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Tutorial 5

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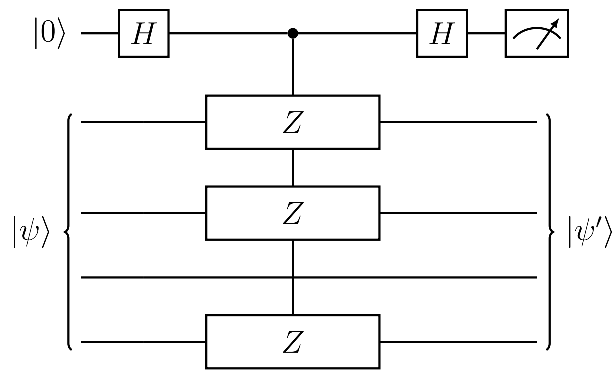
Problem 1: Three-Qubit Parity Check

We want to perform an even/odd parity check on qubits $1, 2, 4$. It's easy to see that the parity operator $P = Z \otimes Z \otimes I \otimes Z$ is both Hermitian and Unitary, so that it can both be regarded as an observable and a quantum gate. Suppose we wish to measure the observable P. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving an updated state after the measurement that is projected to its corresponding eigenspace. We are going to show that the following circuit implements a measurement of P:



a. Derive the action of the three-qubit parity operator $P = Z\otimes Z\otimes I\otimes Z$ on the computational basis state $|x_1x_2x_3x_4\rangle$. What are the eigenvalues of the operator P?

Solution: Recall that the state $|x_1x_2x_3x_4\rangle$ corresponds to the tensor product:

$$|x_1x_2x_3x_4\rangle \equiv |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes |x_4\rangle$$

We can use the property:

$$\begin{gathered} P\,|x_1x_2x_3x_4\rangle = (Z \otimes Z \otimes I \otimes Z)\,(|x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes |x_4\rangle) \\ = Z\,|x_1\rangle \otimes Z\,|x_2\rangle \otimes I\,|x_3\rangle \otimes Z\,|x_4\rangle = (-1)^{x_1}\,|x_1\rangle \otimes (-1)^{x_2}\,|x_2\rangle \otimes |x_3\rangle \otimes (-1)^{x_4}\,|x_4\rangle \\ \implies P\,|x_1x_2x_3x_4\rangle = (-1)^{x_1+x_2+x_4}\,|x_1x_2x_3x_4\rangle \end{gathered}$$

We can see that when P acts on a computational basis, it is scaled by a factor of -1 or $+1$ depending on the bits x_i. This means that the computational basis states are the eigenvectors of P with eigenvalues ± 1.

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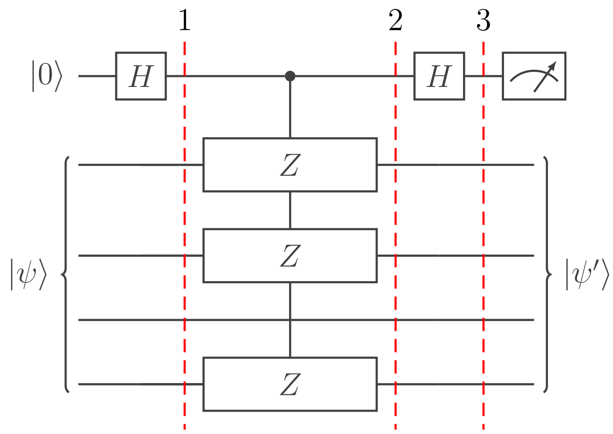
b. Derive the global state right before the measurement of the upper-qubit when the input state reads $|0\rangle \otimes |\psi\rangle$, where $|\psi\rangle = \sum_{x\in\{0,1\}^4} \gamma_x |x\rangle$ is a four qubit arbitrary input state and x is a four bit string.

Solution: First of all, we are going to divide the quantum circuits into subsequent steps and calculate the composite state in each one of them.



The initial state of the composite system of 5 qubits is:

$$|\psi\rangle_0 = |0\rangle \otimes |\psi\rangle = \sum_{x\in\{0,1\}^4} \gamma_x |0\rangle |x\rangle$$

Step 1: On the first step, we act with the Hadamard operator on the first qubit and get:

$$(H \otimes I) |\psi\rangle_0 = \sum_{x\in\{0,1\}^4} \gamma_x H |0\rangle |x\rangle = \sum_{x\in\{0,1\}^4} \gamma_x \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) |x\rangle$$
$$\sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle |x\rangle$$

and so the state $|\psi\rangle_1$ at step 1 is:

$$|\psi\rangle_1 = \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle |x\rangle$$

Step 2: On the second step, we act with the controlled-P operator and get:

$$CP |\psi\rangle_1 = \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle P |x\rangle$$

Note that x is the bitstring $x_1x_2x_3x_4$. Thus, by using the answer of question (a.) we get that the state $|\psi\rangle_2$ at step 2 is:

$$|\psi\rangle_2 = \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle (-1)^{x_1+x_2+x_4} |x\rangle$$
$$= \sum_{x_1+x_2+x_4=even} \gamma_x |+\rangle |x\rangle + \sum_{x_1+x_2+x_4=odd} \gamma_x |-\rangle |x\rangle$$

Step 3: On the third step, we act again with the Hadamard operator on the first qubit and get:

$$|\psi\rangle_3 = \sum_{x_1+x_2+x_4=even} \gamma_x |0\rangle |x\rangle + \sum_{x_1+x_2+x_4=odd} \gamma_x |1\rangle |x\rangle$$

$$|\psi\rangle_3 = |0\rangle \otimes \left(\sum_{x_1+x_2+x_4=even} \gamma_x |x\rangle \right) + |1\rangle \otimes \left(\sum_{x_1+x_2+x_4=odd} \gamma_x |x\rangle \right)$$

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c. Using the rules of partial measurement, show that the measurement of the upper-qubit projects the state of the lower four qubits to its odd or even parity subspaces, depending on the outcome being 0 or 1.

Solution: The partial measurement of the first qubit can be described as the linear operator $P_i \otimes I = |i\rangle \langle i| \otimes I$ with $i \in \{0, 1\}$. If we perform the measurement on the first qubit and find it in the $|0\rangle$ state, then the system after the measurement will be in the state:

$$|\psi\rangle = \frac{P_0 \otimes I |\psi\rangle_3}{||P_0 \otimes I |\psi\rangle_3||} = |0\rangle \otimes \frac{1}{(\sum_{x_1+x_2+x_4=even} |\gamma_x|^2)^{1/2}} \left(\sum_{x_1+x_2+x_4=even} \gamma_x |x\rangle \right)$$

On the other hand, if we measure it to be in the state $|1\rangle$ then the state of the system after the measurement will be:

$$|\psi\rangle = \frac{P_1 \otimes I |\psi\rangle_3}{||P_1 \otimes I |\psi\rangle_3||} = |1\rangle \otimes \frac{1}{(\sum_{x_1+x_2+x_4=odd} |\gamma_x|^2)^{1/2}} \left(\sum_{x_1+x_2+x_4=odd} \gamma_x |x\rangle \right)$$

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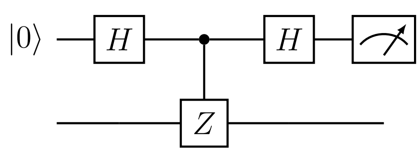
d. Prove that the two circuits below are equivalent:



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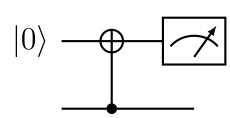
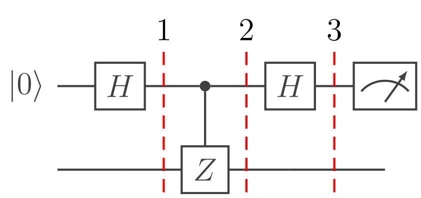


Solution: Consider the second qubit to be in the general state $|\psi\rangle = a|0\rangle + b|1\rangle$. We split the first circuit into three parts.



The initial state of the composite system is:

$$|\psi\rangle_0 = a|00\rangle + b|01\rangle$$

Step 1:

$$|\psi\rangle_1 = (H \otimes I)|\psi\rangle_0 = \frac{a}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{b}{\sqrt{2}}(|01\rangle + |11\rangle)$$

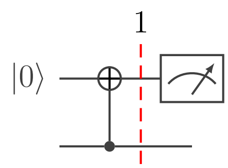
Step 2:

$$\begin{aligned} |\psi\rangle_2 = CZ|\psi\rangle_1 &= \frac{a}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{b}{\sqrt{2}}(|01\rangle - |11\rangle) \\ &= a|+\rangle|0\rangle + b|-\rangle|1\rangle \end{aligned}$$

Step 3:

$$|\psi\rangle_3 = (H \otimes I)|\psi\rangle_2 = a|0\rangle|0\rangle + b|1\rangle|1\rangle$$

We consider again the same input on the second circuit:



We can see that if we start with the same input:

$$|\psi\rangle_0 = a|00\rangle + b|01\rangle$$

then after the action of the controlled-NOT with the control being the second qubit we have:

$$|\psi\rangle_1 = a|00\rangle + b|11\rangle$$

We can thus conclude that the two circuits are equivalent.

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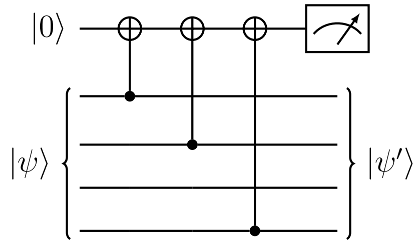
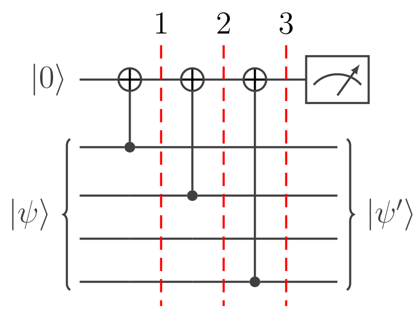
e. Prove that we can achieve the same result with the circuit:



Tutorial 5

Solution: In the same manner, we break the circuit into subsequent steps:



The initial state of the system is:

$$|\psi\rangle_0 = |0\rangle \otimes |\psi\rangle = \sum_{x\in\{0,1\}^4} \gamma_x |0\rangle |x\rangle$$

Step 1:

$$|\psi\rangle_1 = \sum_{x\in\{0,1\}^4} \gamma_x |0 \oplus x_1\rangle |x\rangle$$

Step 2:

$$|\psi\rangle_2 = \sum_{x\in\{0,1\}^4} \gamma_x |0 \oplus x_1 \oplus x_2\rangle |x\rangle$$

Step 3:

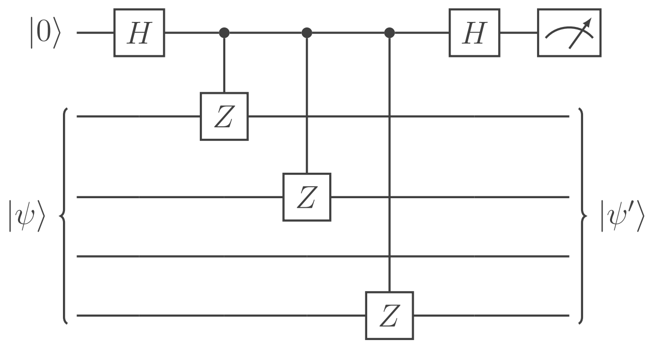
$$|\psi\rangle_3 = \sum_{x\in\{0,1\}^4} \gamma_x |0 \oplus x_1 \oplus x_2 \oplus x_4\rangle |x\rangle$$

$$\implies |\psi\rangle_3 = |0\rangle \otimes \left(\sum_{x_1+x_2+x_4=even} \gamma_x |x\rangle \right) + |1\rangle \otimes \left(\sum_{x_1+x_2+x_4=odd} \gamma_x |x\rangle \right)$$

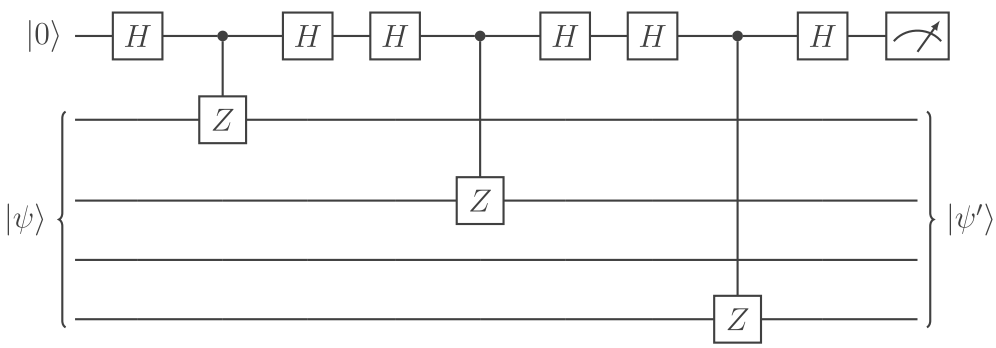
We can see that in both cases the output state is the same. We can thus conclude that the two circuits are equivalent.

Alternative solution:

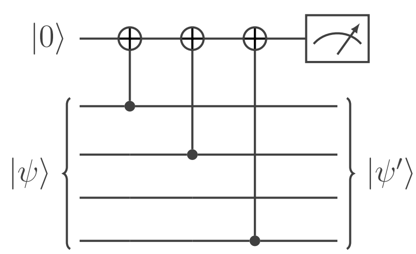
We can rewrite the original circuit by splitting the controlled-multi-Z gate into individual gates (as they are independent of each other, and have the same control qubit):



Now we can insert double Hadamard gates in between controlled-Z gates, as they are equivalent to identity:



Using the results from **d**, this is equivalent to:



Problem 2: SWAP Test

Given the two-qubit SWAP gate:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the two single-qubit states $|\phi_1\rangle = a\,|0\rangle + b\,|1\rangle$ and $|\phi_2\rangle = c\,|0\rangle + d\,|1\rangle$.

a. Show that $U_{\text{SWAP}}\,|\phi_1\rangle \otimes |\phi_2\rangle = |\phi_2\rangle \otimes |\phi_1\rangle$

Solution: We have $|\phi_1\rangle \otimes |\phi_2\rangle = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle$. The action of the SWAP gate is to exchange $|10\rangle$ and $|01\rangle$, leading to

$$\begin{aligned} U_{\text{SWAP}}\,|\phi_1\rangle \otimes |\phi_1\rangle &= ac\,|00\rangle + bc\,|01\rangle + ad\,|10\rangle + bd\,|11\rangle && (1)\\ &= c\,|0\rangle \otimes (a\,|0\rangle + b\,|1\rangle) + d\,|1\rangle \otimes (a\,|0\rangle + b\,|1\rangle) && (2)\\ &= (c\,|0\rangle + d\,|0\rangle) \otimes (a\,|0\rangle + b\,|1\rangle) && (3)\\ &= |\phi_2\rangle \otimes |\phi_1\rangle && (4) \end{aligned}$$

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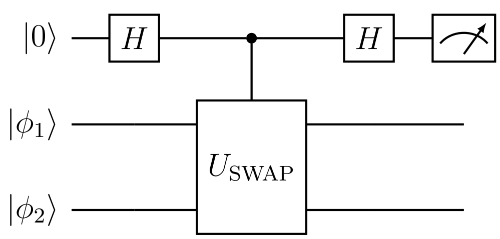
b. Consider the following SWAP test circuit acting on the two states $|\phi_1\rangle$ and $|\phi_2\rangle$.



Give the quantum state of the three qubit system at each step of the circuit.

Solution: The input to the circuit reads $|\psi_0\rangle = |0\rangle \otimes |\phi_1\rangle \otimes |\phi_2\rangle$. After the first Hadamard gate, the state reads

$$\begin{aligned} |\psi_1\rangle &= H \otimes I \otimes I\,|0\rangle \otimes |\phi_1\rangle \otimes |\phi_2\rangle && (5)\\ &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |\phi_1\rangle \otimes |\phi_2\rangle && (6) \end{aligned}$$

After the controlled SWAP, the state reads

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle \otimes |\phi_1\rangle \otimes |\phi_2\rangle + |1\rangle \otimes |\phi_2\rangle \otimes |\phi_1\rangle\right] \quad (7)$$

After the last Hadamard gate, the state reads

$$\begin{aligned} |\psi_3\rangle &= \frac{1}{2}\left[(|0\rangle + |1\rangle) \otimes |\phi_1\rangle \otimes |\phi_2\rangle + (|0\rangle - |1\rangle) \otimes |\phi_2\rangle \otimes |\phi_1\rangle\right] \\ &= |0\rangle \otimes \frac{1}{2}\left[|\phi_1\rangle \otimes |\phi_2\rangle + |\phi_2\rangle \otimes |\phi_1\rangle\right] + |1\rangle \otimes \frac{1}{2}\left[|\phi_1\rangle \otimes |\phi_2\rangle - |\phi_2\rangle \otimes |\phi_1\rangle\right] \end{aligned}$$

c. Compute the probability $P(0)$ of obtaining the outcome 0 at the top qubit, the probability $P(1)$ of obtaining the outcome 1, and their bias $P(0) - P(1)$.

Solution: The probability of outcome result correspond to $||\tilde{\Pi}_0 |\psi_3\rangle ||^2 = \langle\psi_3| \tilde{\Pi}_0 |\psi_3\rangle$. It is easy to see that

$$\tilde{\Pi}_0 |\psi_3\rangle = |0\rangle \otimes \frac{1}{2} [|\phi_1\rangle \otimes |\phi_2\rangle + |\phi_2\rangle \otimes |\phi_1\rangle] . \quad (8)$$

Then the probability of its outcome reads

$$\begin{aligned} \langle\psi_3| \tilde{\Pi}_0 |\psi_3\rangle &= \frac{1}{4} [\langle\phi_1| \otimes \langle\phi_2| + \langle\phi_2| \otimes \langle\phi_1|] [|\phi_1\rangle \otimes |\phi_2\rangle + |\phi_2\rangle \otimes |\phi_1\rangle] \\ &= \frac{1}{4} + \frac{1}{4}\langle\phi_2|\phi_1\rangle\langle\phi_1|\phi_2\rangle + \frac{1}{4}\langle\phi_1|\phi_2\rangle\langle\phi_2|\phi_1\rangle + \frac{1}{4} \quad (9) \\ &= \frac{1}{2} + \frac{1}{2}\langle\phi_1|\phi_2\rangle\langle\phi_1|\phi_2\rangle^* \quad (10) \\ &= \frac{1}{2} + \frac{1}{2}|\langle\phi_2|\phi_1\rangle|^2 \quad (11) \end{aligned}$$

A similar calculation as above leads to:

$$P(1) = \frac{1}{2} - \frac{1}{2}|\langle\phi_2|\phi_1\rangle|^2. \quad (12)$$

Therefore, the bias of probabilities reads $P(0) - P(1) = |\langle\phi_2|\phi_1\rangle|^2$.

Problem 3: Quantum Fourier Transform

As you have seen in the lectures, we can represent any integer z in its binary form as:

$$z = z_1 z_2 \dots z_n$$

where $z_1, z_2, \dots, z_n$ are such so that:

$$z = z_1 2^{n-1} + \dots + z_{n-1} 2^1 + z_n$$

a. How many qubits at least would we need to encode the integer states $|14\rangle$ and $|9\rangle$? What is their binary representation when using qubits to encode the integers?

Solution: In order to represent an integer state $|N\rangle$, one would require at least $n =$

$\lceil \log(N+1) \rceil$ qubits. This implies that for both cases we require 4 qubits. The binary representation of these four-qubit integer states is:

$$|14\rangle = |1110\rangle$$
$$|9\rangle = |1001\rangle$$

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b. Recall that:

$$0.z_l z_{l+1} \dots z_m \equiv \frac{z_l}{2} + \frac{z_{l+1}}{2^2} + \dots + \frac{z_m}{2^{m-l+1}}$$

Calculate:

1. $2^3 \cdot 0.z_1z_2z_3$, $2^2 \cdot 0.z_1z_2z_3$ and $2 \cdot 0.z_1z_2z_3$, where $z_i \in \{0,1\}$.
2. $e^{2\pi i \cdot 2^2 \cdot 0.j_1j_2j_3}$ where $j_i \in \{0,1\}$.

Solution: We start by writing down the expression for $0.z_1z_2z_3$:

$$0.z_1z_2z_3 = \frac{z_1}{2} + \frac{z_2}{4} + \frac{z_3}{8}$$

Then it is easy to calculate the expressions above. For the first case, we have:

$$2^3 \cdot 0.z_1z_2z_3 = 4z_1 + 2z_2 + z_3$$
$$2^2 \cdot 0.z_1z_2z_3 = 2z_1 + z_2 + \frac{z_3}{2}$$
$$2 \cdot 0.z_1z_2z_3 = z_1 + \frac{z_2}{2} + \frac{z_3}{4}$$

For the second case:

$$e^{2\pi i \cdot 2^2 \cdot 0.j_1j_2j_3} = e^{2\pi i(2j_1+j_2+j_3/2)} = e^{2\pi i(2j_1+j_2)} e^{2\pi i j_3/2} = e^{2\pi i 0.j_3},$$

where in the second equality we used the fact that $2j_1 + j_2$ is an integer and therefore $e^{2\pi i(2j_1+j_2)} = 1$.

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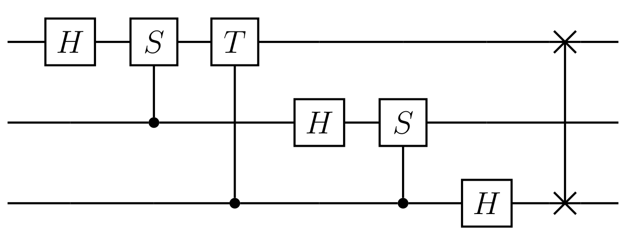
c. Now consider the quantum Fourier circuit for three qubits:

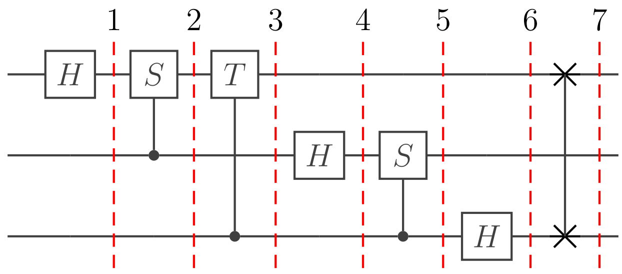


with S and T being the gates:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Suppose that we input the state $|j\rangle = |j_1 j_2 j_3\rangle$. What will be the output state?

Solution: We start as usual by dividing the quantum circuit into subsequent steps:



Initially, the system is in the state:

$$|\psi\rangle_0 = |j_1 j_2 j_3\rangle$$

Then we act with the Hadamard operator on the first qubit and use the fact that $e^{2\pi i 0.j_1}$ is $+1$ if $j_1 = 0$ and -1 if $j_1 = 1$. Thus the state at step 1 is transformed to:

$$|\psi\rangle_1 = \frac{1}{2^{1/2}}(|0\rangle + e^{2\pi i 0.j_1}|1\rangle)|j_2 j_3\rangle$$

Recall that the unitary operator R_k is defined as:

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

It's easy to see that both S and T are special cases of the operator R_k for two different choices of k. S corresponds to R_2 while T corresponds to R_3.

On the next step, applying the S operator on the first qubit controlled by the second qubits produces the state:

$$|\psi\rangle_2 = \frac{1}{2^{1/2}}(|0\rangle + e^{2\pi i 0.j_1} e^{2\pi i 0.0 j_2}|1\rangle)|j_2 j_3\rangle = \frac{1}{2^{1/2}}(|0\rangle + e^{2\pi i 0.j_1 j_2}|1\rangle)|j_2 j_3\rangle$$

Next, we perform the controlled-T operation and so we get:

$$|\psi\rangle_3 = \frac{1}{2^{1/2}}(|0\rangle + e^{2\pi i 0.j_1 j_2} e^{2\pi i 0.00 j_3}|1\rangle)|j_2 j_3\rangle = \frac{1}{2^{1/2}}(|0\rangle + e^{2\pi i 0.j_1 j_2 j_3}|1\rangle)|j_2 j_3\rangle$$

If we work with the exact same way for the rest of the steps we will get:
Step 4:

$$|\psi\rangle_4 = \frac{1}{2}(|0\rangle + e^{2\pi i 0.j_1 j_2 j_3} |1\rangle)(|0\rangle + e^{2\pi i 0.j_2}) |j_3\rangle$$

Step 5:

$$|\psi\rangle_4 = \frac{1}{2}(|0\rangle + e^{2\pi i 0.j_1 j_2 j_3} |1\rangle)(|0\rangle + e^{2\pi i 0.j_2 j_3}) |j_3\rangle$$

Step 6:

$$|\psi\rangle_4 = \frac{1}{2^{3/2}}(|0\rangle + e^{2\pi i 0.j_1 j_2 j_3} |1\rangle)(|0\rangle + e^{2\pi i 0.j_2 j_3})(|0\rangle + e^{2\pi i 0.j_3} |1\rangle)$$

At the final step, we swap the state of the first and third qubit and recover the *quantum Fourier transformation*:

$$|\psi\rangle_4 = \frac{1}{2^{3/2}}(|0\rangle + e^{2\pi i 0.j_3} |1\rangle)(|0\rangle + e^{2\pi i 0.j_2 j_3})(|0\rangle + e^{2\pi i 0.j_1 j_2 j_3} |1\rangle)$$
