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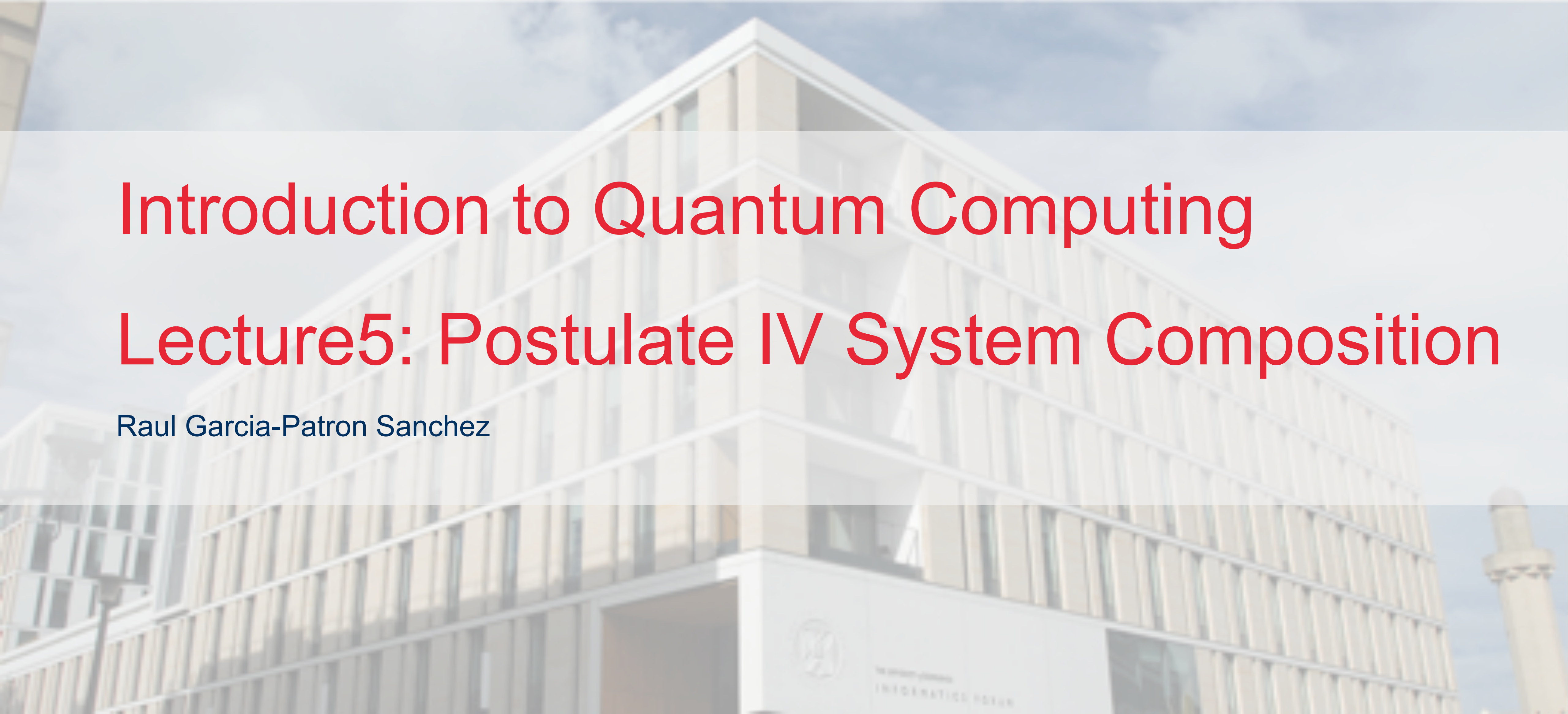
informatics



Introduction to Quantum Computing

Lecture5: Postulate IV System Composition

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Composition of classical systems

- Two independent perfect coins: $p_{AB}(a,b) = p_A(a)p_B(b)$

$p(0,1)$

A\B	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	1

$\} \; p(a)$

$p(b)$

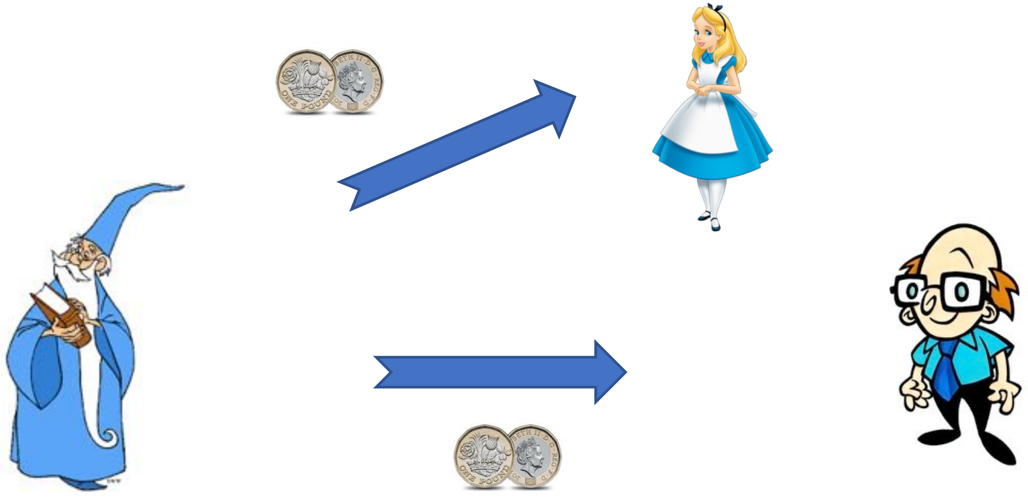
- Composition of independent coins:

$$\bar{u} \in \mathcal{C}_A, \bar{v} \in \mathcal{C}_B \text{ leads to } \bar{u} \otimes \bar{v} \in \mathcal{C}_A \otimes \mathcal{C}_B$$

$$\text{Vector notation: } \bar{u} \otimes \bar{v} \equiv \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \otimes \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_0 v_0 \\ u_0 v_1 \\ u_1 v_0 \\ u_1 v_1 \end{bmatrix} = \begin{bmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{bmatrix}$$

Composition of classical systems

- Correlated coins: $p_{AB}(a,b) \neq p_A(a)p_B(b)$



$p(1,1)$

A\B	0	1	
0	1/2	0	1/2
1	0	1/2	1/2
	1/2	1/2	1

$p(a)$

$p(b)$

- Correlations: $\bar{p} \neq \bar{u} \otimes \bar{v}$ but $\bar{p} = \sum_i p_i \bar{u}_i \otimes \bar{v}_i$
- A basis of probability space $\bar{p} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = p(0)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + p(1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = p(0)\bar{0} + p(1)\bar{1}$
- $\bar{a} \otimes \bar{b}$ is a basis for the composite system: $\bar{p}_{AB} = \sum_{a,b} p(a,b)\bar{a} \otimes \bar{b}$

Composition of quantum systems

Two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ can form a new Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ s.t. $\dim \mathcal{H}_{AB} = \dim \mathcal{H}_A \times \dim \mathcal{H}_B$.

A basis of $\mathcal{H}_{AB}$ is built via tensor product of basis of $\mathcal{H}_A$ and $\mathcal{H}_B$.

An operator $\mathcal{L}(\mathcal{H}_{AB})$ is built via tensor product of operators $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$.

Tensor product of vectors/states

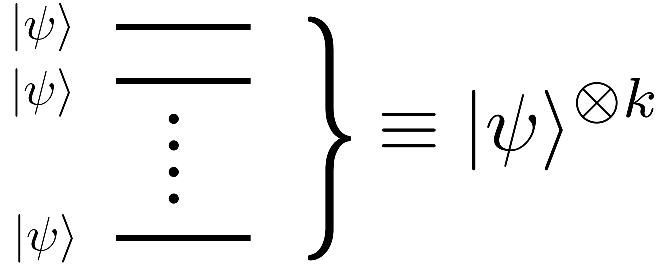
- Tensor product of vectors: $\mathcal{H}_A \times \mathcal{H}_B \to \mathcal{H}_{A,B} = \mathcal{H}_A \otimes \mathcal{H}_B$
  - $|u\rangle \in \mathcal{H}_A, |v\rangle \in \mathcal{H}_B$ leads to $|u\rangle \otimes |v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
  - $c(|u\rangle \otimes |v\rangle) = (c|u\rangle) \otimes |v\rangle = |u\rangle \otimes (c|v\rangle)$
  - $|u\rangle \otimes (|v_1\rangle + |v_2\rangle) = |u\rangle \otimes |v_1\rangle + |u\rangle \otimes |v_2\rangle$

$$|+\rangle \otimes |+\rangle = (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \otimes (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) = \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

  - Vector notation: $|u\rangle \otimes |v\rangle \equiv \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \otimes \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_0 v_0 \\ u_0 v_1 \\ u_1 v_0 \\ u_1 v_1 \end{bmatrix} = \begin{bmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{bmatrix}$

Tensor product of vectors/states

- Inner-product: $\mathcal{H}_A \otimes \mathcal{H}_B \times \mathcal{H}_A \otimes \mathcal{H}_B \to \mathbb{C}$
  - $(|v\rangle_A \otimes |w\rangle_B, |v'\rangle_A \otimes |w'\rangle_B) = \langle v|v'\rangle_A \langle w|w'\rangle_B$
  - $(\sum_i a_i |v_i\rangle_A \otimes |w_i\rangle_B, \sum_j b_j |v'_j\rangle_A \otimes |w'_j\rangle_B) = \sum_{i,j} a_i^* b_j \langle v_i|v'_j\rangle_A \langle w_i|w'_j\rangle_B$
- Additional notation:
  - $|\psi\rangle^{\otimes k} = \underbrace{|\psi\rangle \otimes |\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{k}$
  - $|0^k\rangle = |0\rangle^{\otimes k} = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle$
  - $|\bar{x}\rangle = |x_1, x_2, \ldots., x_n\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \ldots \otimes |x_n\rangle$



Entanglement

- $\exists|\psi_{AB}\rangle \in \mathcal{H}_{AB}$ s.t. $\nexists|\phi_A\rangle \in \mathcal{H}_A$ and $|\varphi_B\rangle \in \mathcal{H}_B : |\psi_{AB}\rangle = |\phi_A\rangle \otimes |\varphi_B\rangle$

Bell states: a basis of $\mathcal{H}_Q \otimes \mathcal{H}_Q$ composed of entangled states.

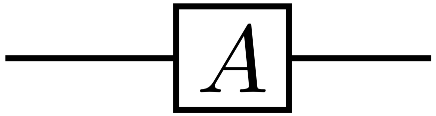
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

- Maximally entangled states of 2 qubits.
- Bell states are entanglement units. Similarly as an unbiased coins is a unit of randomness.

- $\forall|\psi_{AB}\rangle \in \mathcal{H}_{AB} : |\psi_{AB}\rangle = \sum_{i,j} \psi_{i,j}|i\rangle \otimes |j\rangle$

Tensor product of operators



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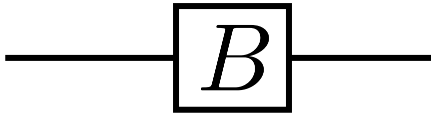
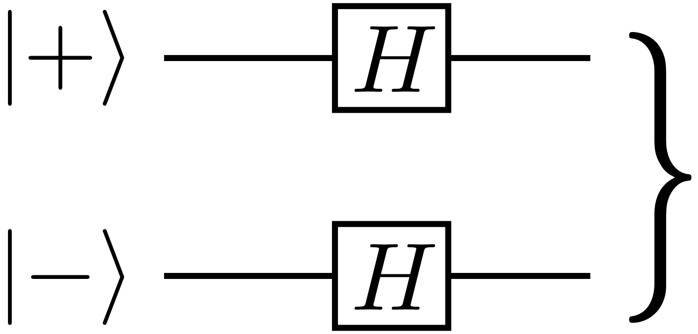


Tensor product of operators

- Tensor product of operators: $\mathcal{L}(\mathcal{H}_A) \times \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_{AB}) = \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$
  - $(A \otimes B)(|v\rangle \otimes |w\rangle) = (A|v\rangle) \otimes (B|w\rangle) = A|v\rangle \otimes B|w\rangle$
  - $(A \otimes B)(\sum_i a_i |v_i\rangle \otimes |w_i\rangle) = \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$
  - $(\sum_i c_i A_i \otimes B_i)|v\rangle \otimes |w\rangle = \sum_i c_i A_i|v\rangle \otimes B_i|w\rangle$
  - Matrix notation: $A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nn}B \end{bmatrix}$

Example (Dirac notation)



$|\pm\rangle$ basis:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

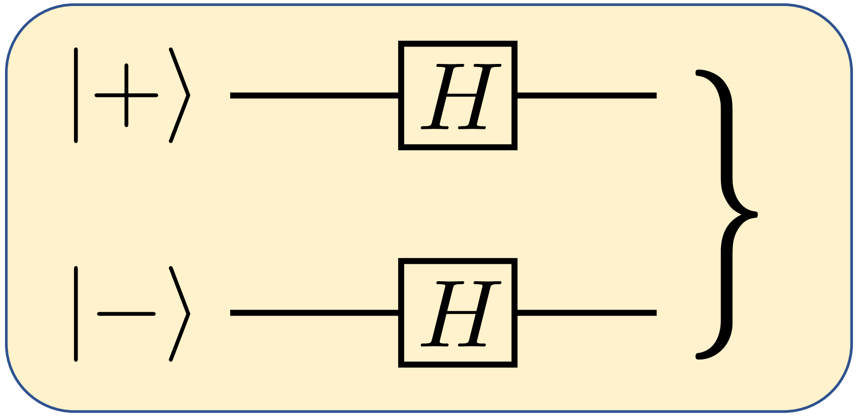
$$(H \otimes H)(|+\rangle \otimes |-\rangle) = (H|+\rangle) \otimes (H|-\rangle)$$

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

$$= |0\rangle \otimes |1\rangle$$

Example (matrix notation)

- $|+\rangle \otimes |-\rangle \equiv \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

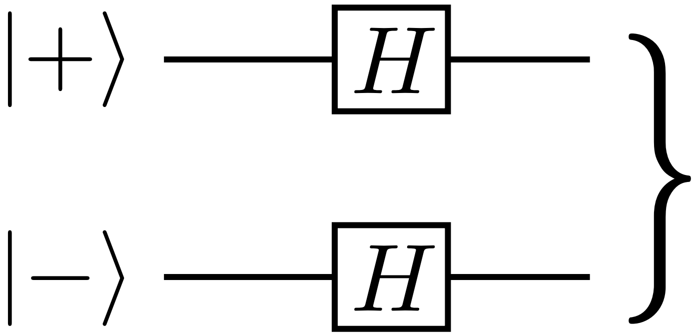


- $H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nn}B \end{bmatrix}$$

- $|\psi\rangle = (H \otimes H)(|+\rangle \otimes |-\rangle) \equiv \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |0\rangle \otimes |1\rangle$

Example (Dirac notation)

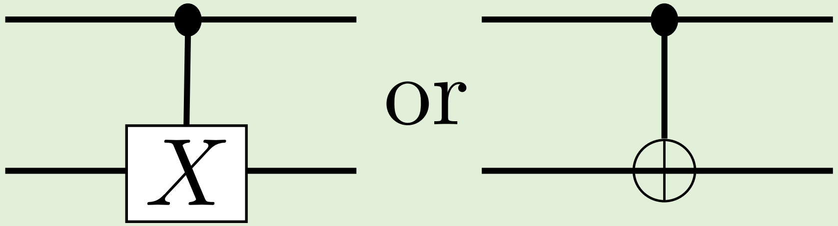


$$(H \otimes H)(|+\rangle \otimes |-\rangle) = (H|+\rangle) \otimes (H|-\rangle) = |0\rangle \otimes |1\rangle$$

Entangling operations

- $\exists U_{AB} \in \mathcal{L}(\mathcal{H}_{AB})$ s.t. $\nexists U_A \in \mathcal{L}(\mathcal{H}_A)$ and $U_B \in \mathcal{L}(\mathcal{H}_B) : U_{AB} = U_A \otimes U_B$

Controlled-NOT gate (CNOT gate):



$$U_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes X$$

- $\forall U \in \mathcal{L}(\mathcal{H}_{AB}), \exists \{V_i\} \in \mathcal{L}(\mathcal{H}_A)$ and $\{W_i\} \in \mathcal{L}(\mathcal{H}_B) : U = \sum_i V_i \otimes W_i$

Outer-products (Dirac notation)

$$\begin{bmatrix} a_{01} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} = a_{00} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{01} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{10} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{11} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

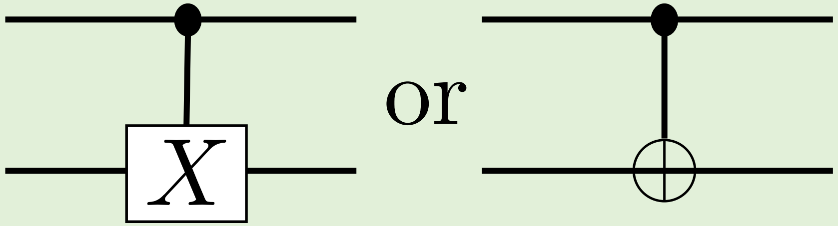
$$= a_{00}|0\rangle\langle 0| + a_{01}|0\rangle\langle 1| + a_{10}|1\rangle\langle 0| + a_{11}|1\rangle\langle 1|$$

$|0\rangle\langle 0| \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$|0\rangle\langle 1| \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
$|1\rangle\langle 0| \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$|1\rangle\langle 1| \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Entangling operations

- $\exists U_{AB} \in \mathcal{L}(\mathcal{H}_{AB})$ s.t. $\nexists U_A \in \mathcal{L}(\mathcal{H}_A)$ and $U_B \in \mathcal{L}(\mathcal{H}_B) : U_{AB} = U_A \otimes U_B$

Controlled-NOT gate (CNOT gate):

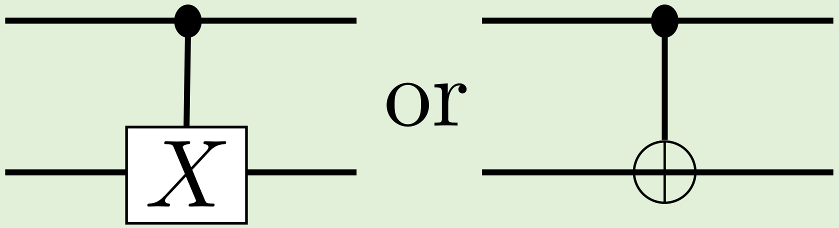


$$U_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes X$$

- $\forall U \in \mathcal{L}(\mathcal{H}_{AB}), \exists \{V_i\} \in \mathcal{L}(\mathcal{H}_A)$ and $\{W_i\} \in \mathcal{L}(\mathcal{H}_B) : U = \sum_i V_i \otimes W_i$

Entangling operations

Controlled-not gate (cnot gate):

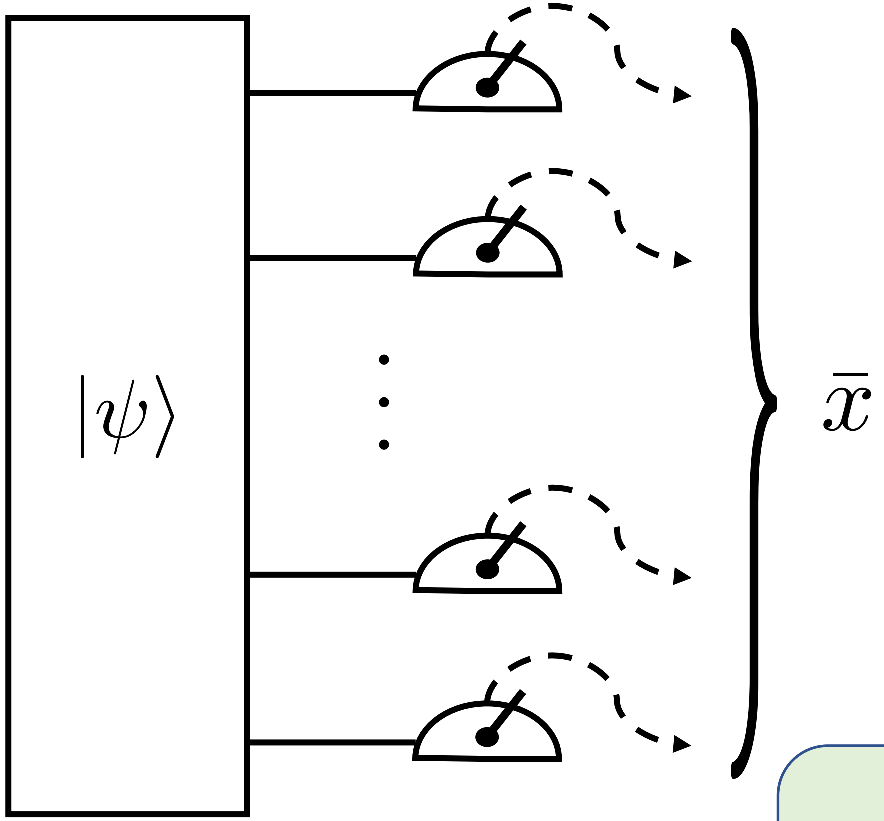


$$U_{CNOT} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes X = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$\|0\rangle\langle 0\| \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\|0\rangle\langle 1\| \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
$\|1\rangle\langle 0\| \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\|1\rangle\langle 1\| \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Composition of measurement: computational basis

- Let's $\bar{x}$ encode the outcome of n bits

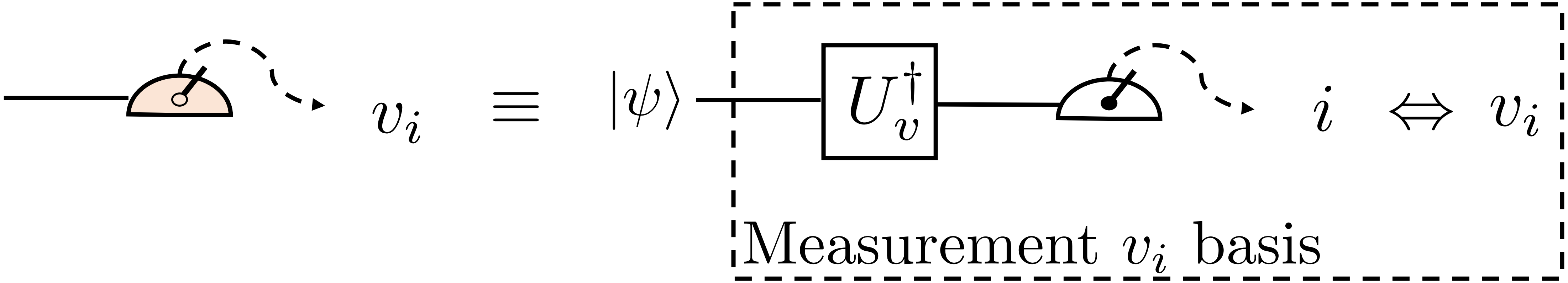


$$|\bar{x}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes .... \otimes |x_n\rangle$$

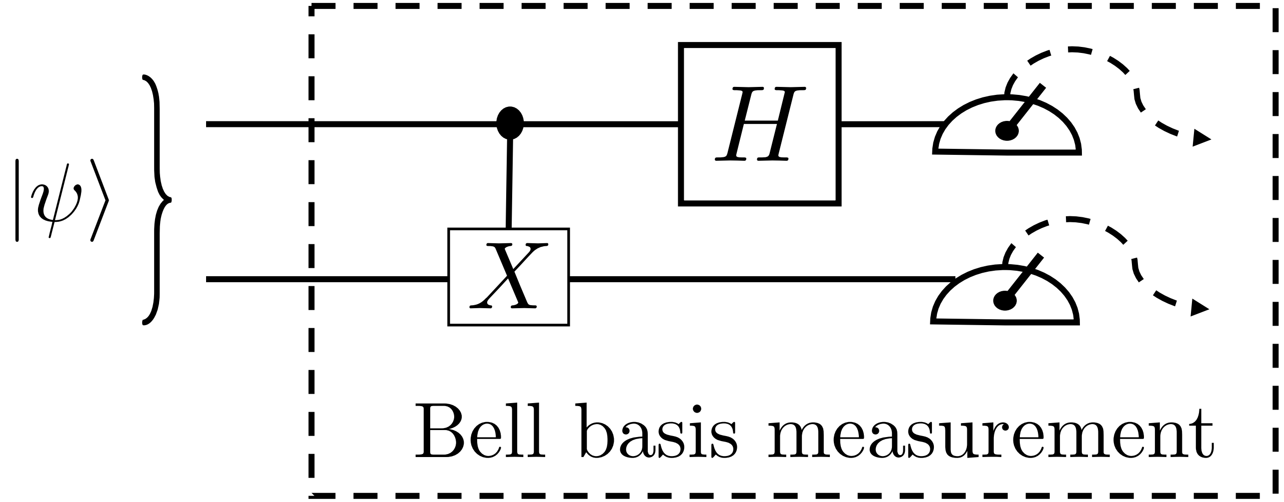
$$|\langle\bar{x}|\psi\rangle|^2 = |\langle\bar{x}| \sum_{\bar{y}\in\{0,1\}^n} \psi_{\bar{y}}|\bar{y}\rangle|^2 = |\psi_{\bar{x}}|^2$$

General multi-qubit basis measurement

- Measurement basis $\{|v_i\rangle\}$:



- Bell basis measurement:



Bell basis

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

References

Reading references

1. Tensor product NC 2.1.7
2. Outer-product NC 2.1.4 page 67

NC ≣ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information
Cambridge University Press (2010)