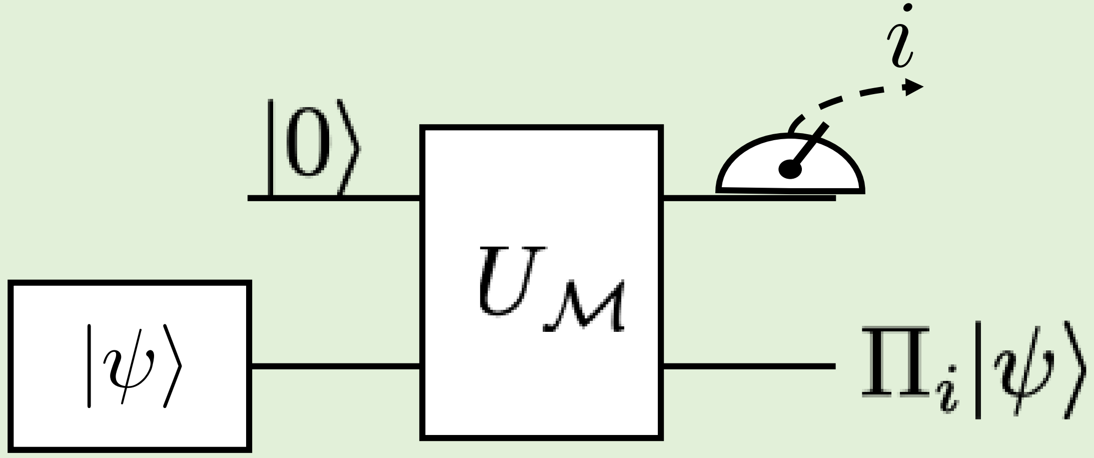


From the math definition to an actual circuit

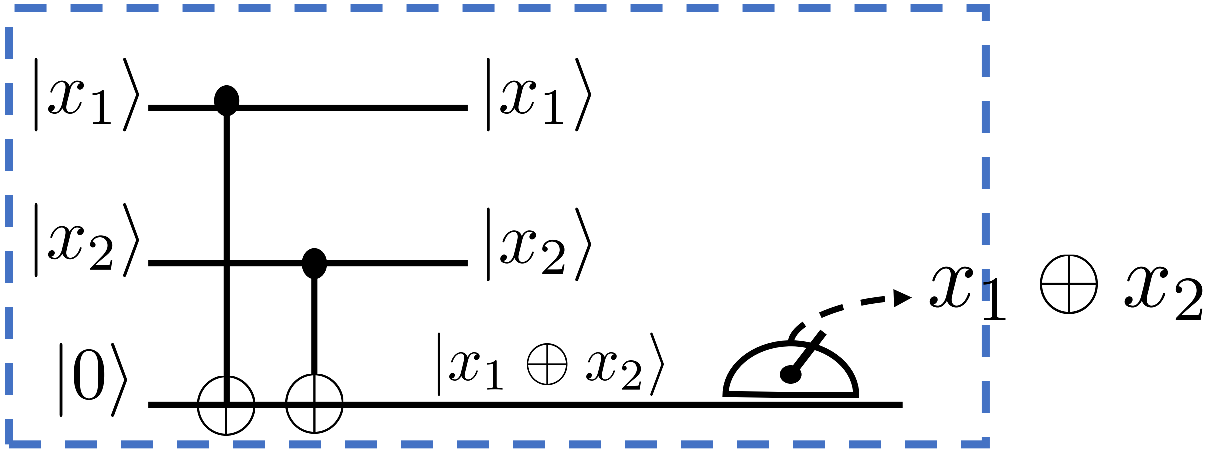
$$\sum_{i=0}^{l} \Pi_l = I_{2^n} \qquad P(i) = ||\Pi_i|\psi\rangle||^2 \qquad \text{Update: } \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$



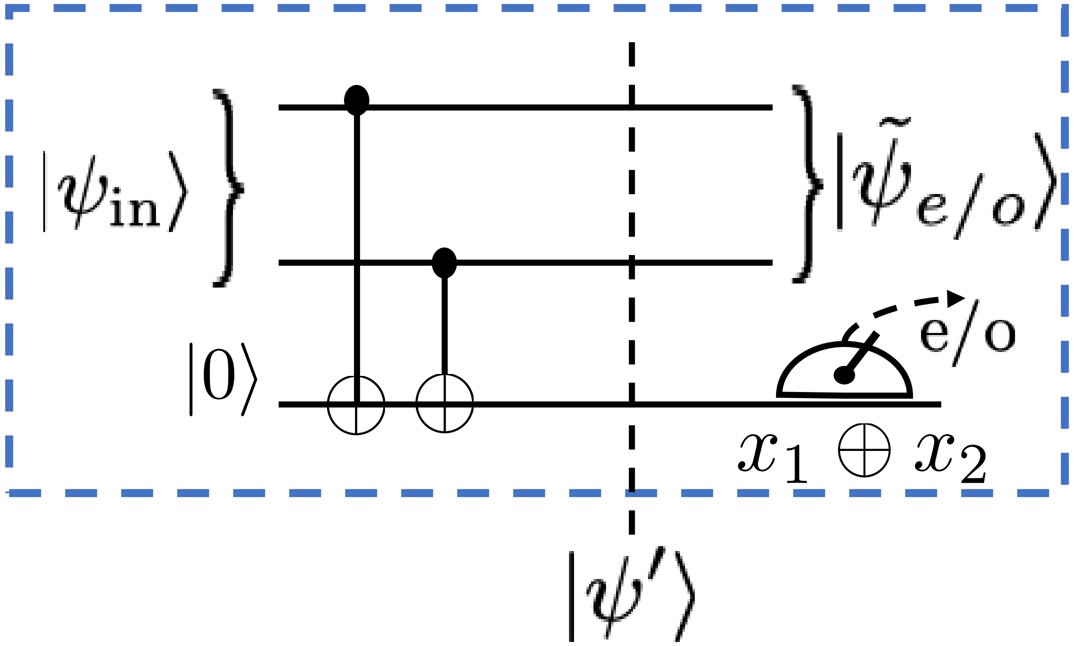
We need to find the circuit $U_{\mathcal{M}}$



Parity check circuit



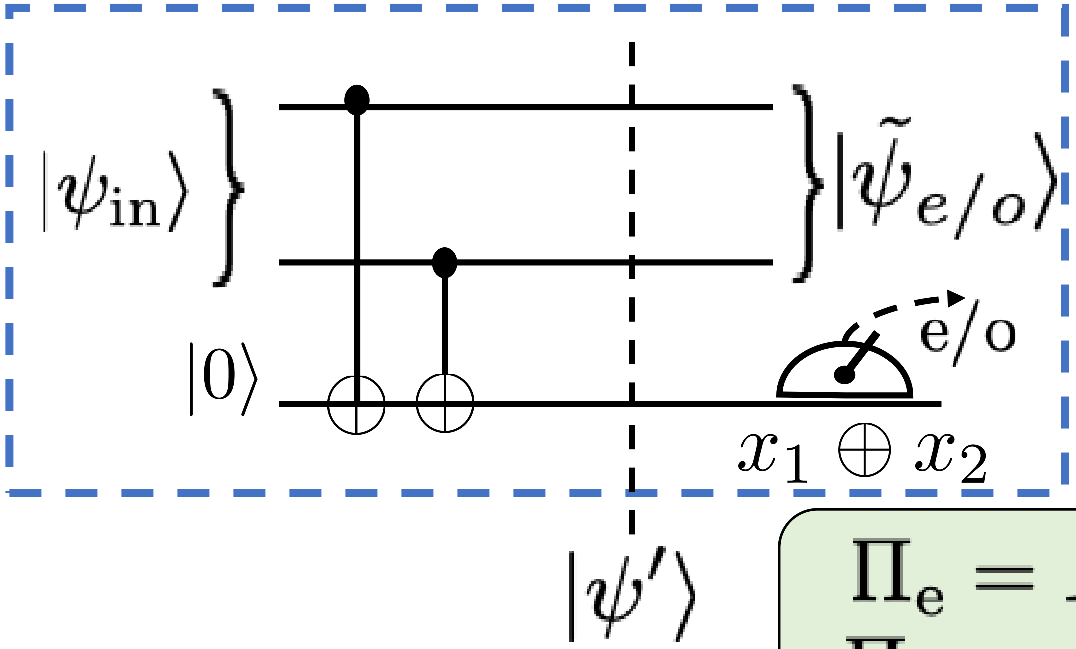
Parity check circuit – general input



$$|\psi_{\text{in}}\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2} |x_1, x_2\rangle \otimes |0\rangle$$

$$|\psi'\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2} |x_1, x_2\rangle \otimes |x_1 \oplus x_2\rangle$$

Parity check circuit – general input



$$|\psi_{\text{in}}\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2}|x_1,x_2\rangle \otimes |0\rangle$$

$$|\psi'\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2}|x_1,x_2\rangle \otimes |x_1 \oplus x_2\rangle$$

$$\Pi_{\text{e}} = I \otimes I \otimes |0\rangle\langle 0|$$
$$\Pi_{\text{o}} = I \otimes I \otimes |1\rangle\langle 1|$$

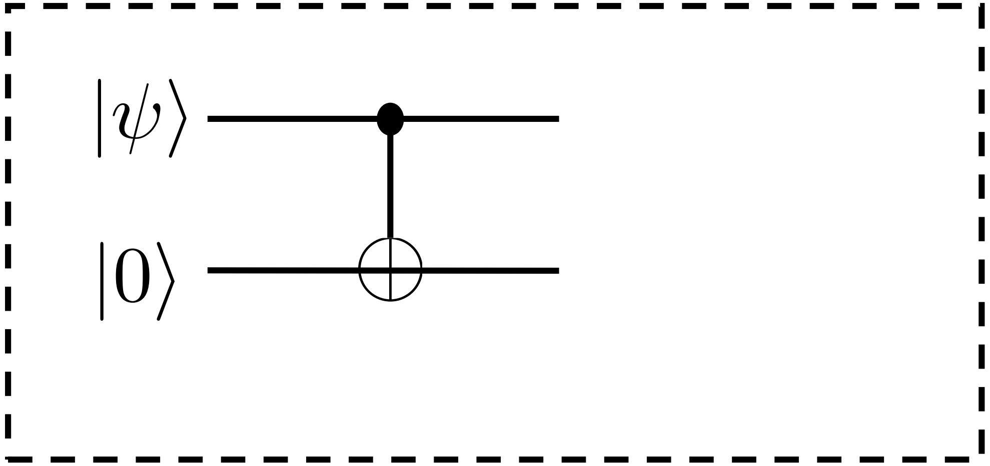
$$\Pi_{\text{e}} + \Pi_{\text{o}} = I_{\mathcal{H}^{\otimes 3}}$$

$$\Pi_o|\psi\rangle = (\psi_{0,0}|0,1\rangle + \psi_{1,0}|1,0\rangle) \otimes |1\rangle$$

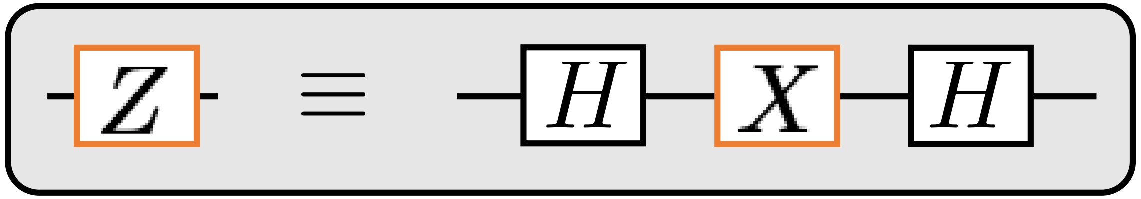
$$||\Pi_{\text{o}}|\psi\rangle||^2 = |\psi_{0,1}|^2 + |\psi_{1,0}|^2$$

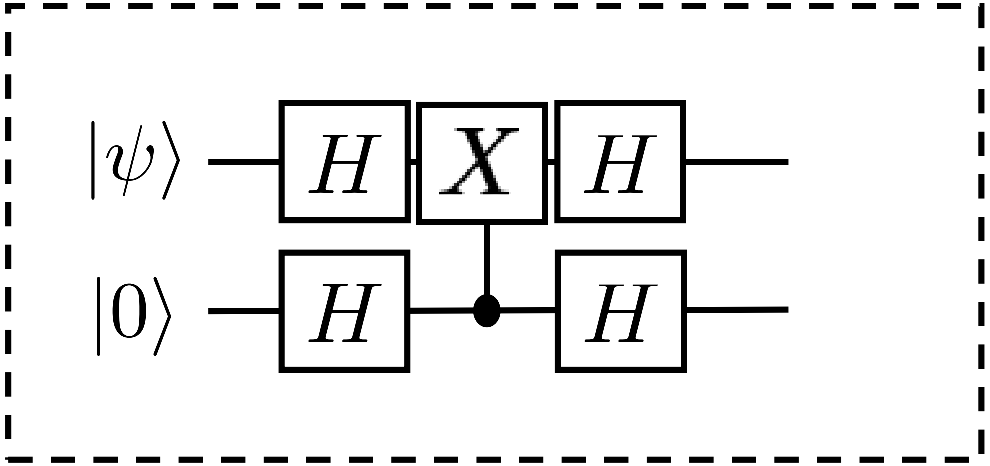
$$\frac{\Pi_{\text{o}}|\psi\rangle}{||\Pi_{\text{o}}|\psi\rangle||} = \frac{1}{\sqrt{|\psi_{0,1}|^2 + |\psi_{1,0}|^2}} (\psi_{0,1}|0,1\rangle + \psi_{1,0}|1,0\rangle) \otimes |1\rangle = |\tilde{\psi}_{\text{o}}\rangle \otimes |1\rangle$$

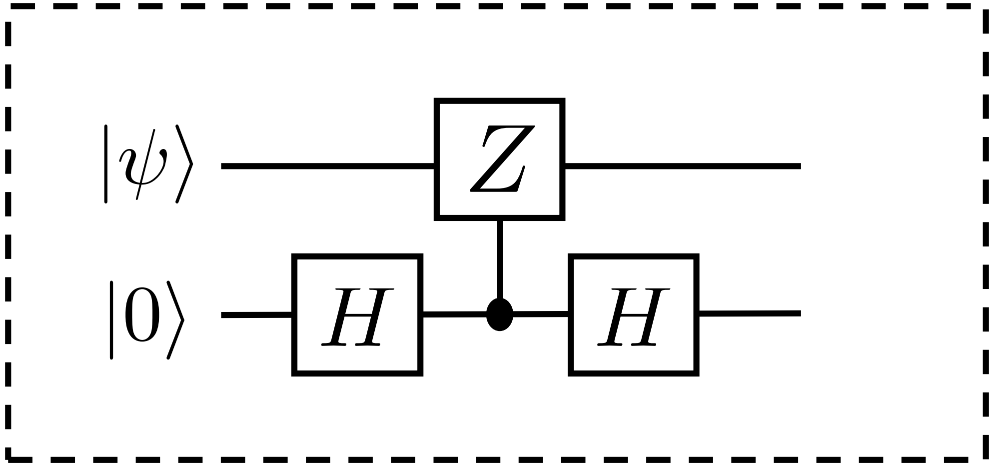
An equivalent circuit



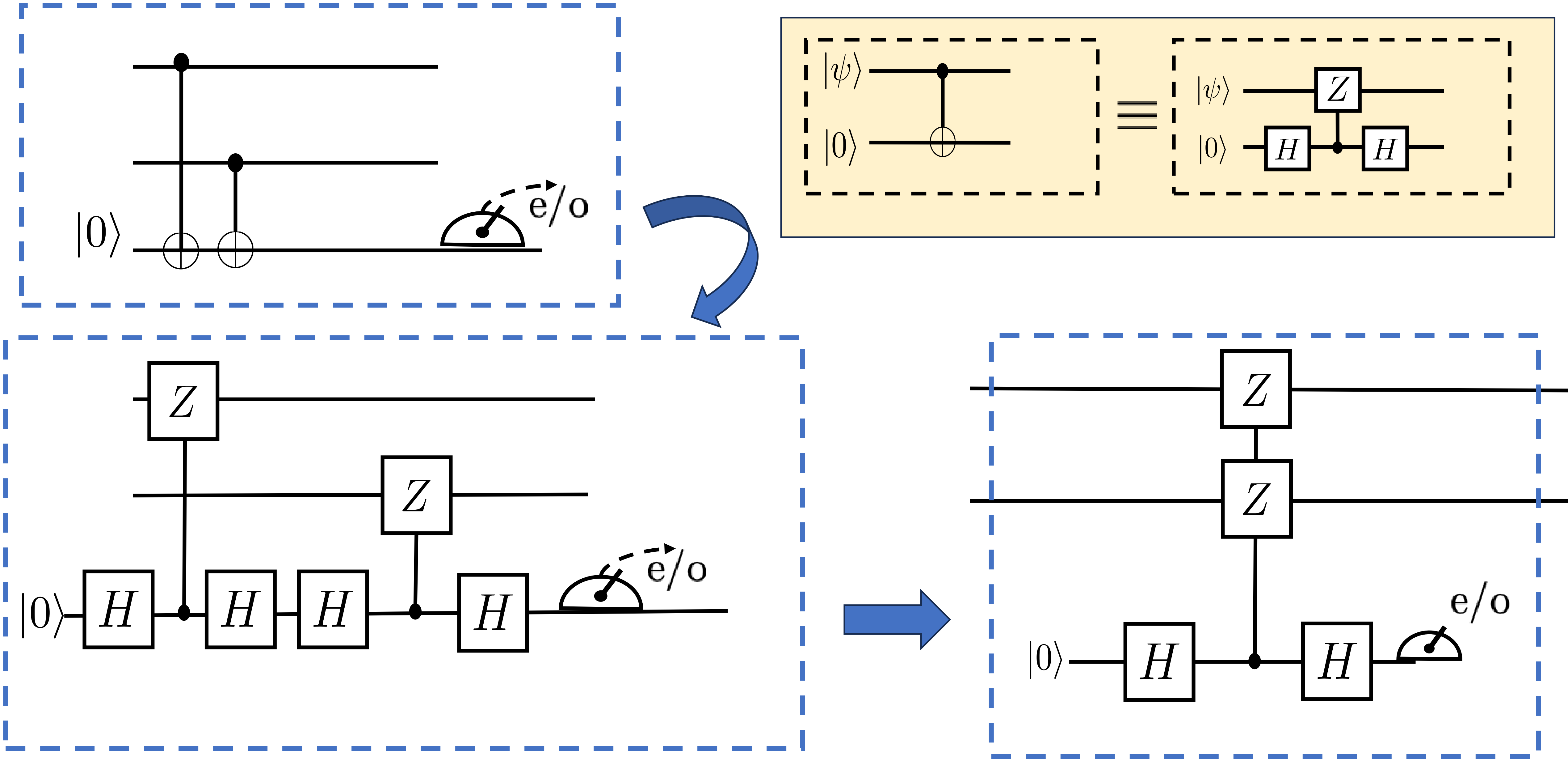
Tutorial 2 problem 2



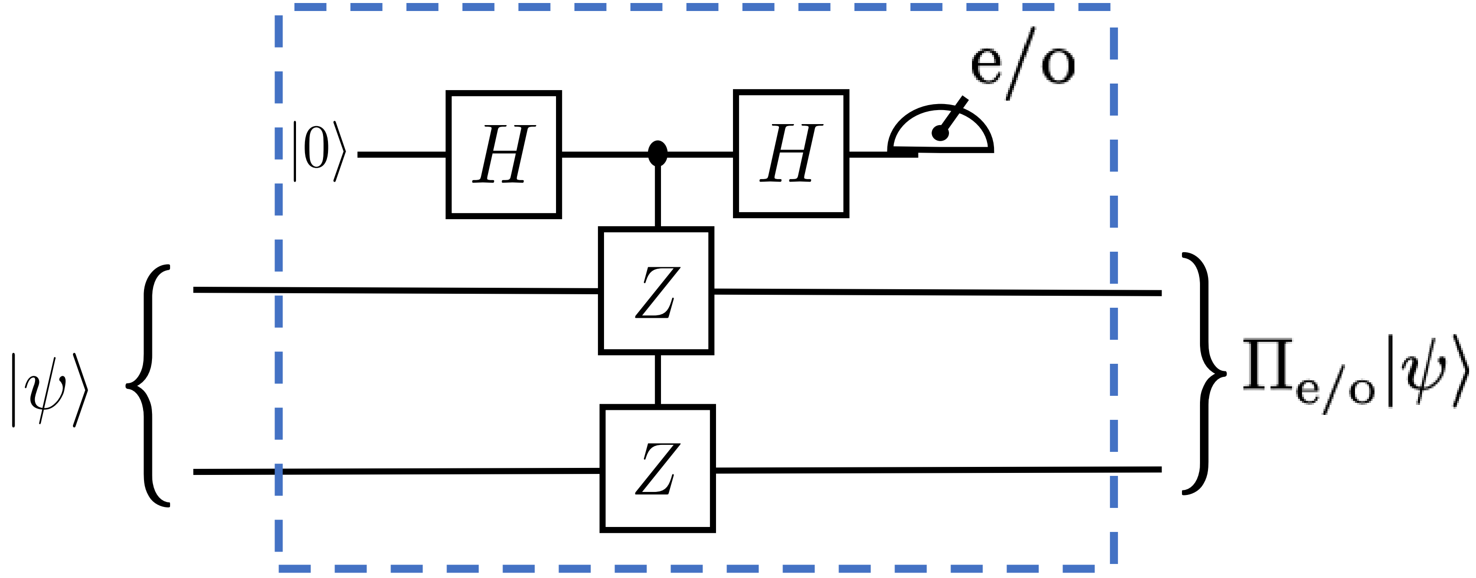




Hadamard test circuit for parity check



Hadamard test circuit for parity check



$$Z_1 \otimes Z_2 |x_1 x_2\rangle = (-1)^{x_1 \oplus x_2} |x_1 x_2\rangle$$
$$\Pi_{\mathrm{e}} = |00\rangle\langle 00| + |11\rangle\langle 11|$$
$$\Pi_{\mathrm{o}} = |01\rangle\langle 01| + |10\rangle\langle 10|$$
$$(Z \otimes Z)\Pi_{\mathrm{e/o}} = \pm \Pi_{\mathrm{e/o}}$$

The operator ZZ:

Eigenspace eigenvalue 1 is Even parity

Eigenspace eigenvalue -1 is Odd parity

This will generalize to the Hadamard Test, see next lecture

Parity check of phase

The intuition

$$X \equiv H\,Z\,H \qquad H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) = |\pm\rangle$$

Bit flip error

X

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|x\rangle = |x \oplus 1\rangle$$

- Basis $Z|x\rangle = (-1)^x|x\rangle$
- Parity check $Z \otimes Z$

$$\Pi_{\mathrm{e}} = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\Pi_{\mathrm{o}} = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$(Z \otimes Z)\Pi_{\mathrm{e/o}} = \pm\Pi_{\mathrm{e/o}}$$

H (↔)

Phase flip error

Z

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|\pm\rangle = |\mp\rangle$$

$$X|\pm\rangle = \pm|\pm\rangle$$

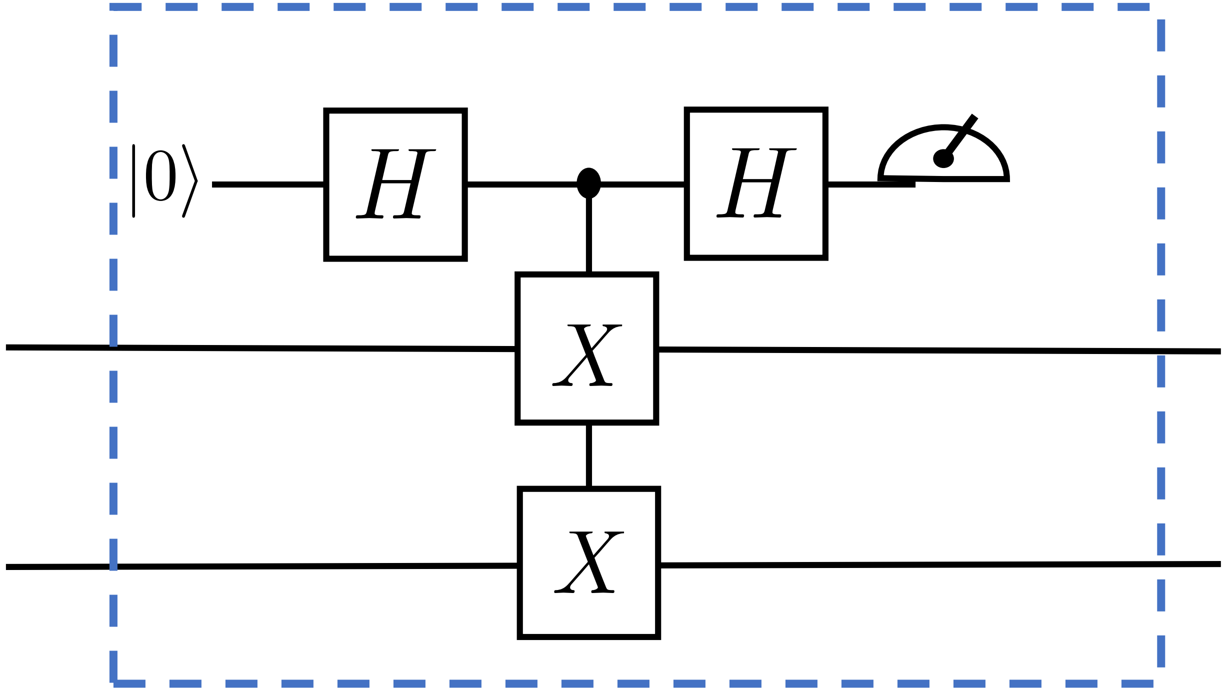
$$X_i \otimes X_j$$

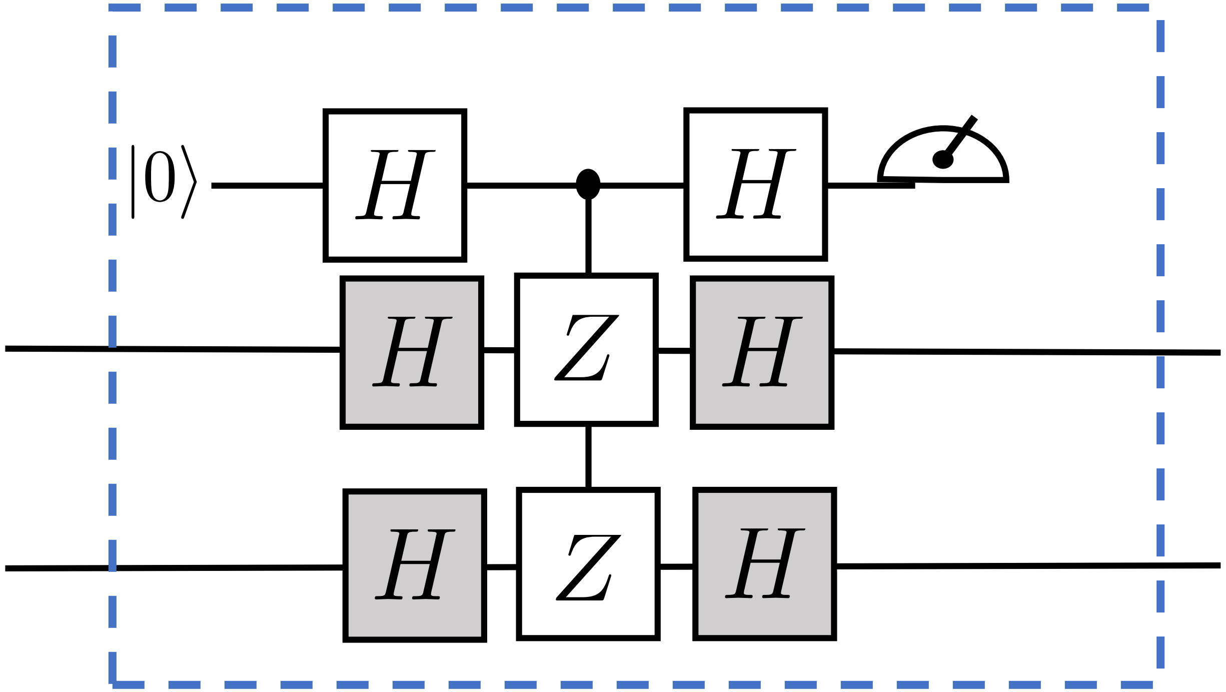
$$\tilde{\Pi}_{\mathrm{e}} = |{++}\rangle\langle{++}| + |{--}\rangle\langle{--}|$$

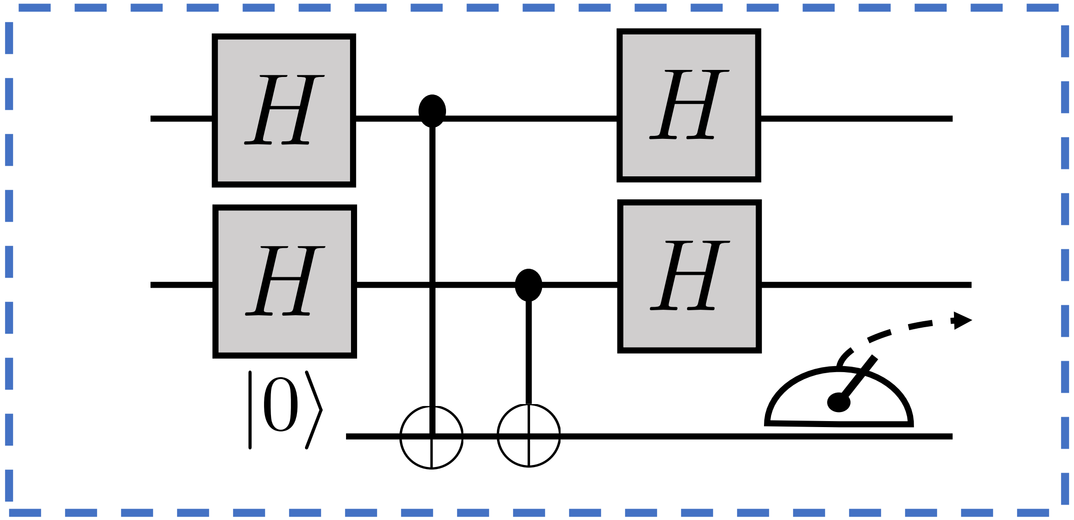
$$\tilde{\Pi}_{\mathrm{o}} = |{+-}\rangle\langle{+-}| + |{-+}\rangle\langle{-+}|$$

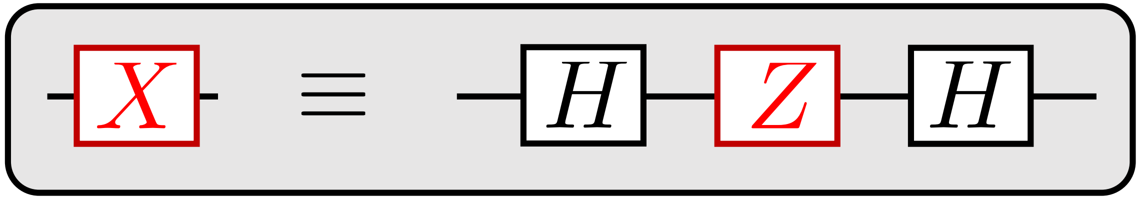
$$(X \otimes X)\tilde{\Pi}_{\mathrm{e/o}} = \pm\tilde{\Pi}_{\mathrm{e/o}}$$

Parity on X basis









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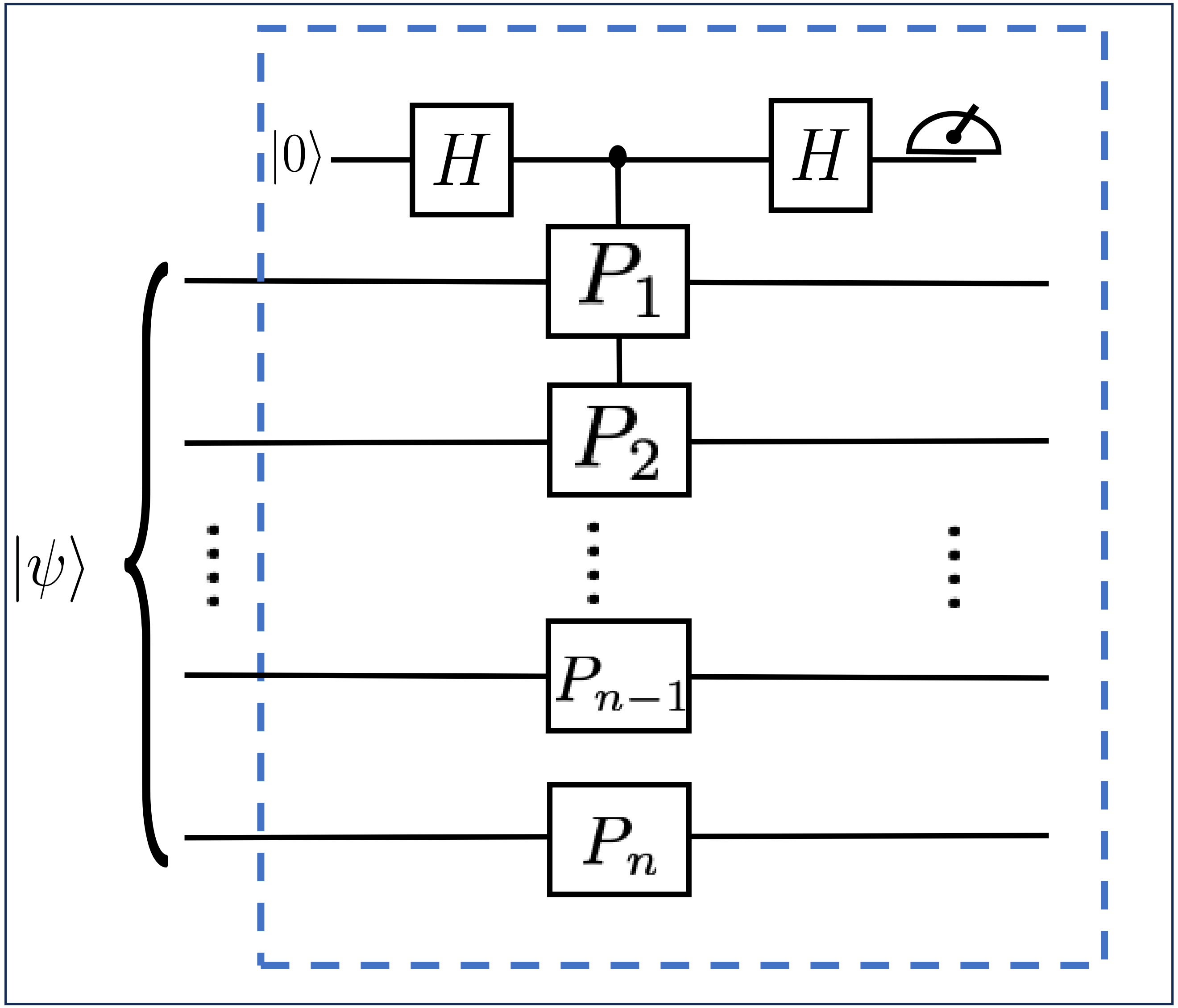
General Parity Check – QEC lectures

Generalized parity check



Parity-check operator (syndrome)

$$P = P_1 \otimes P_2 \otimes \ldots \otimes P_{n-1} \otimes P_n$$

A string of Pauli matrices

$$P_i \in \{I, X, Y, Z\}$$

$$P\Pi_{\pm 1} = \pm \Pi_{\pm 1}$$

$\Pi_{\pm 1}$ eigenspace with eigenvalue ± 1

$$\Pi_{+1} + \Pi_{-1} = I_{2^n}$$

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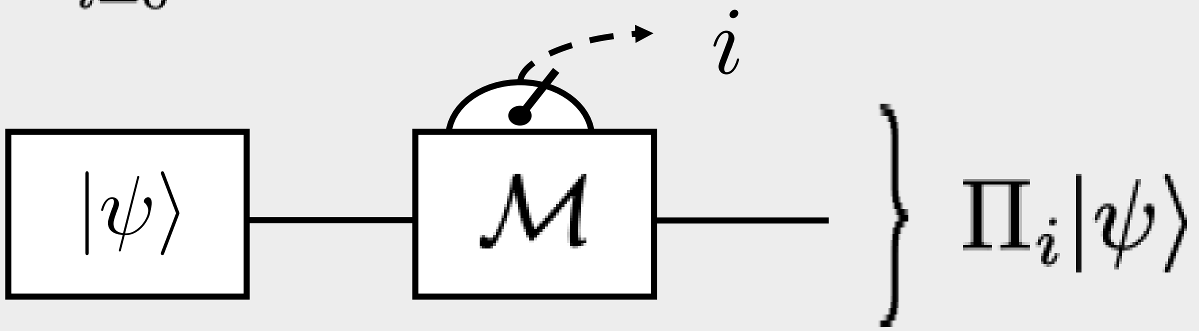
Non demolition measurement of 1 qubit

From the math definition to an actual circuit

$$\sum_{i=0}^{l} \Pi_l = I_{2^n} \qquad P(i) = ||\Pi_i|\psi\rangle||^2 \qquad \text{Update: } \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$



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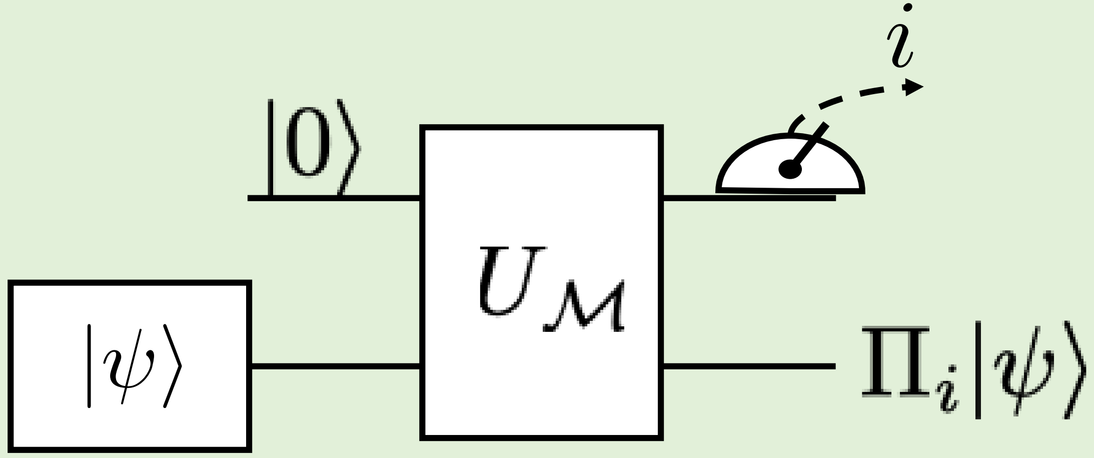
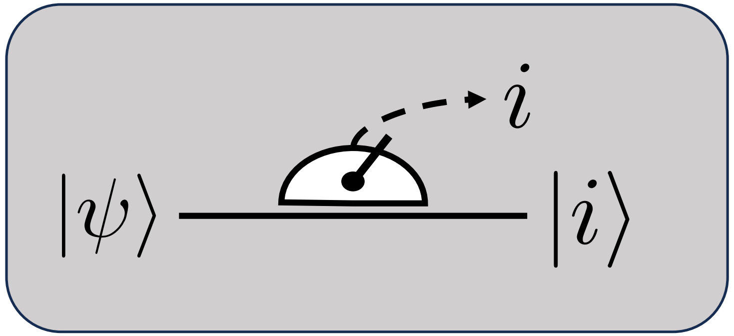


A quantum register for the measurement apparatus

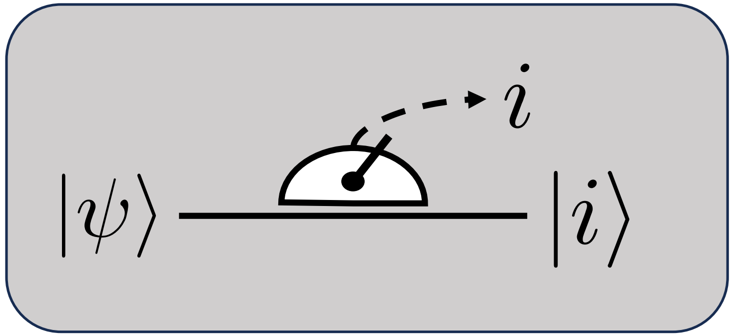


Projective measurement

$$\Pi_i^2 = \Pi_i \qquad \sum_{i=0}^{l} \Pi_i = I$$

$$P(i) = ||\Pi_i|\psi\rangle||^2 \qquad \text{Update: } \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$

A quantum register for the measurement apparatus

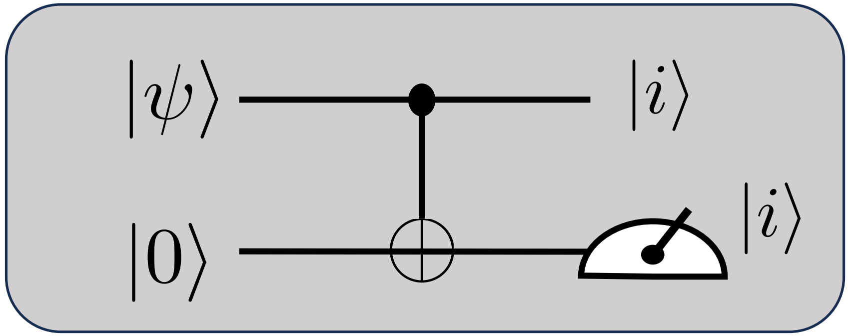


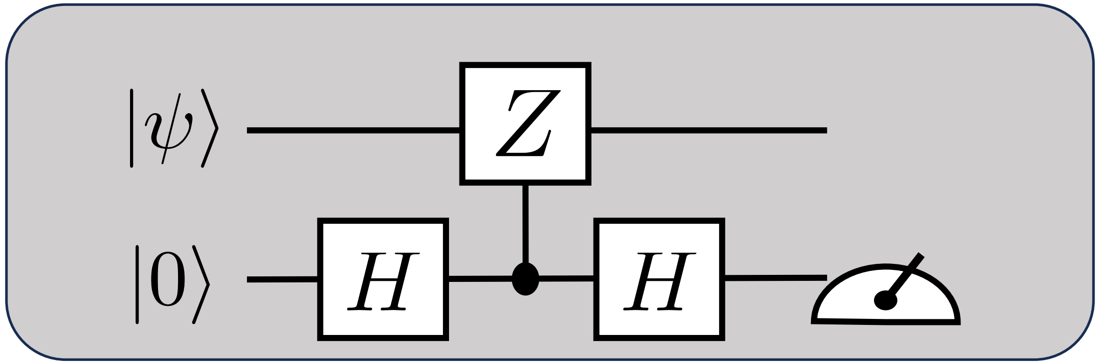
Projective measurement

$$\Pi_i^2 = \Pi_i \qquad \sum_{i=0}^{l} \Pi_i = I$$

$$P(i) = ||\Pi_i|\psi\rangle||^2 \qquad \text{Update: } \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$

- Quantum non-demolition measurement: lower register model measurement apparatus





Hadamard Test of Z

References

Reading references

1. Operators and Observables: NC 2.1.2, 2.15, 2.1.6,
2. Projective measurement: NC 2.2.5
3. Parity measurement: NC 10.5.8

NC ≣ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information
Cambridge University Press (2010)