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Introduction to Quantum Computing

Lecture 16: Hadamard Test

Raul Garcia-Patron Sanchez

Hadamard Test – binary outcome measurements

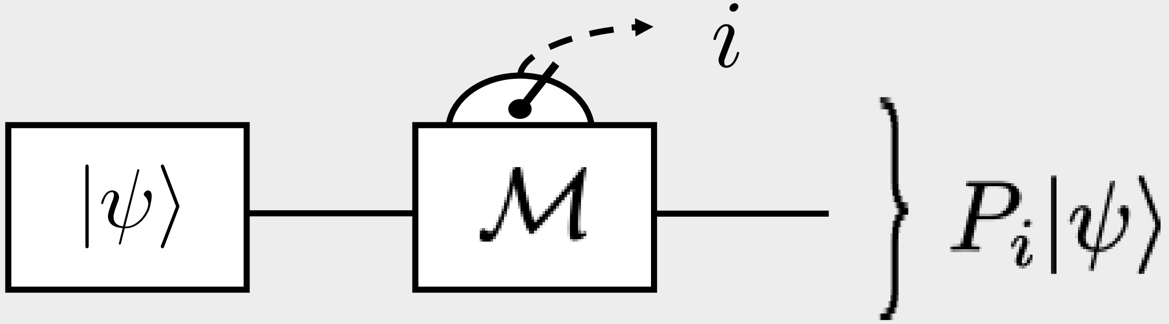
We want to ask the question: "Are you in subspace Π_0 or Π_1?

$$\Pi_0 + \Pi_1 = I_{2^n}$$

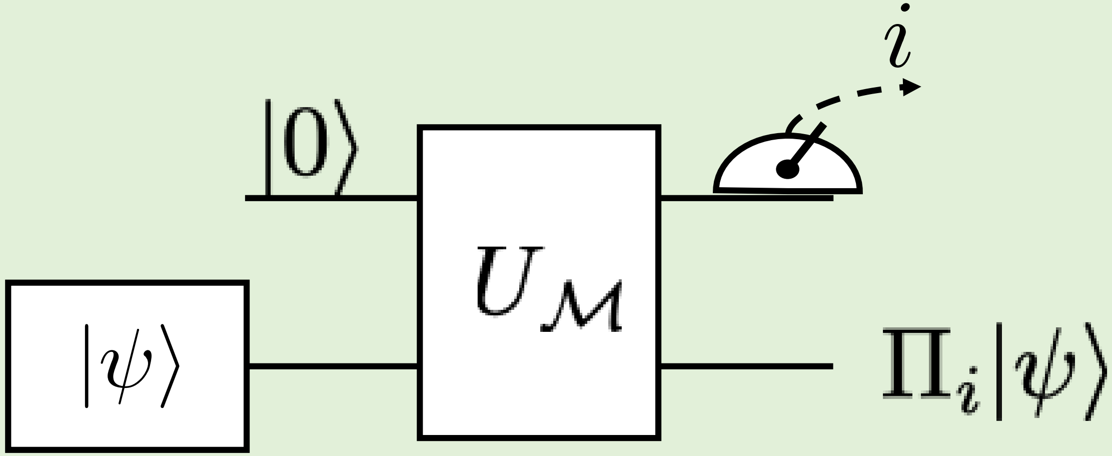
- Syndrome measurement in quantum error correction
- SWAP test: overlap between quantum states, measurement of purity with two copies

From the definition to designing a circuit implementing it

$$\Pi_0 + \Pi_1 = I_d \qquad P(i) = ||\Pi_i|\psi\rangle||^2 \qquad \text{Update: } \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$



We need to find the circuit $U_{\mathcal{M}}$

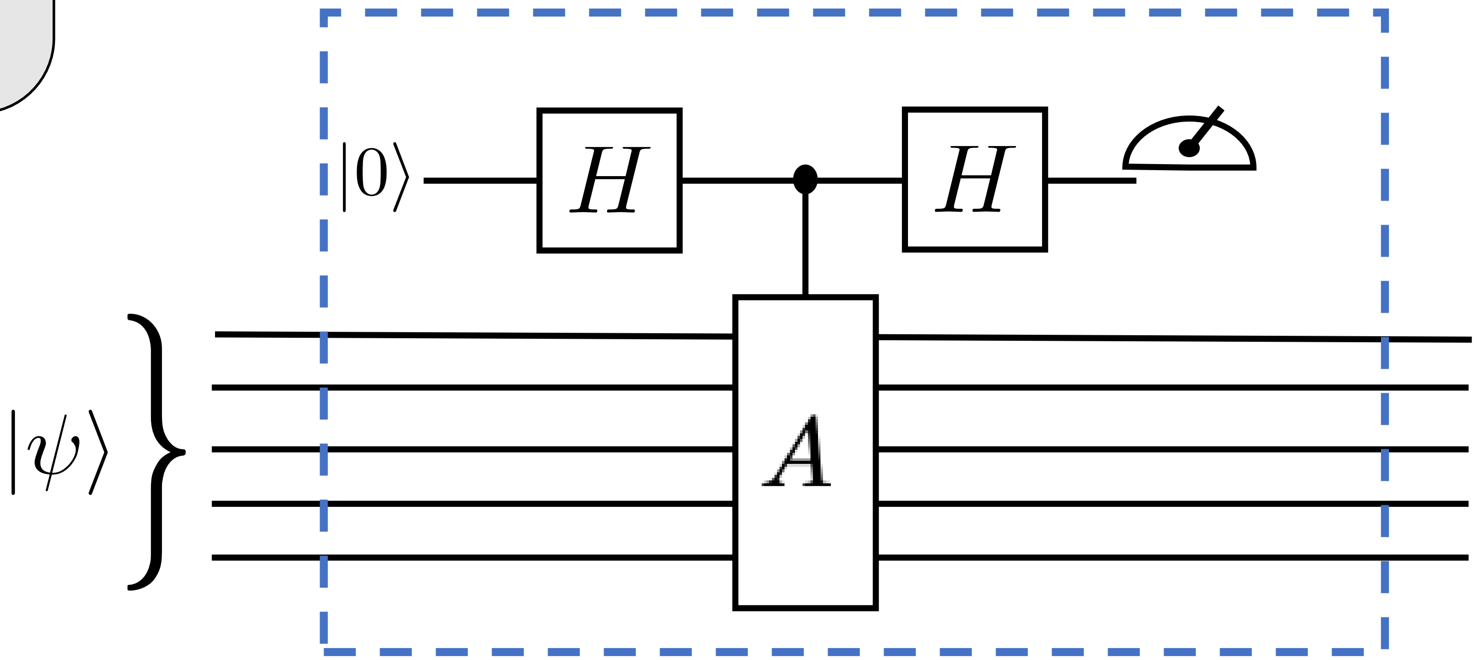


Hadamard Test – binary outcome measurements

Exist unitary A:

$$A = \Pi_0 - \Pi_1$$

$$\Pi_0 + \Pi_1 = I_{2^n}$$



- Syndrome measurement in quantum error correction
- SWAP test: overlap between quantum states, measurement of purity with two copies

Hadamard Test – unitary is observable with ±1 eigenvalues

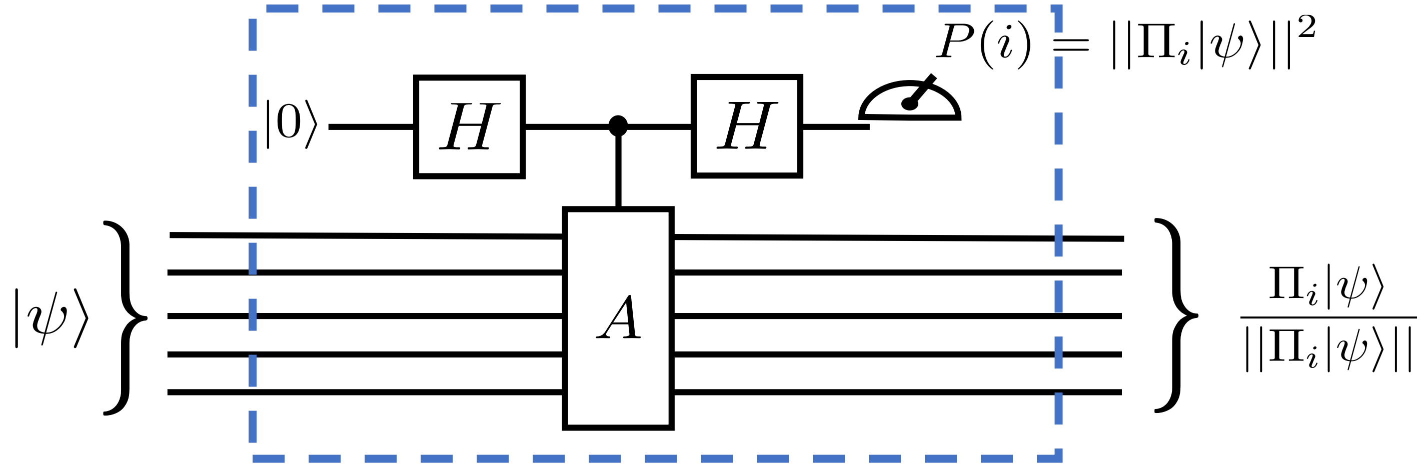
Observable ±1 eigenvalues

$$A = \Pi_0 - \Pi_1$$

$$\Pi_0 + \Pi_1 = I_{2^n}$$

- A is also a unitary matrix

$$AA^\dagger = (\Pi_0 - \Pi_1)(\Pi_0 - \Pi_1) = \Pi_0^2 - \Pi_0\Pi_1 - \Pi_1\Pi_0 + \Pi_1^2 = \Pi_0 + \Pi_1 = I$$



Hadamard Test – unitary is observable with ±1 eigenvalues

Observable ±1 eigenvalues

$$A = \Pi_0 - \Pi_1$$

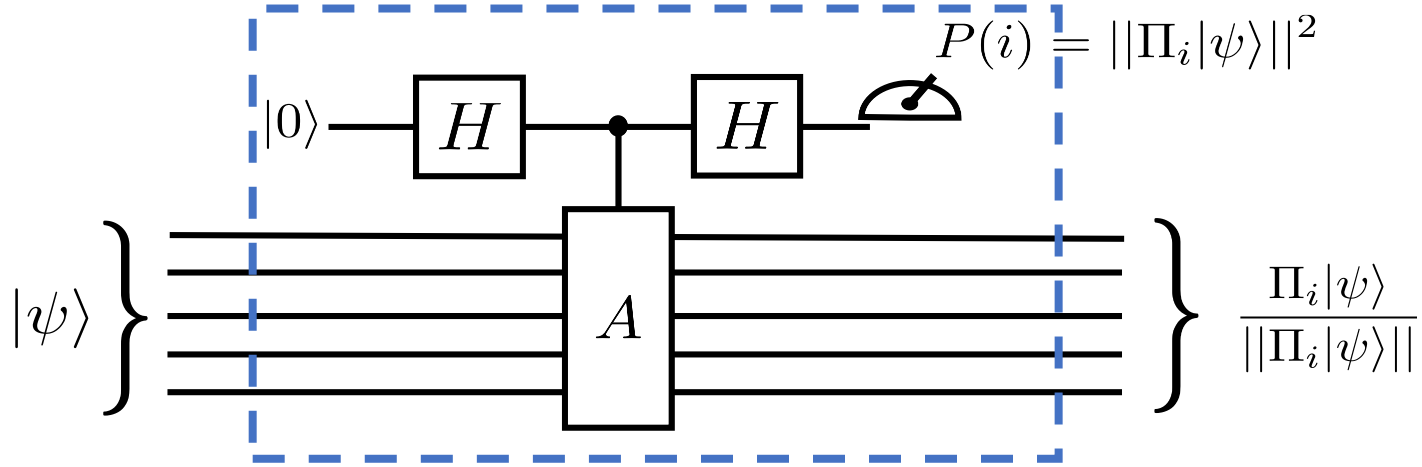
$$\Pi_0 + \Pi_1 = I_{2^n}$$

$$A\Pi_{0/1} = \pm\Pi_{0/1}$$

$$\Pi_0 = \sum_k |v_k^0\rangle\langle v_k^0|$$

$$\Pi_1 = \sum_l |v_l^1\rangle\langle v_l^1|$$

$$A\Pi_1 = (\Pi_0 - \Pi_1)\Pi_1 = \Pi_0\Pi_1 - \Pi_1^2 = -\Pi_1$$



Hadamard Test – binary outcome measurements

Exist unitary A:

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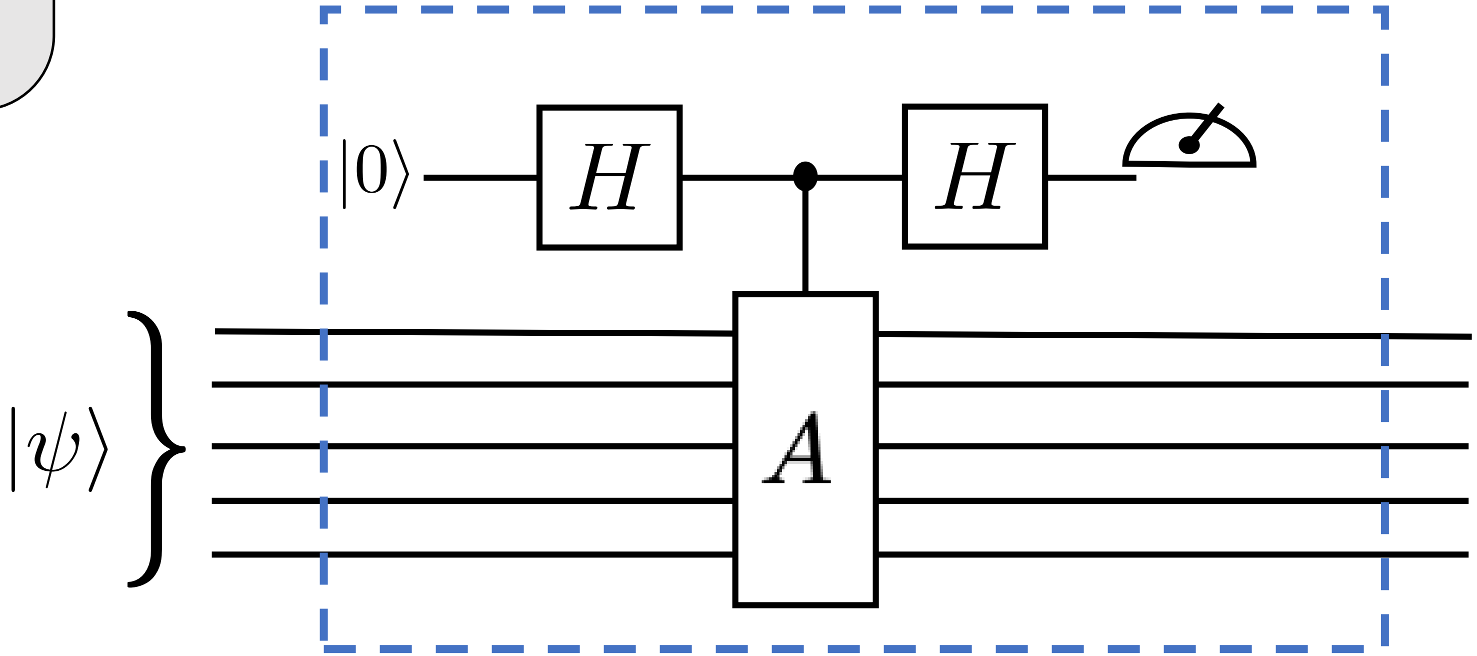

- Syndrome measurement in quantum error correction
- SWAP test: overlap between quantum states, measurement of purity with two copies

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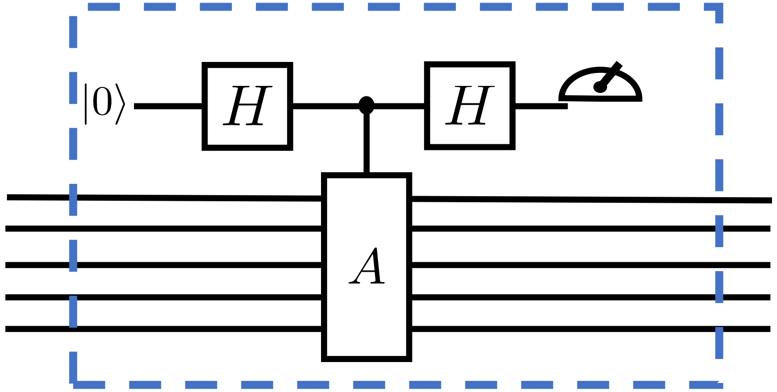
Parity check

Hadamard test circuit for parity check

$$A = \Pi_0 - \Pi_1$$

$$\Pi_0 + \Pi_1 = I_{2^n}$$

$$A\Pi_{0/1} = \pm\Pi_{0/1}$$



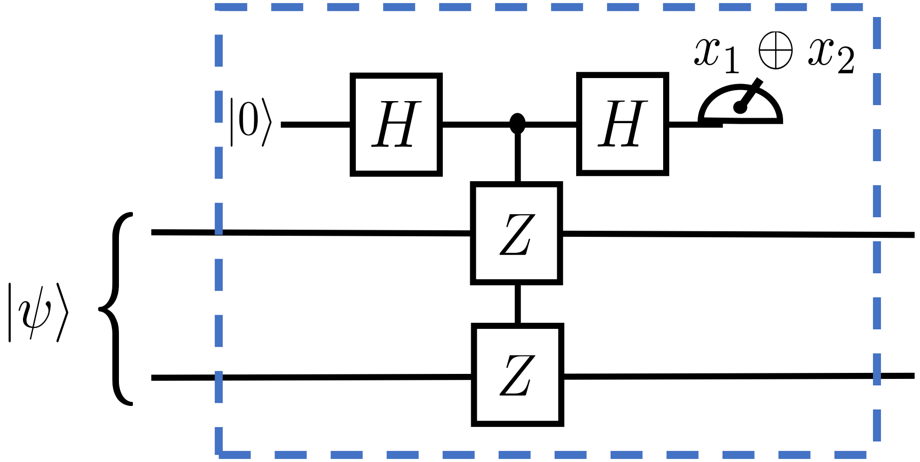
$$A = Z \otimes Z = \Pi_\mathrm{e} - \Pi_\mathrm{o}$$

$$Z_1 \otimes Z_2 |x_1 x_2\rangle = (-1)^{x_1 \oplus x_2} |x_1 x_2\rangle$$

$$\Pi_\mathrm{e} = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\Pi_\mathrm{o} = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$(Z \otimes Z)\Pi_{\mathrm{e/o}} = \pm\Pi_{\mathrm{e/o}}$$

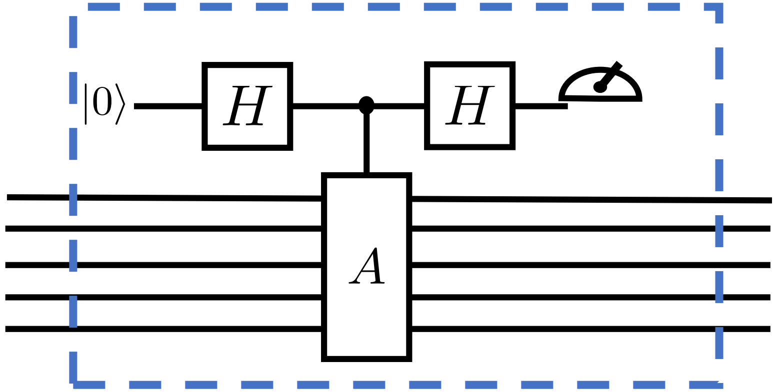


Hadamard test circuit for general check

$$A = \Pi_0 - \Pi_1$$

$$\Pi_0 + \Pi_1 = I_{2^n}$$

$$A\Pi_{0/1} = \pm\Pi_{0/1}$$

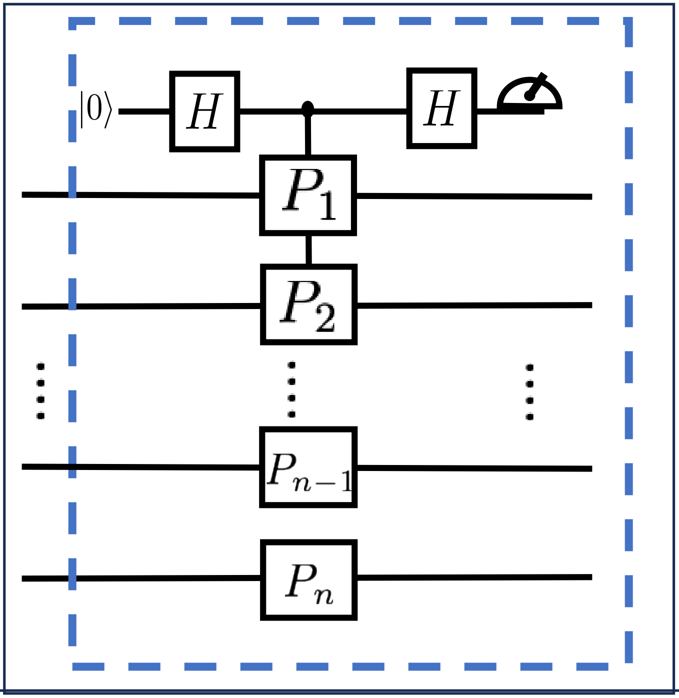


$$P = P_1 \otimes \dots \otimes P_n = \Pi_{+1} - \Pi_{-1}$$

$$P_i \in \{I, X, Y, Z\}$$

$$P\Pi_{\pm 1} = \pm\Pi_{\pm 1}$$

$$\Pi_{+1} + \Pi_{-1} = I_{2^n}$$



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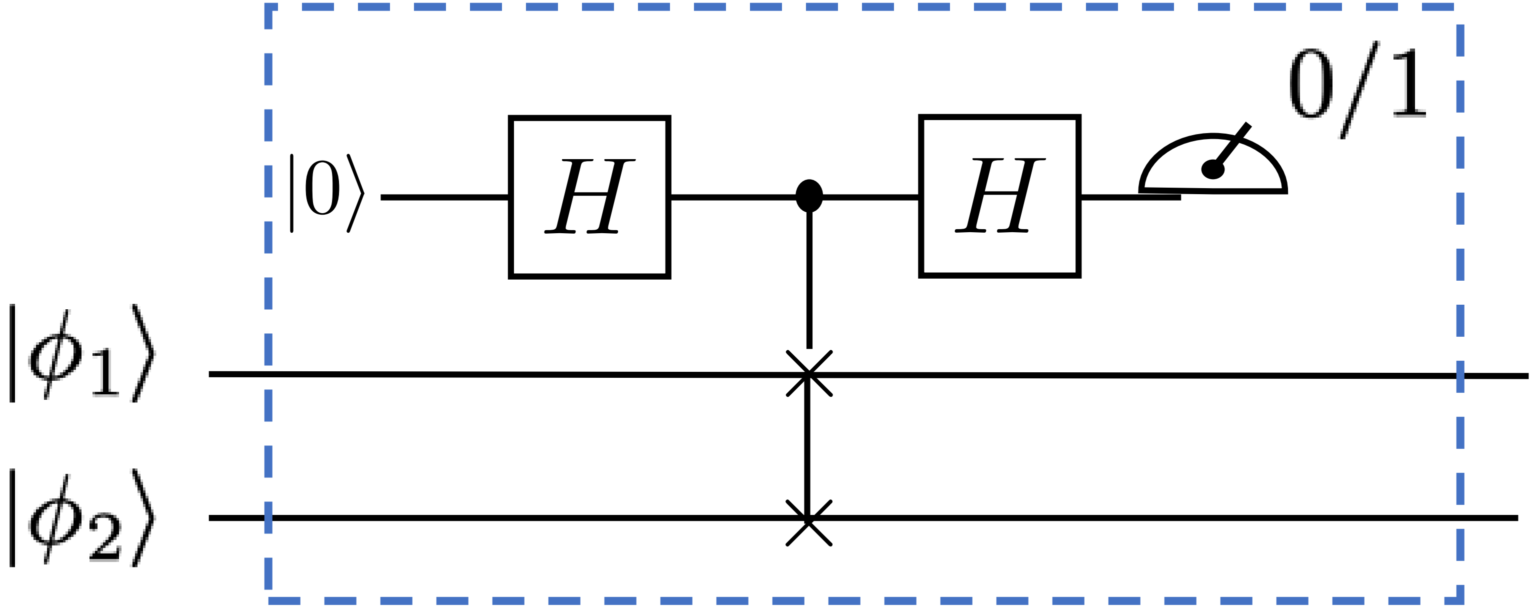
Two-qubits SWAP Test

SWAP Test



$$P(0) - P(1) = |\langle\phi_1|\phi_2\rangle|^2$$

We need to run multiple experiments to estimate $P(0) - P(1)$

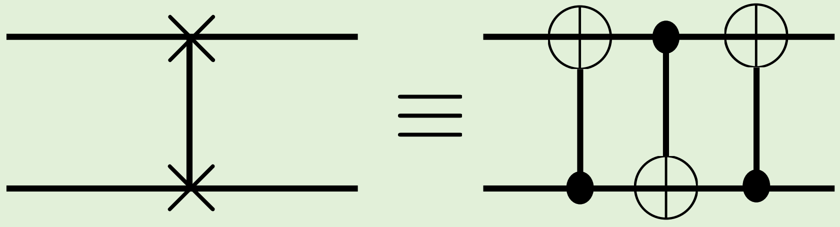
For precision ϵ we need $O(1/\epsilon^2)$ circuit runs

Hidden assumption: We are assuming we can prepare same copies every time (not specific to quantum)

SWAP gate

SWAP (permutation) gate

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Does exisits$\{a, b, c, d\}$ s.t.

$$U_{\mathrm{SWAP}} = a|\Phi^+\rangle\langle\Phi^+| + b|\Phi^-\rangle\langle\Phi^-| + c|\Psi^+\rangle\langle\Psi^+| + d|\Phi^-\rangle\langle\Phi^-|?$$

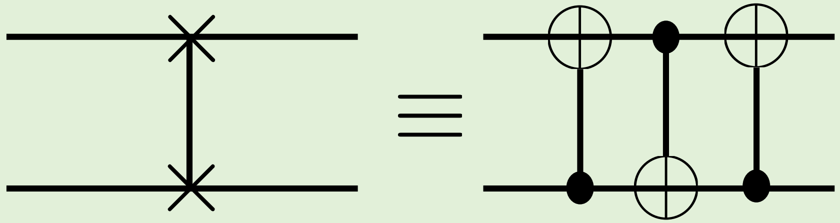
Bell basis $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$ $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$

$$|\Phi^\pm\rangle\langle\Phi^\pm| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix} \qquad |\Psi^\pm\rangle\langle\Psi^\pm| = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

SWAP gate

SWAP (permutation) gate

$$U_{\mathrm{SWAP}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$U_{\mathrm{SWAP}} = |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| - |\Phi^-\rangle\langle\Phi^-|$$

$$\Pi_0 = |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+|$$

$$\Pi_1 = |\Psi^-\rangle\langle\Psi^-|$$

$$U_{\mathrm{SWAP}} = \Pi_0 - \Pi_1$$

Bell basis

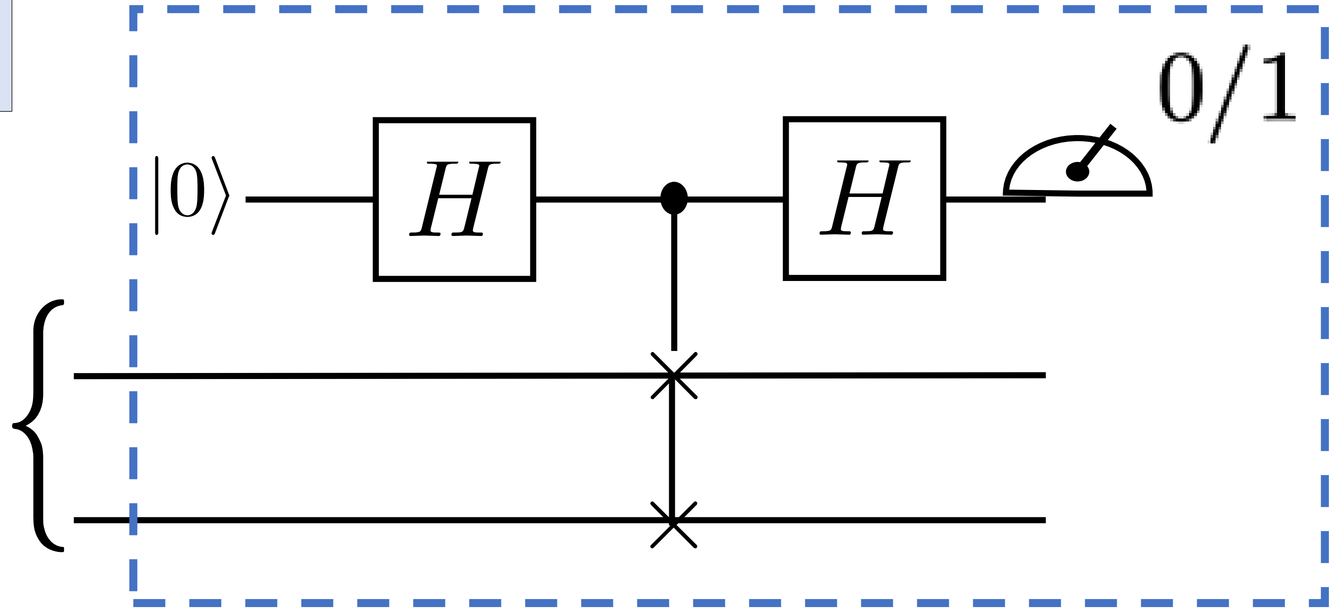
$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

$$|\Phi^\pm\rangle\langle\Phi^\pm| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix} \quad |\Psi^\pm\rangle\langle\Psi^\pm| = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

SWAP Test

$$U_{\text{SWAP}} = \Pi_0 - \Pi_1$$
$$\Pi_0 + \Pi_1 = I_4$$



$$|\psi\rangle = \Pi_0|\psi\rangle + \Pi_1|\psi\rangle$$

$$|0\rangle \otimes |\psi\rangle \xrightarrow{H\otimes I\otimes I} (|0\rangle + |1\rangle)/\sqrt{2} \otimes (\Pi_0|\psi\rangle + \Pi_1|\psi\rangle)$$

$$\xrightarrow{\text{c-SAWP}} 1/\sqrt{2}(|0\rangle \otimes (\Pi_0|\psi\rangle + \Pi_1|\psi\rangle) + |1\rangle \otimes U_{\text{SWAP}}(\Pi_0|\psi\rangle + \Pi_1|\psi\rangle)$$

$$= 1/\sqrt{2}(|0\rangle \otimes (\Pi_0|\psi\rangle + \Pi_-|\psi\rangle) + |1\rangle \otimes (\Pi_0|\psi\rangle - \Pi_1|\psi\rangle)$$

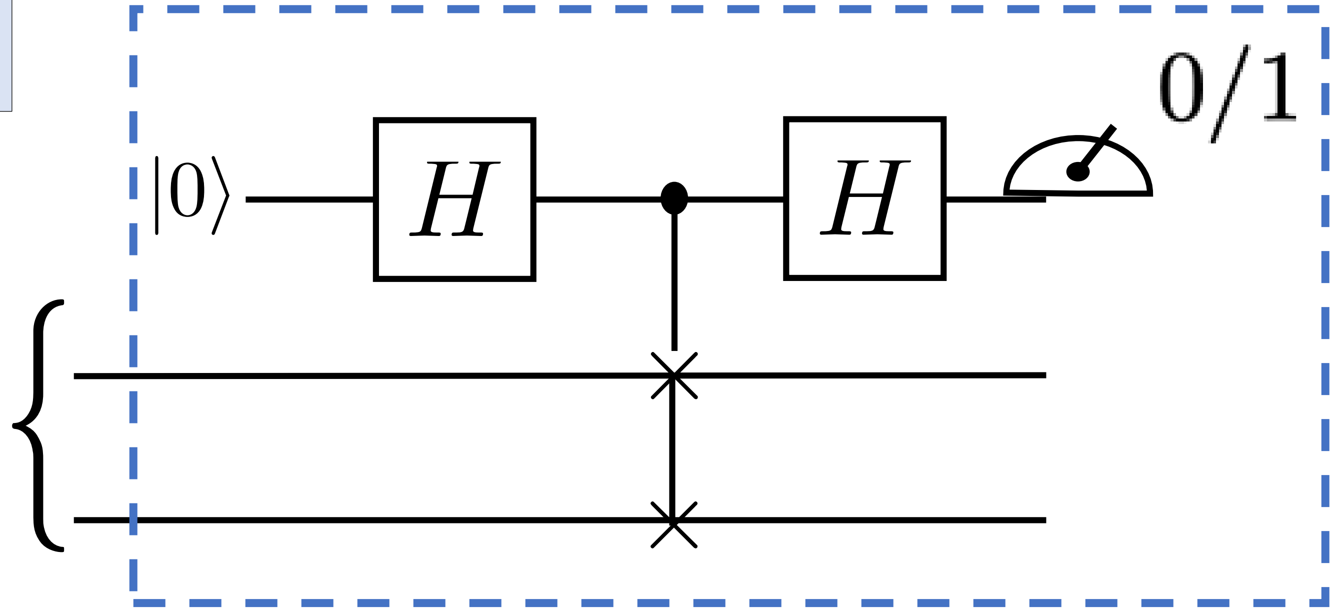
$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \Pi_0|\psi\rangle + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \Pi_1|\psi\rangle = |+\rangle \otimes \Pi_0|\psi\rangle + |-\rangle \otimes \Pi_1|\psi\rangle$$

$$\xrightarrow{H\otimes I\otimes I} |0\rangle \otimes \Pi_0|\psi\rangle + |1\rangle \otimes \Pi_1|\psi\rangle$$

SWAP Test

$$U_{\text{SWAP}} = \Pi_0 - \Pi_1$$
$$\Pi_0 + \Pi_1 = I_4$$

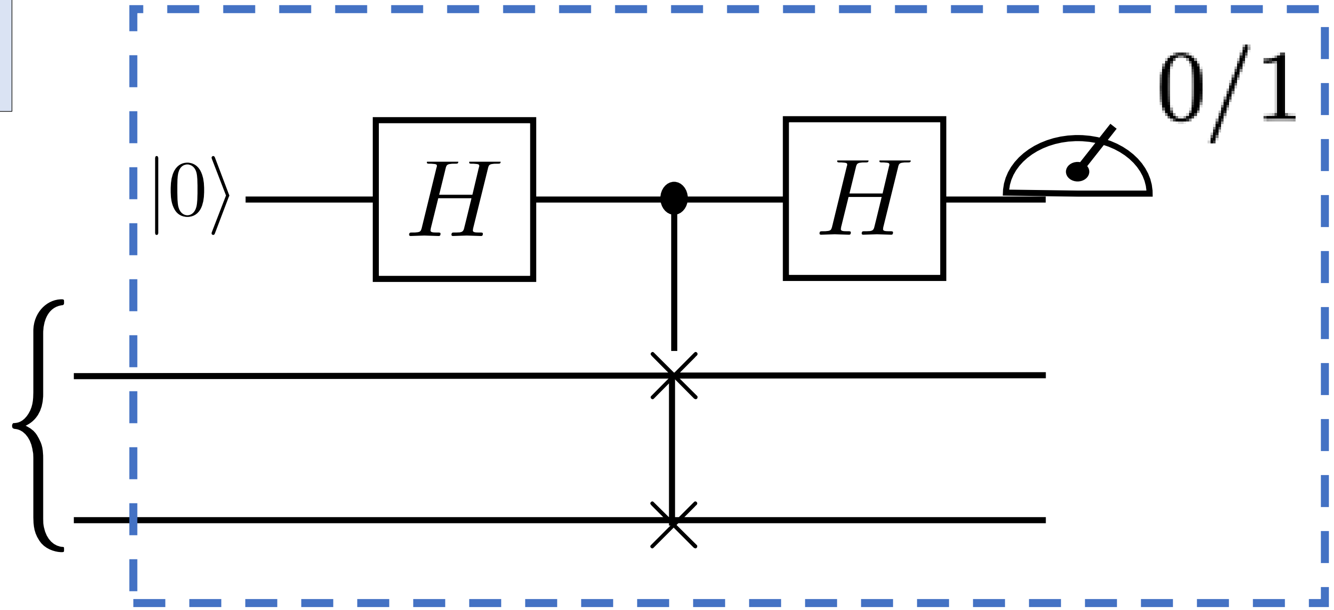
$$|\psi\rangle = \Pi_0|\psi\rangle + \Pi_1|\psi\rangle$$



$$|0\rangle \otimes |\psi\rangle \to |0\rangle \otimes \Pi_0|\psi\rangle + |1\rangle \otimes \Pi_1|\psi\rangle$$

SWAP Test

$$U_{\text{SWAP}} = \Pi_0 - \Pi_1$$
$$\Pi_0 + \Pi_1 = I_4$$



$$|\psi\rangle = \Pi_0|\psi\rangle + \Pi_1|\psi\rangle$$

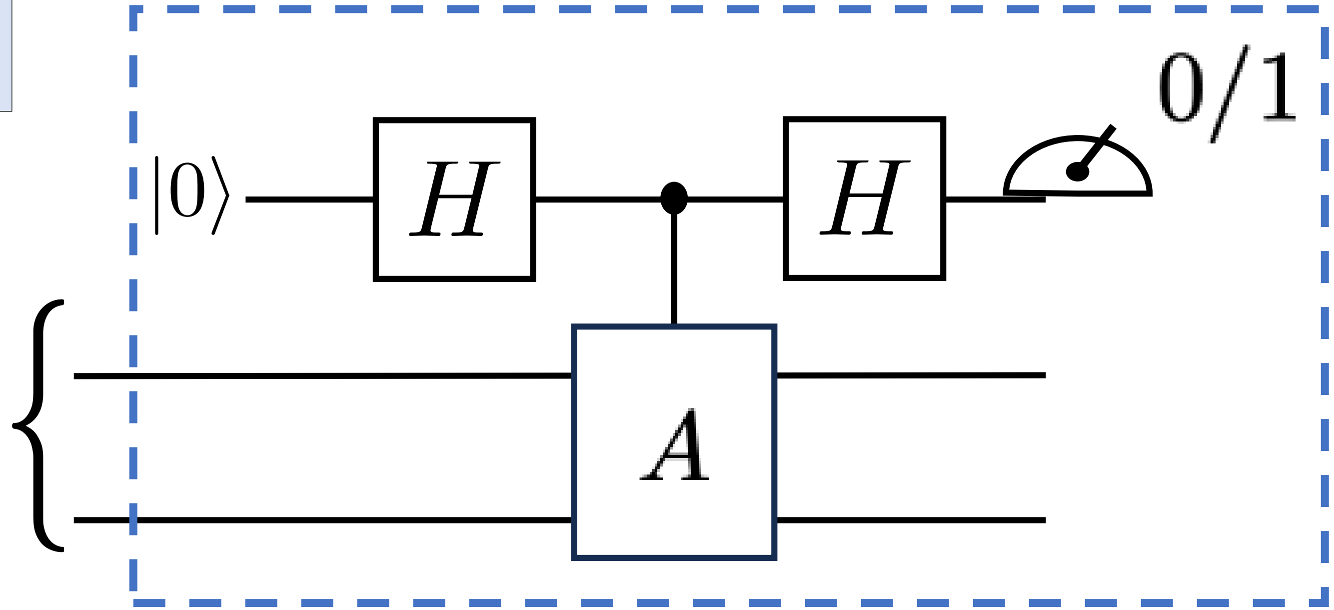
$$|0\rangle \otimes |\psi\rangle \rightarrow |0\rangle \otimes \Pi_0|\psi\rangle + |1\rangle \otimes \Pi_1|\psi\rangle = |\tilde{\phi}\rangle$$

$$\tilde{\Pi}_y = |y\rangle\langle y| \otimes I_4$$
$$\tilde{\Pi}_0 + \tilde{\Pi}_1 = I_8$$

$$\tilde{\Pi}_0 = |0\rangle\langle 0| \otimes I$$
$$\tilde{\Pi}_0|\tilde{\phi}\rangle = (|0\rangle\langle 0| \otimes I)(|0\rangle \otimes \Pi_0|\psi\rangle + |1\rangle \otimes \Pi_1|\psi\rangle) = |0\rangle \otimes \Pi_0|\psi\rangle$$
$$P(0) = ||\tilde{\Pi}_0|\tilde{\phi}\rangle||^2 = (\langle\tilde{\phi}|\tilde{\Pi}_0)(\tilde{\Pi}_0|\tilde{\phi}\rangle) = (\langle 0| \otimes \langle\psi|\Pi_0)(|0\rangle \otimes \Pi_0|\psi\rangle)$$
$$= \langle 0|0\rangle\langle\psi|\Pi_0^2|\psi\rangle = \langle 0|0\rangle\langle\psi|\Pi_0|\psi\rangle = ||\Pi_0|\psi\rangle||^2$$

Hadamard Test

$$A = \Pi_0 - \Pi_1$$
$$\Pi_0 + \Pi_1 = I_4$$



$$|\psi\rangle = \Pi_0|\psi\rangle + \Pi_1|\psi\rangle$$

$$|0\rangle \otimes |\psi\rangle \rightarrow |0\rangle \otimes \Pi_0|\psi\rangle + |1\rangle \otimes \Pi_1|\psi\rangle = |\tilde{\phi}\rangle$$

$$\tilde{\Pi}_y = |y\rangle\langle y| \otimes I_4$$
$$\tilde{\Pi}_0 + \tilde{\Pi}_1 = I_8$$

$$\tilde{\Pi}_0 = |0\rangle\langle 0| \otimes I$$
$$\tilde{\Pi}_0|\tilde{\phi}\rangle = |0\rangle \otimes \Pi_0|\psi\rangle$$
$$P(0) = ||\tilde{\Pi}_0|\tilde{\phi}\rangle||^2 = ||\Pi_0|\psi\rangle||^2$$

The proof works for is general and works for any A s.t.:

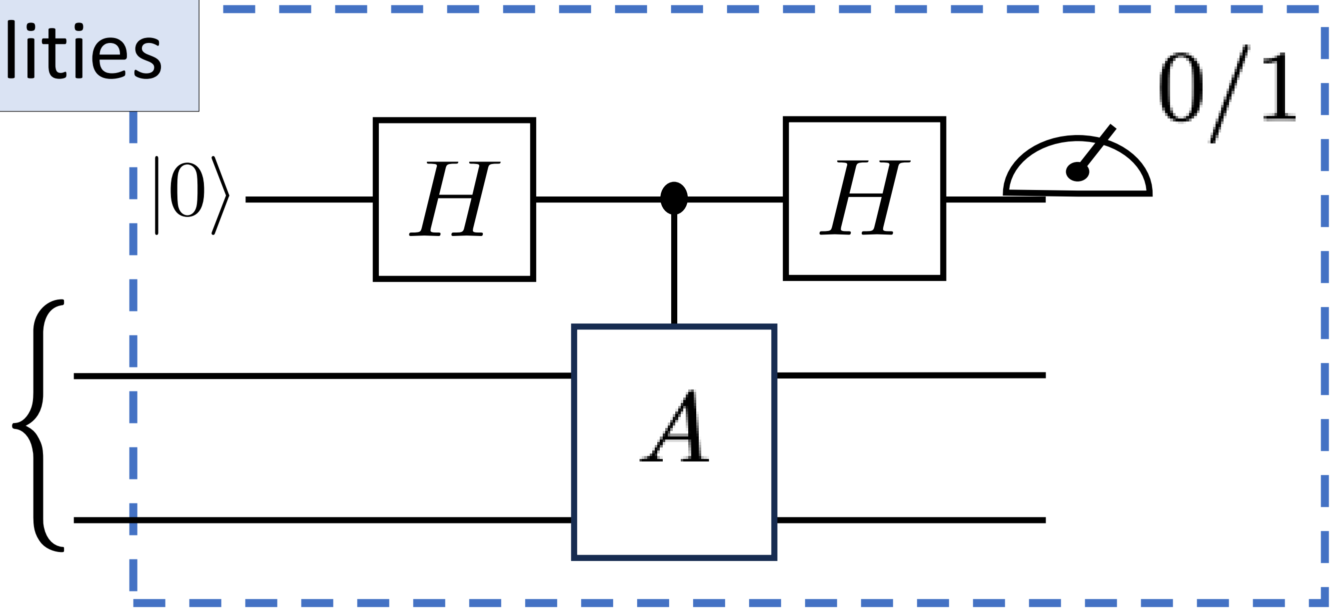
$$A = \Pi_0 - \Pi_1$$

It also works for any A and arbitrary size register.

Hadamard Test – Bias of probabilities

$$A = \Pi_0 - \Pi_1$$
$$\Pi_0 + \Pi_1 = I_4$$

$$|\psi\rangle = \Pi_0|\psi\rangle + \Pi_1|\psi\rangle$$



$$P(0) = ||\tilde{\Pi}_0|\tilde{\phi}\rangle||^2 = ||\Pi_0|\psi\rangle||^2 \qquad P(1) = ||\tilde{\Pi}_1|\tilde{\phi}\rangle||^2 = ||\Pi_1|\psi\rangle||^2$$

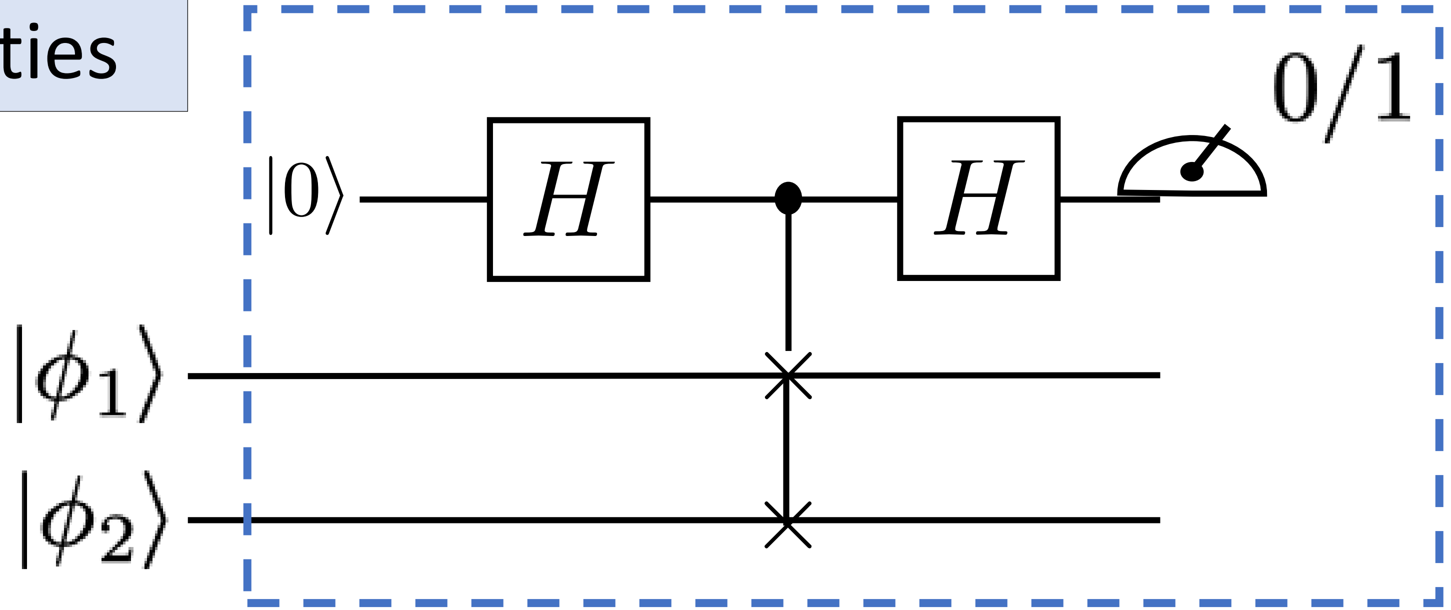
$$\begin{aligned} P(0) - P(1) &= ||\Pi_0|\psi\rangle||^2 - ||\Pi_1|\psi\rangle||^2 \\ &= \langle\psi|\Pi_0^\dagger\Pi_0|\psi\rangle - \langle\psi|\Pi_1^\dagger\Pi_1|\psi\rangle \\ &= \langle\psi|\Pi_0|\psi\rangle - \langle\psi|\Pi_1|\psi\rangle = \langle\psi|\Pi_0 - \Pi_1|\psi\rangle \\ &= \langle\psi|A|\psi\rangle \end{aligned}$$

Expectation value of A on state $|\psi\rangle$

SWAP Test – Bias of probabilities

$$U_{\text{SWAP}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U_{\text{SWAP}} = |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| - |\Phi^-\rangle\langle\Phi^-|$$



$$P(0) - P(1) = \langle\psi|U_{\text{SWAP}}|\psi\rangle$$

$$\begin{aligned}
&= \langle\phi_1| \otimes \langle\phi_2|U_{\text{SWAP}}|\phi_1\rangle \otimes |\phi_2\rangle \\
&= (\langle\phi_1| \otimes \langle\phi_2|)(|\phi_2\rangle \otimes |\phi_1\rangle) \\
&= \langle\phi_1|\phi_2\rangle\langle\phi_2|\phi_1\rangle \\
&= \langle\phi_1|\phi_2\rangle\langle\phi_1|\phi_2\rangle^* = |\langle\phi_1|\phi_2\rangle|^2
\end{aligned}$$

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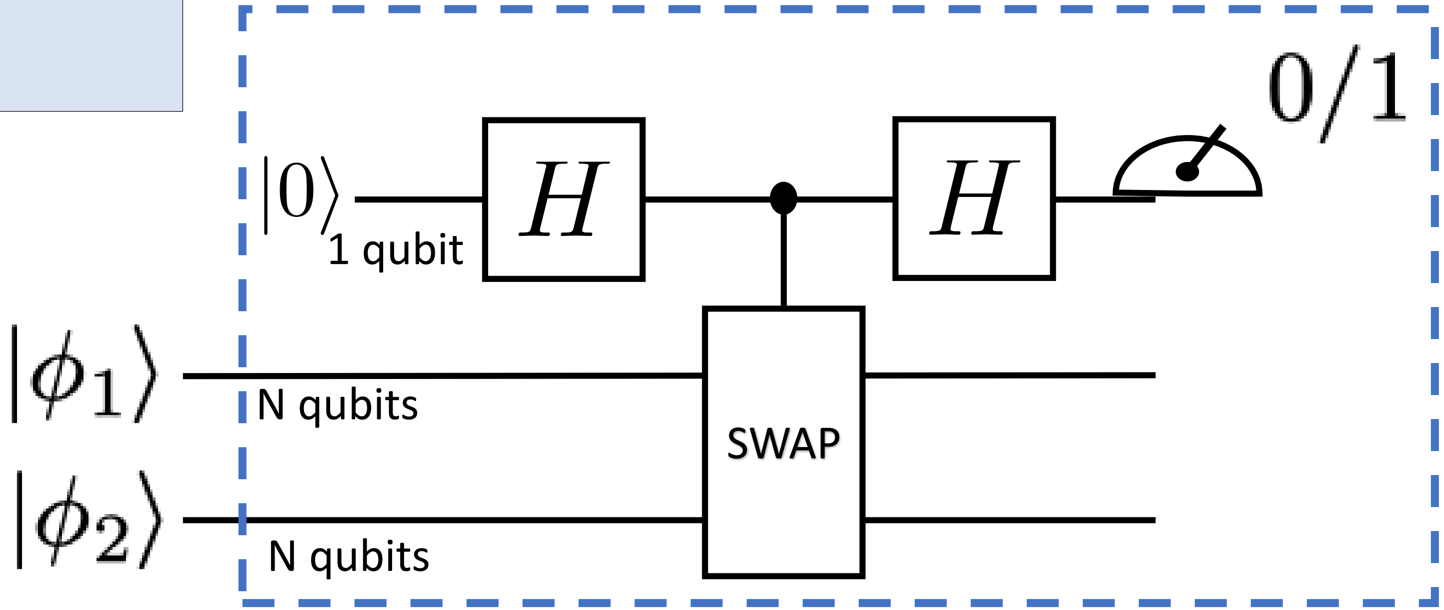
General Swap Test

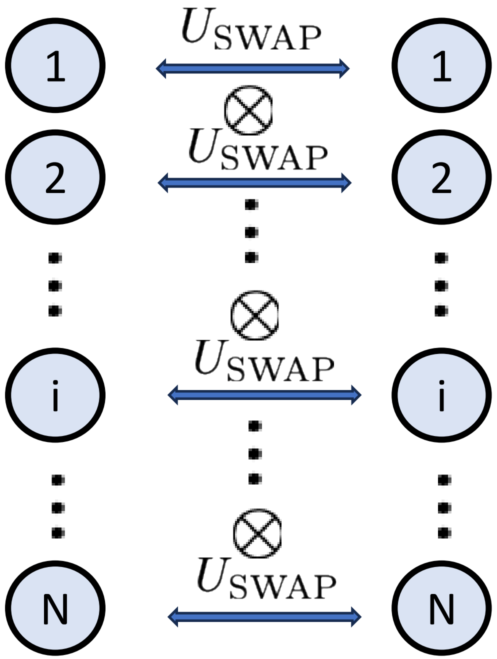
SWAP Test – Multiple qubits



<u>Single qubit registers</u>
The Hadamard Test needs to control the exchange of the two two qubits. Done with SWAP gate.

<u>Multiple-qubit registers</u>
The Hadamard Test needs to control the exchange of the two N qubit registers.

The exchange is done:
1. Pair together the i-th qubits of each register
2. Apply SWAP gate on each pair (controlled by upper-qubit)



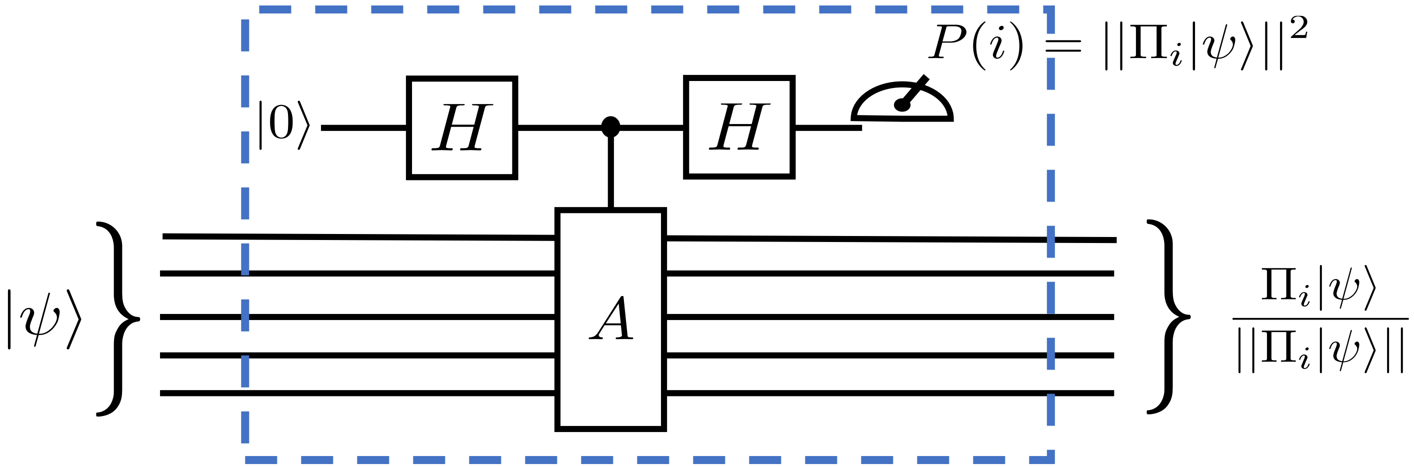
Removing the control qubit

What if we do not need the updated state

Exist unitary A:

$$A = \Pi_0 - \Pi_1$$

$$\Pi_0 + \Pi_1 = I_{2^n}$$



If we do not need to use the updated state $\frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$ later in the computation

we can just measure $|\psi\rangle$ in the eigenbasis of the operator A

What if we do not need the updated state – parity check

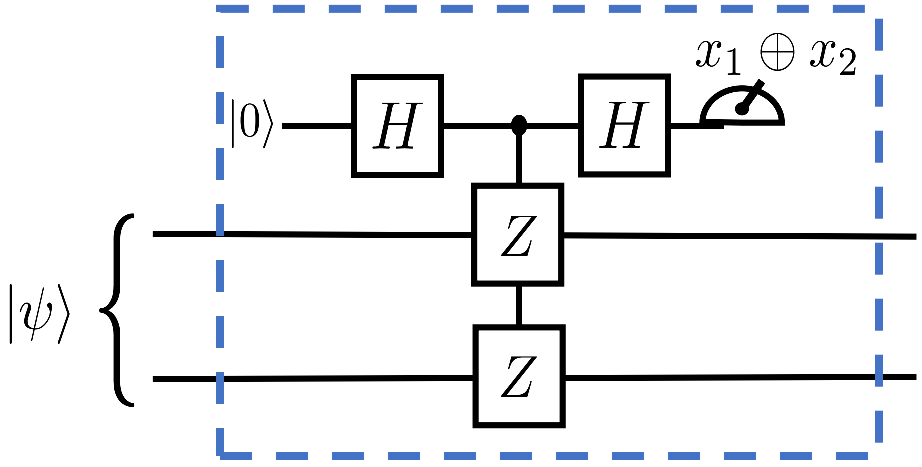
$$A = Z \otimes Z = \Pi_{\mathrm{e}} - \Pi_{\mathrm{o}}$$

$$Z_1 \otimes Z_2 |x_1 x_2\rangle = (-1)^{x_1 \oplus x_2} |x_1 x_2\rangle$$

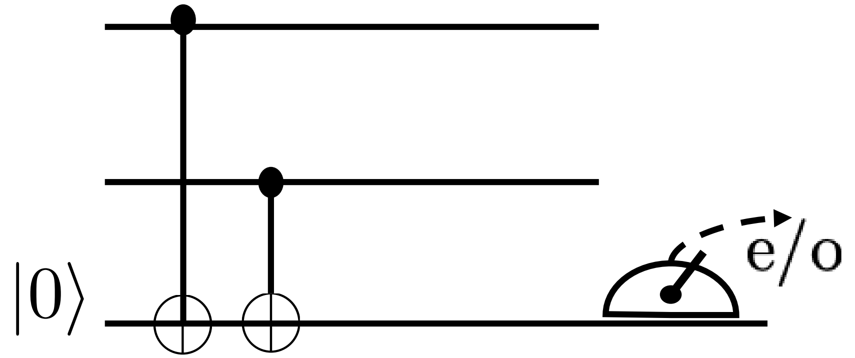
$$\Pi_{\mathrm{e}} = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\Pi_{\mathrm{o}} = |01\rangle\langle 01| + |10\rangle\langle 10|$$

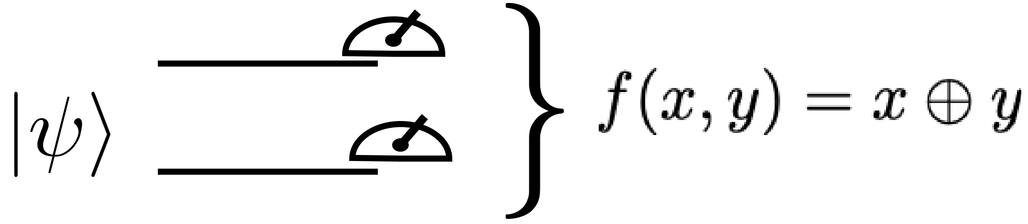
$$(Z \otimes Z)\Pi_{\mathrm{e/o}} = \pm \Pi_{\mathrm{e/o}}$$



Non-demolition measurement



Demolition measurement



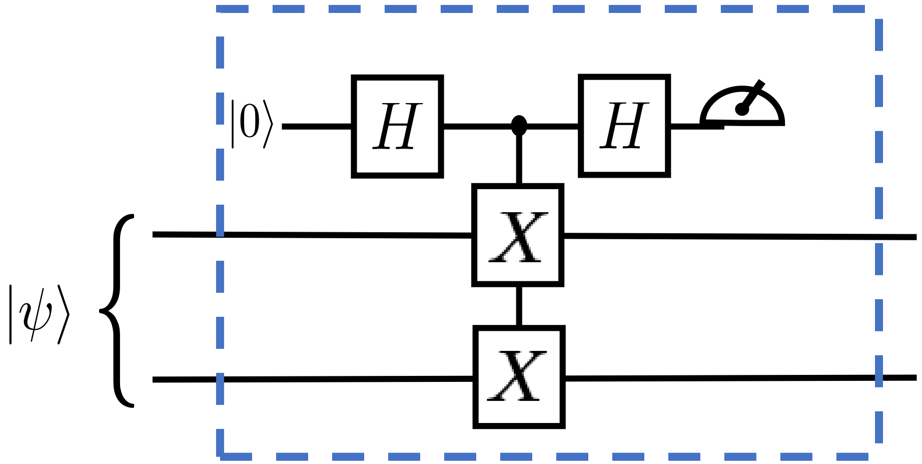
What if we do not need the updated state – parity check

$$A = X \otimes X = \tilde{\Pi}_{\mathrm{e}} - \tilde{\Pi}_{\mathrm{o}}$$

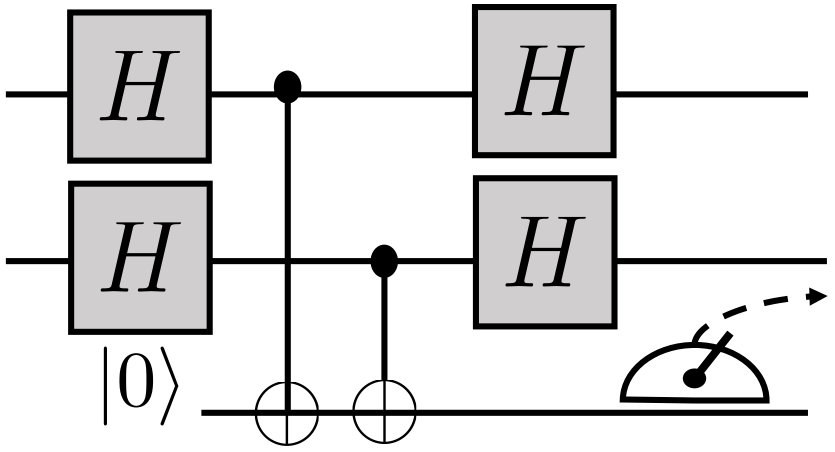
$$\tilde{\Pi}_{\mathrm{e}} = |++\rangle\langle++| + |--\rangle\langle--|$$

$$\tilde{\Pi}_{\mathrm{o}} = |+-\rangle\langle+-| + |-+\rangle\langle-+|$$

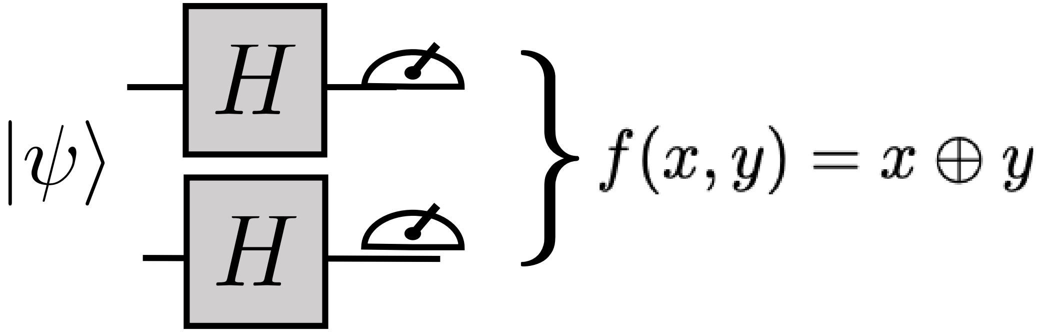
$$(X \otimes X)\tilde{\Pi}_{\mathrm{e/o}} = \pm\tilde{\Pi}_{\mathrm{e/o}}$$



Non-demolition measurement



Demolition measurement



What if we do not need the updated state – 2-qubit SWAP Test

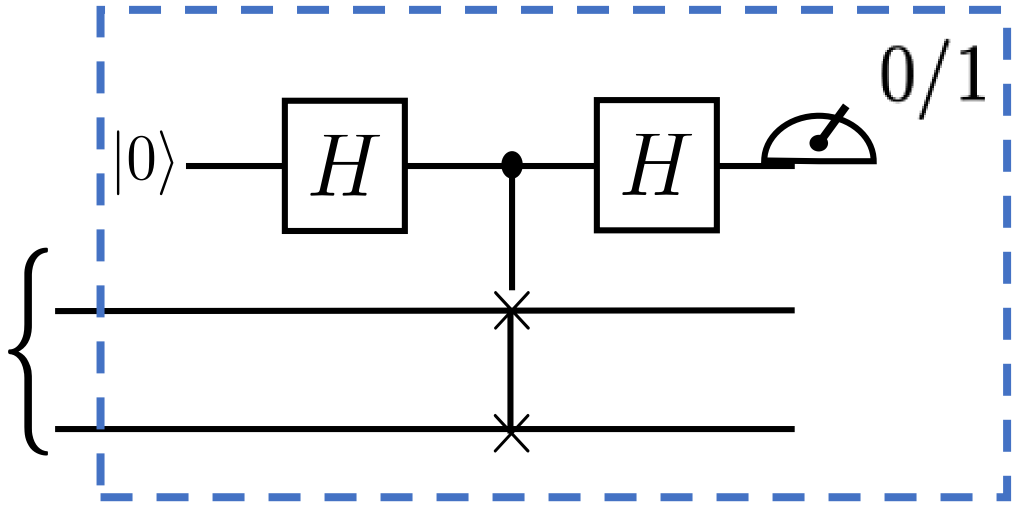
$$U_{\mathrm{SWAP}} = \Pi_0 - \Pi_1$$

Symmetric subspace

$$\Pi_0 = |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+|$$

Anti-symmetric subspace

$$\Pi_1 = |\Psi^-\rangle\langle\Psi^-|$$



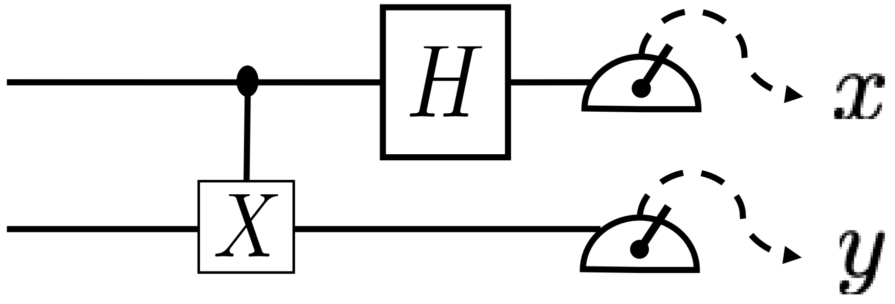
Bell basis

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

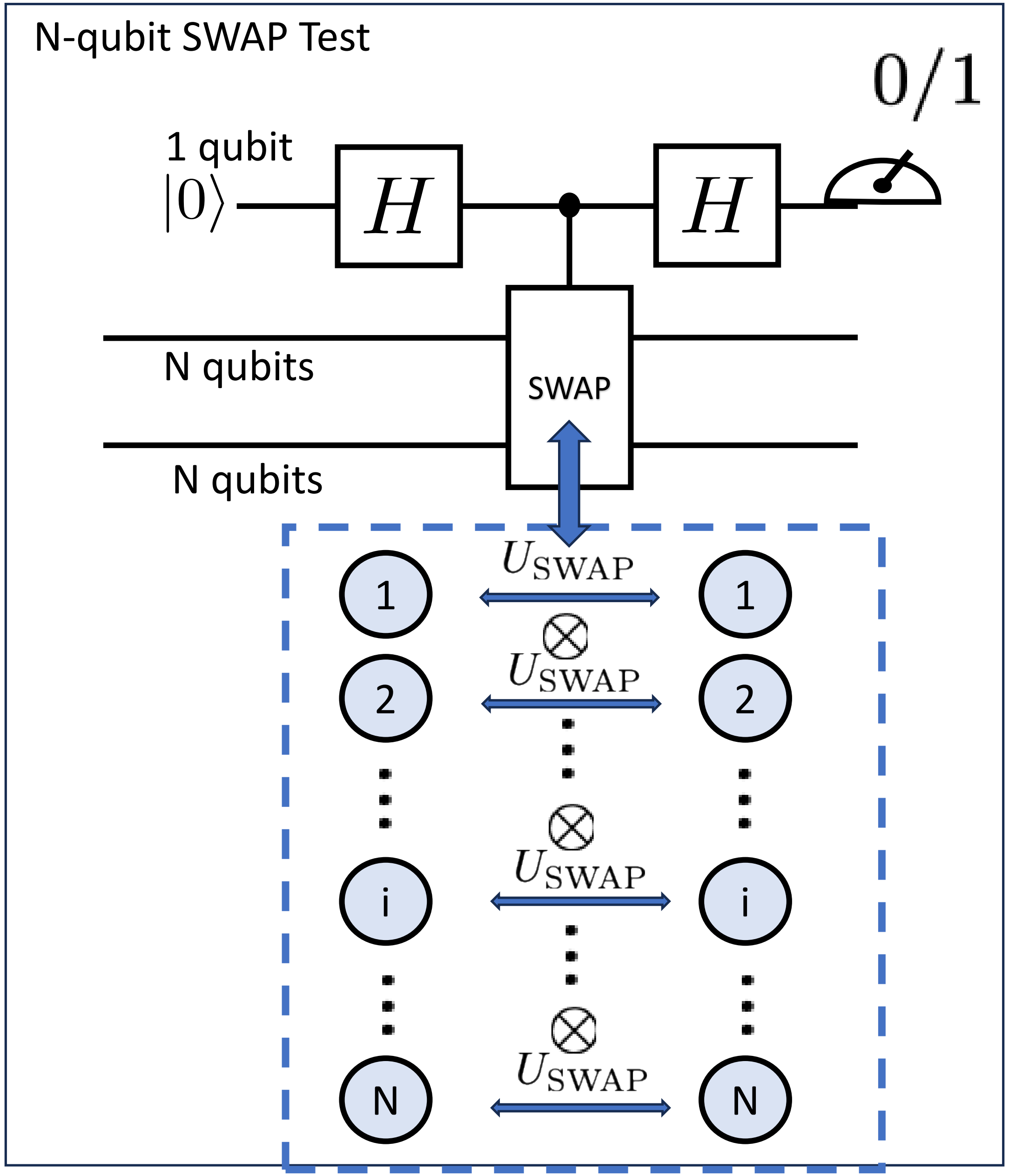
We can replace the SWAP Test by measurement in the Bell basis + postprocessing

Bell measurement + post-processing

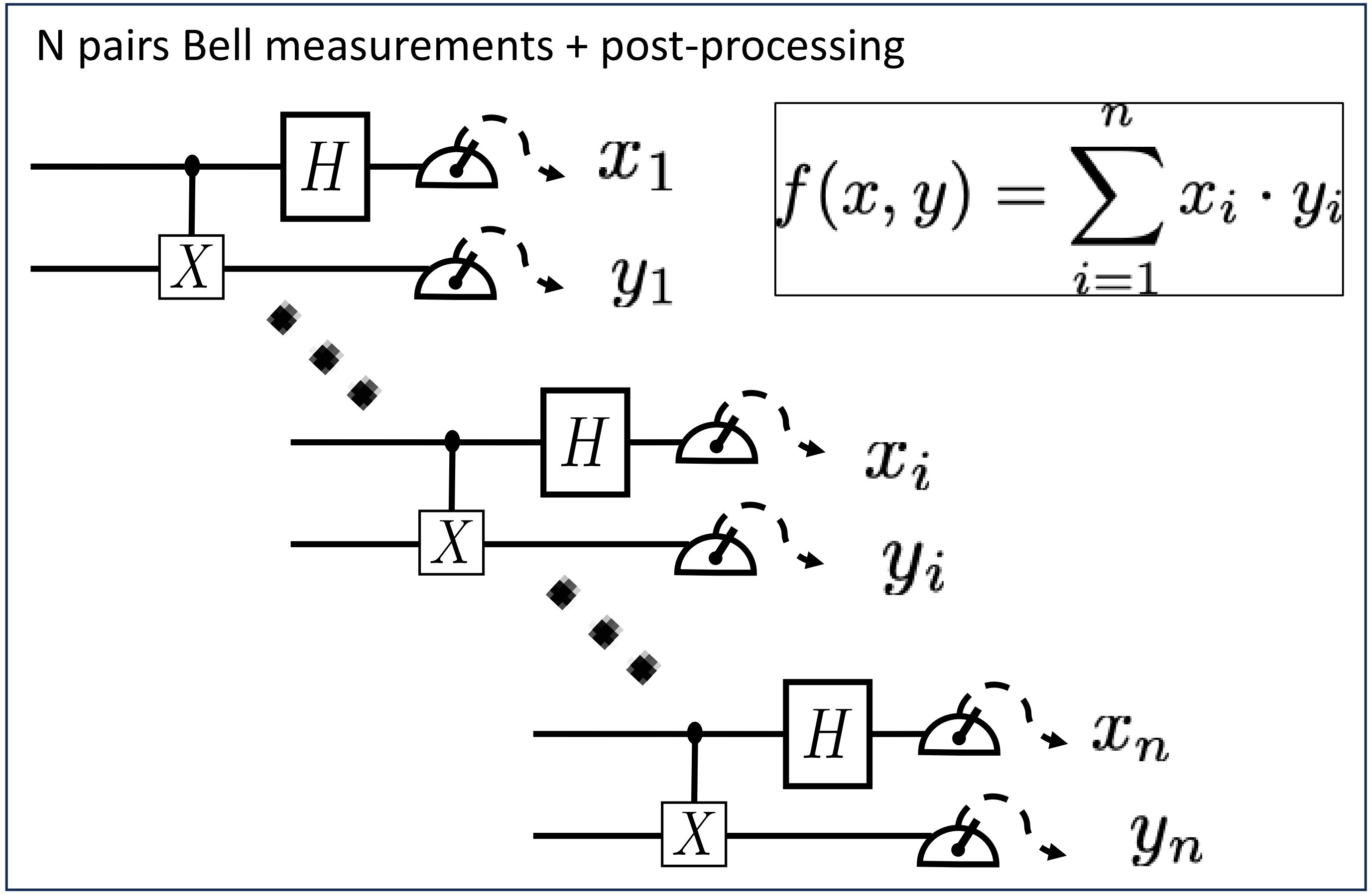


$$f(x, y) = x \cdot y$$

What if we do not need the updated state – N-qubit SWAP Test



We can replace the SWAP Test by measurement in the Bell basis + postprocessing

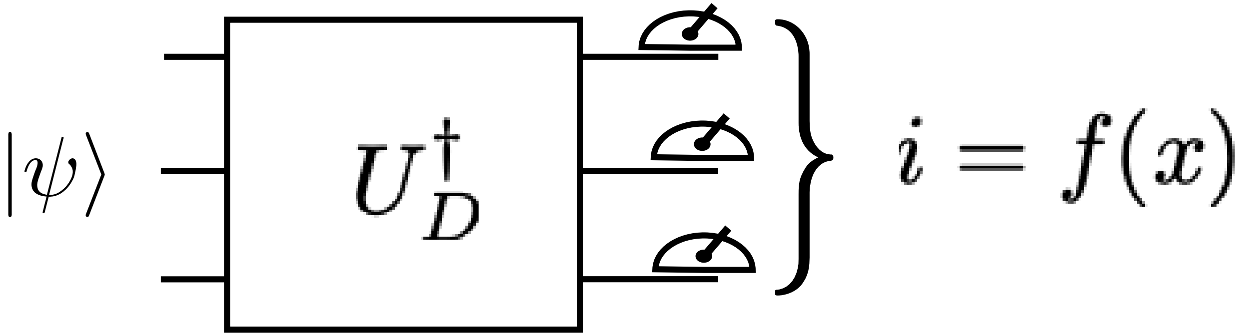


What if we do not need the updated state – general case

Exist unitary A:

$$A = \Pi_0 - \Pi_1$$

$$\Pi_0 + \Pi_1 = I_{2^n}$$



$$\forall A = \Pi_0 - \Pi_1$$

$\exists$ unitary U_D that diagonalizes A, Π_0 and Π_1

$$U_D A U_D^\dagger = D_0 - D_1$$

D_0, D_1 are the diagonalisation of Π_0, Π_1 composed of $\{0, 1\}$