

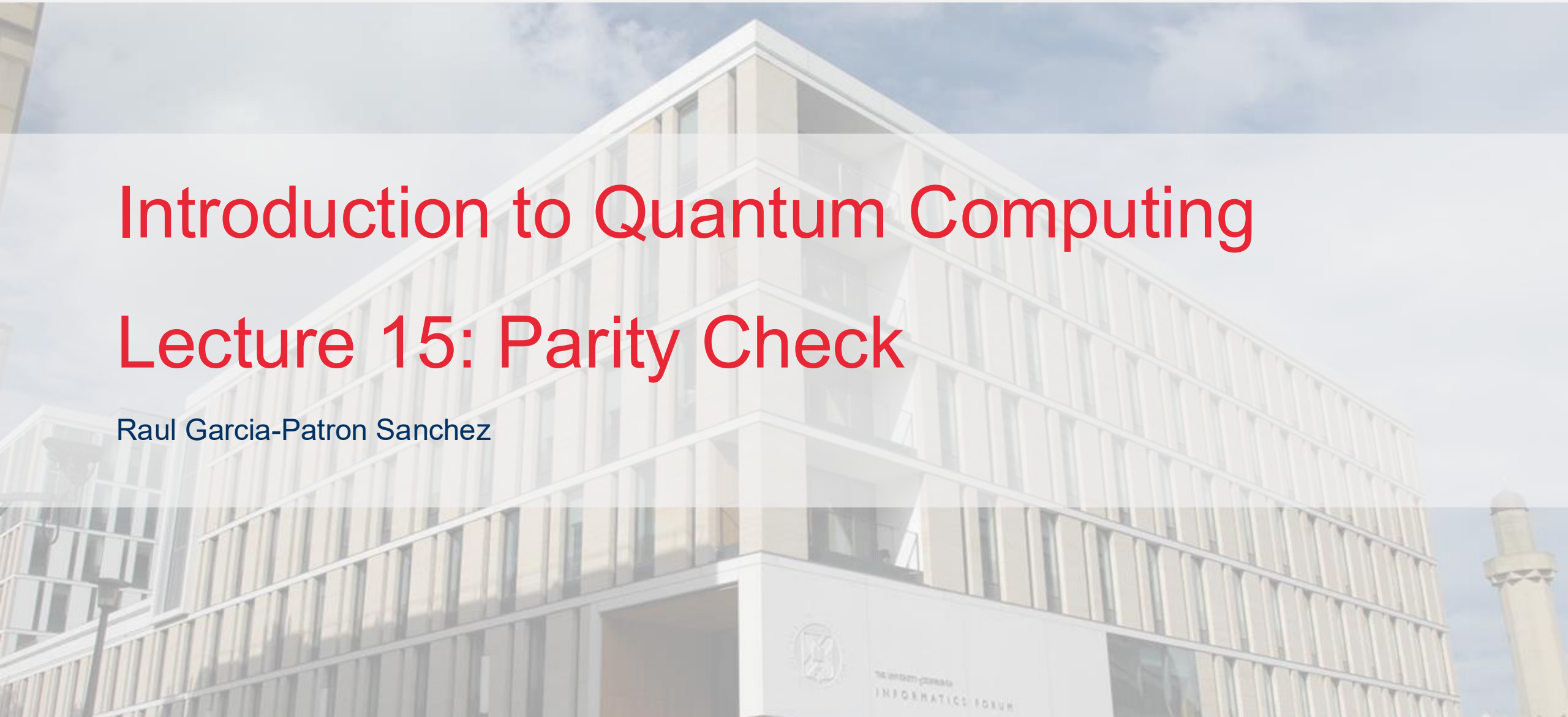


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Introduction to Quantum Computing

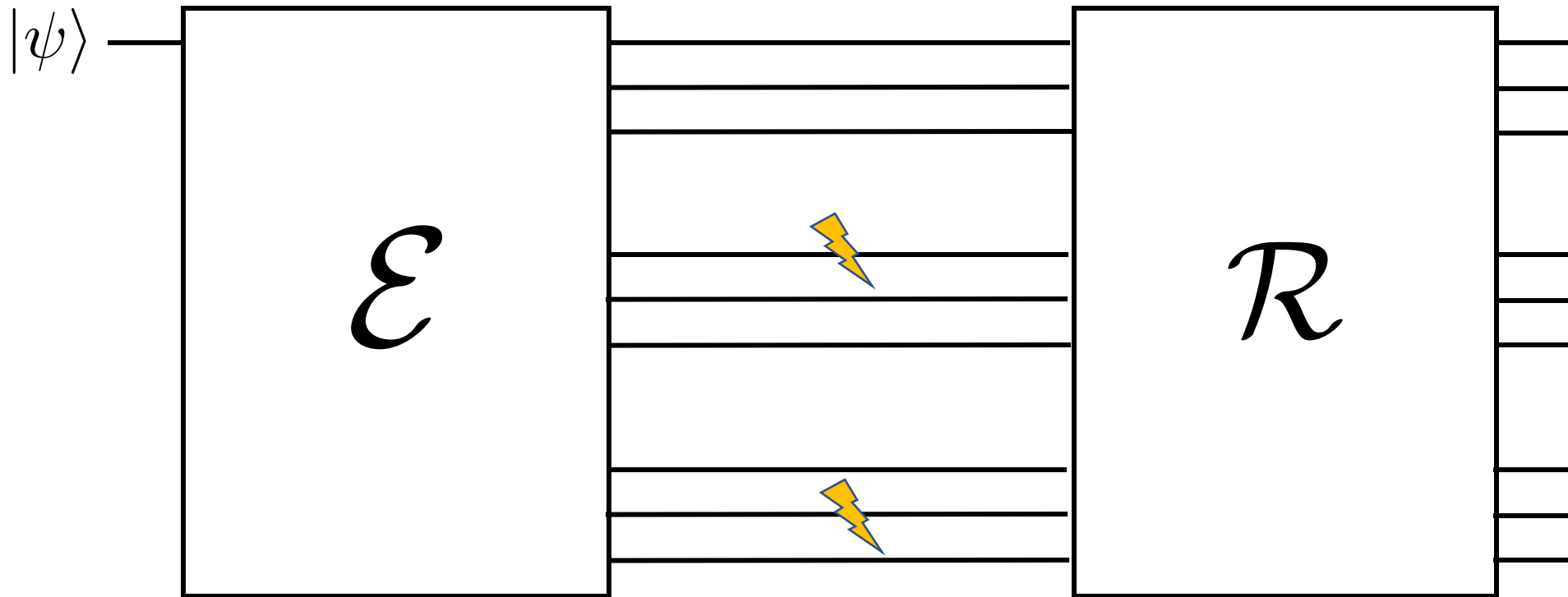
Lecture 15: Parity Check

Raul Garcia-Patron Sanchez



Quantum Error Correction in one slide

- Noise is local: independent for every qubit
- Our operations are perfect: error due to interaction with local environment.



- In reality we have also correlated error
- Our operations have errors $\Rightarrow$ Theory of fault-tolerance



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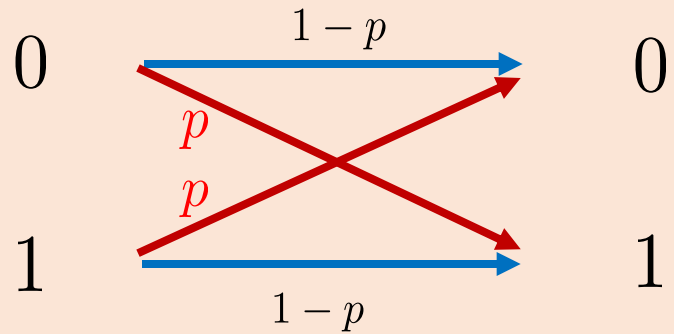
Classical repetition code and parity check



The repetition code

Encoding $0_L \rightarrow 000$ Codeword
 $1_L \rightarrow 111$

Binary symmetric channel



$000 \rightarrow 010$

Error detection: majority vote

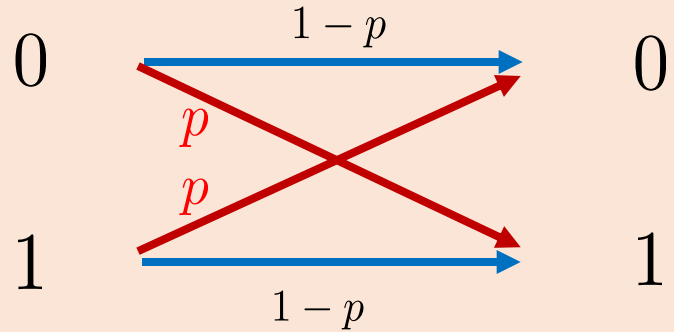
Error correction

$\{000, 100, 010, 001\} \rightarrow 0_L$
 $\{111, 110, 011, 101\} \rightarrow 1_L$

The repetition code

Encoding $0_L \rightarrow 000$ Codeword
 $1_L \rightarrow 111$

Binary symmetric channel



$000 \rightarrow 010$

Error detection: majority vote

Error correction

$\{000, 100, 010, 001\} \rightarrow 0_L$
 $\{111, 110, 011, 101\} \rightarrow 1_L$

$[n, k, d]$ code

n : size of codeword

k : # logical bits

d : Hamming distance

e : correctable error

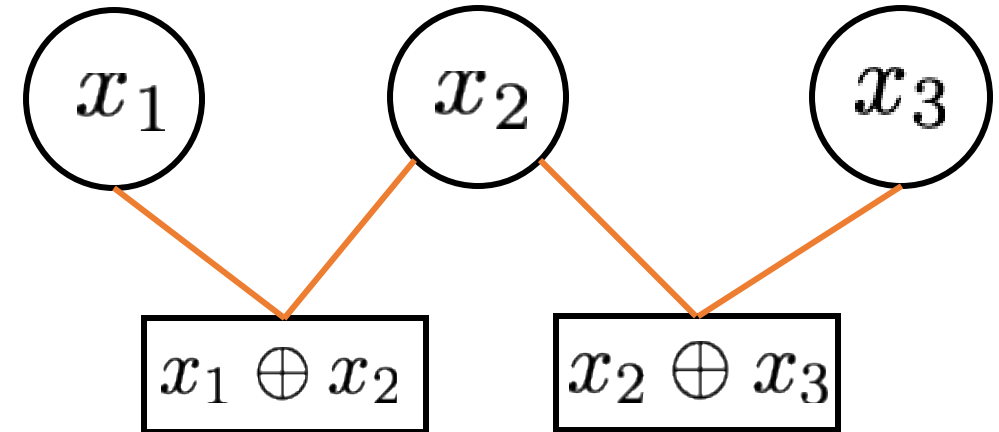
$$d = 2e + 1$$

Repetition code

$[3, 1, 3]$ code

Parity checks

Encoding $0_L \rightarrow 000$ Codeword
 $1_L \rightarrow 111$



Message	Parity Check 1	Parity Check 2	Error location	Output
000	0	0	no	000
001	0	1	3	000
010	1	1	2	000
100	1	0	1	000
011	1	0	1	111
101	1	1	2	111
110	0	1	3	111
111	0	0	no	111



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Quantum parity check

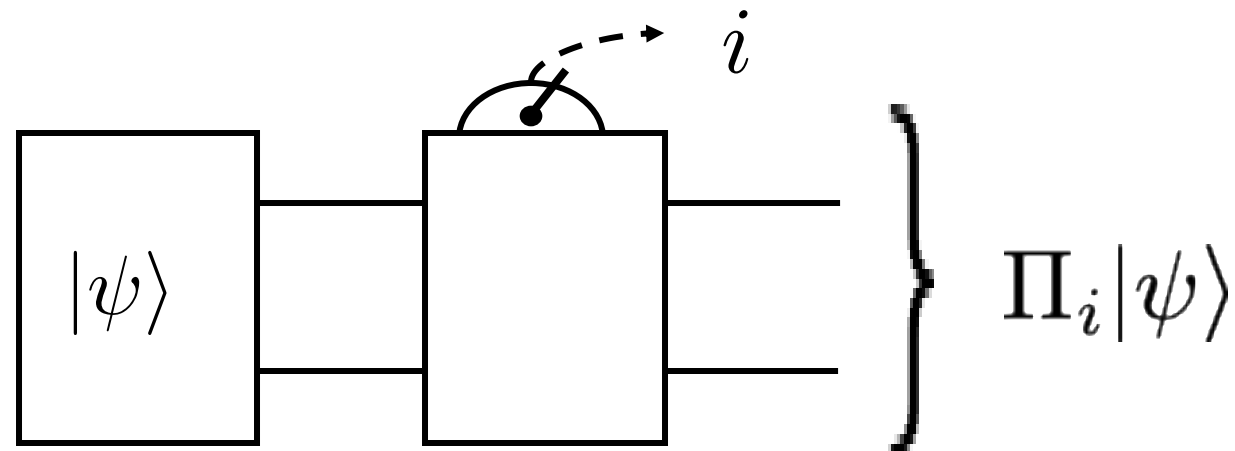


Example 2: Parity of 2 qubits

- $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o = \text{span}\{|00\rangle, |11\rangle\} \oplus \text{span}\{|01\rangle, |10\rangle\}$

$\oplus \equiv$ Direct sum

- $$\left. \begin{aligned} \Pi_e &= |00\rangle\langle 00| + |11\rangle\langle 11| \\ \Pi_o &= |01\rangle\langle 01| + |10\rangle\langle 10| \end{aligned} \right\} \Pi_e + \Pi_o = I_4$$

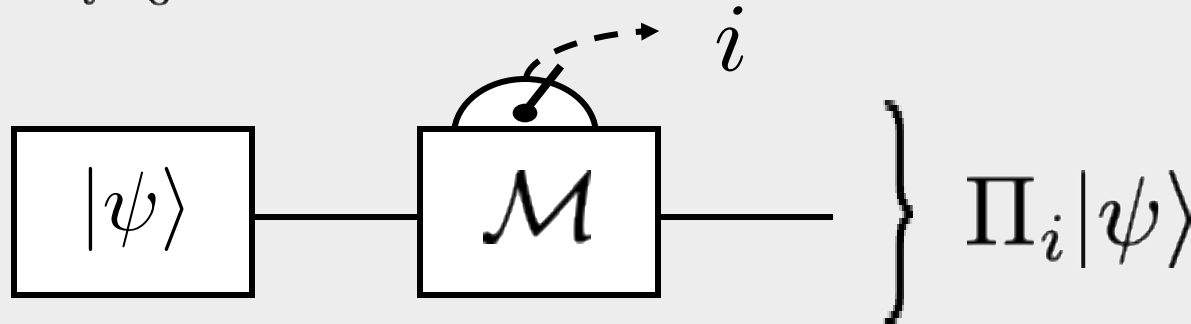


From the math definition to an actual circuit

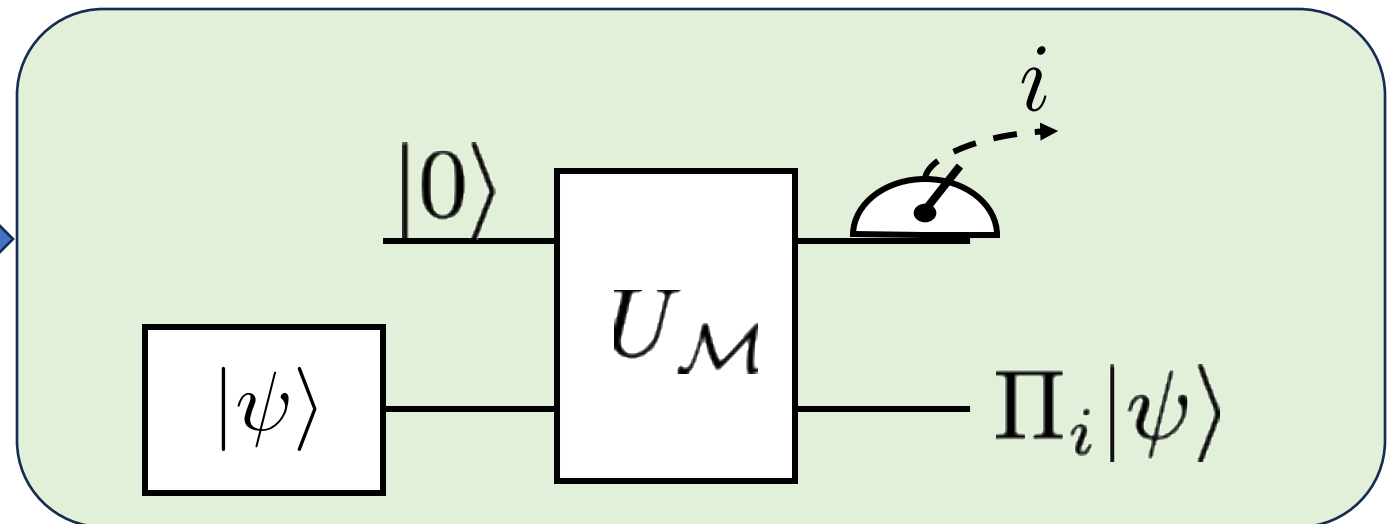
$$\sum_{i=0}^l \Pi_i = I_{2^n}$$

$$P(i) = ||\Pi_i|\psi\rangle||^2$$

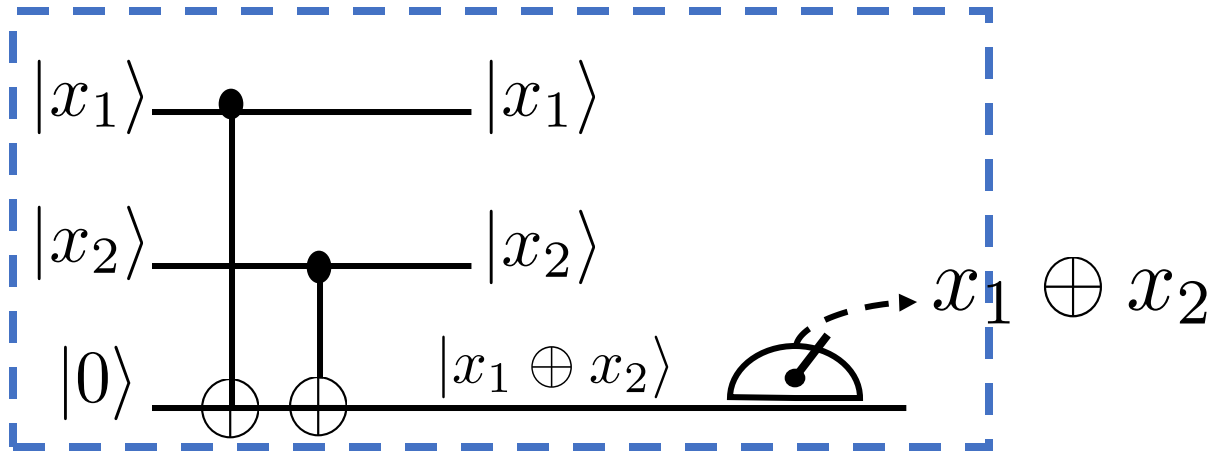
$$\text{Update: } \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$



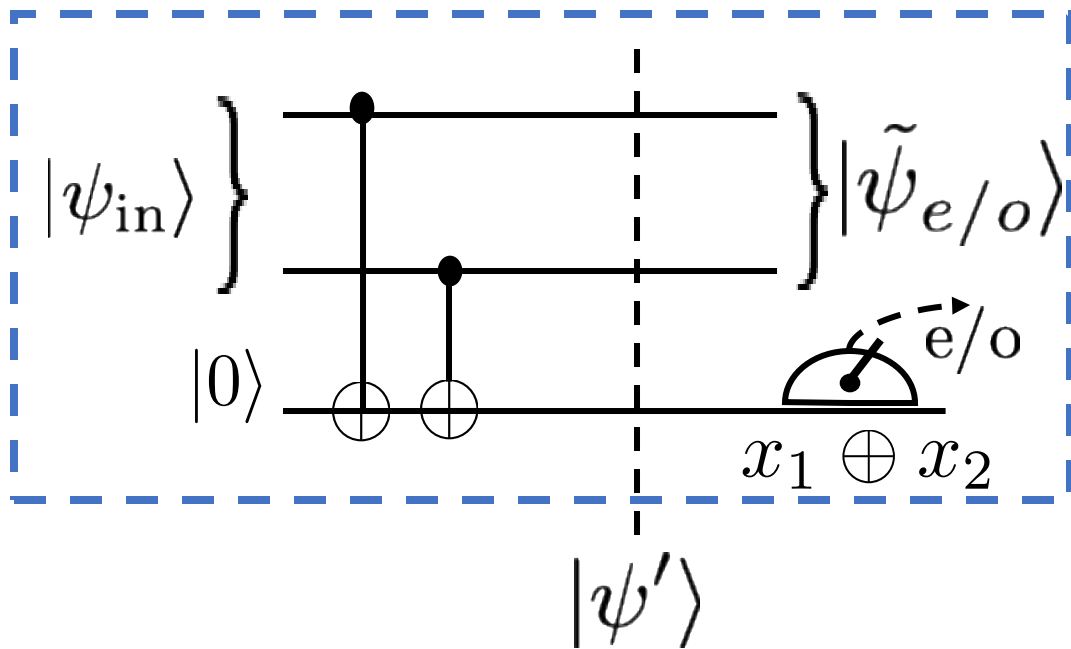
We need to find the circuit $U_{\mathcal{M}}$



Parity check circuit



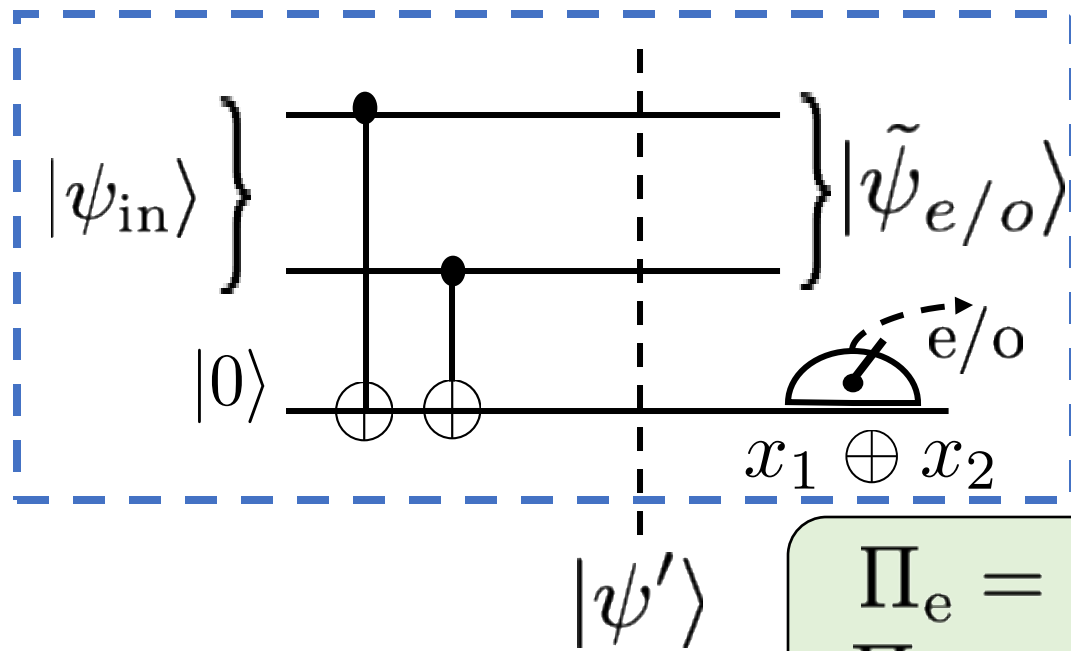
Parity check circuit – general input



$$|\psi_{\text{in}}\rangle = \sum_{x_1, x_2} \psi_{x_1, x_2} |x_1, x_2\rangle \otimes |0\rangle$$

$$|\psi'\rangle = \sum_{x_1, x_2} \psi_{x_1, x_2} |x_1, x_2\rangle \otimes |x_1 \oplus x_2\rangle$$

Parity check circuit – general input



$$|\psi_{\text{in}}\rangle = \sum_{x_1, x_2} \psi_{x_1, x_2} |x_1, x_2\rangle \otimes |0\rangle$$

$$|\psi'\rangle = \sum_{x_1, x_2} \psi_{x_1, x_2} |x_1, x_2\rangle \otimes |x_1 \oplus x_2\rangle$$

$$\begin{aligned} \Pi_e &= I \otimes I \otimes |0\rangle\langle 0| \\ \Pi_o &= I \otimes I \otimes |1\rangle\langle 1| \end{aligned}$$

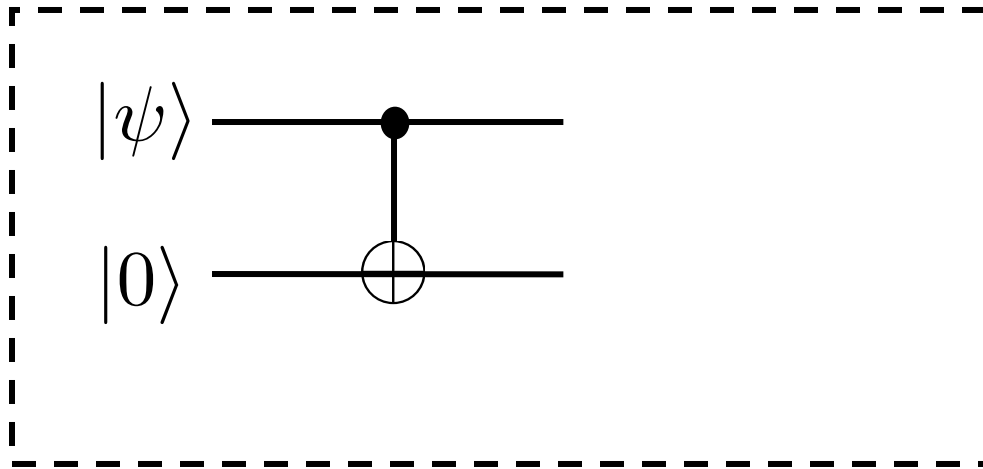
$$\Pi_e + \Pi_o = I_{\mathcal{H}^{\otimes 3}}$$

$$\Pi_o |\psi\rangle = (\psi_{0,0} |0, 1\rangle + \psi_{1,0} |1, 0\rangle) \otimes |1\rangle$$

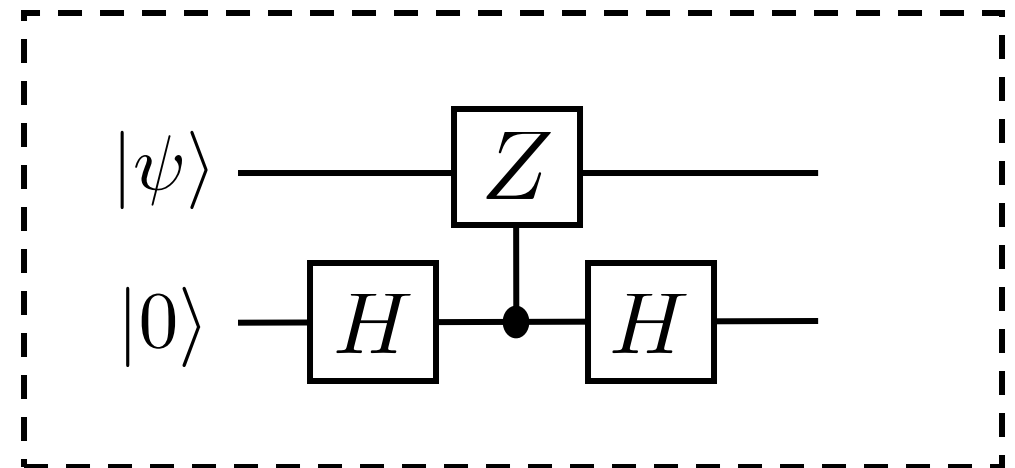
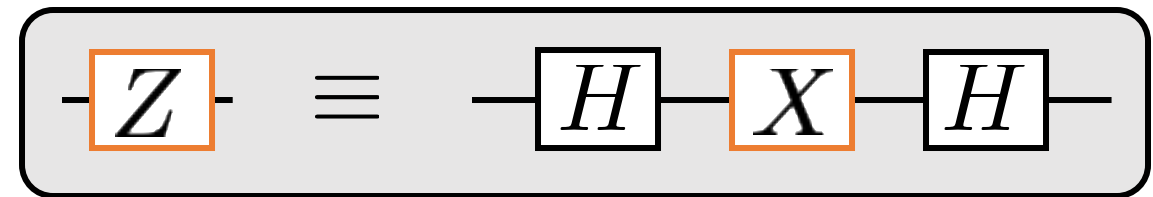
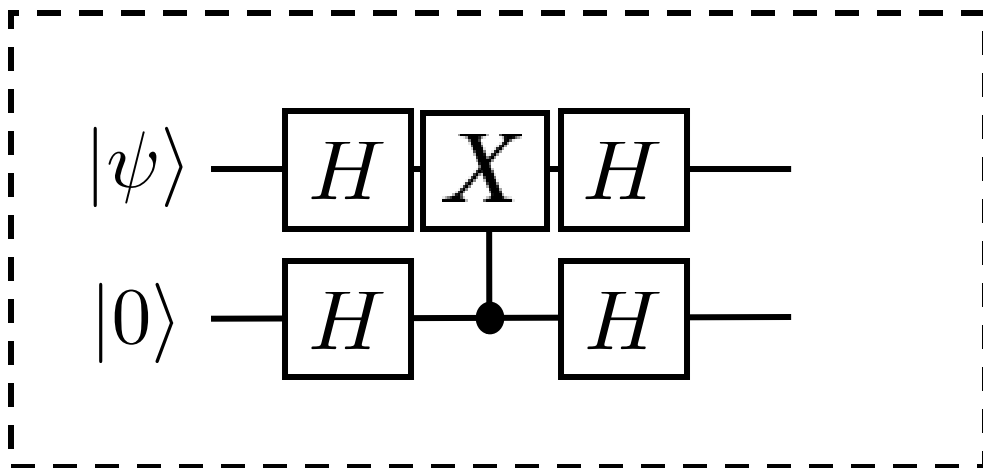
$$||\Pi_o |\psi\rangle||^2 = |\psi_{0,1}|^2 + |\psi_{1,0}|^2$$

$$\frac{\Pi_o |\psi\rangle}{||\Pi_o |\psi\rangle||} = \frac{1}{\sqrt{|\psi_{0,1}|^2 + |\psi_{1,0}|^2}} (\psi_{0,1} |0, 1\rangle + \psi_{1,0} |1, 0\rangle) \otimes |1\rangle = |\tilde{\psi}_o\rangle \otimes |1\rangle$$

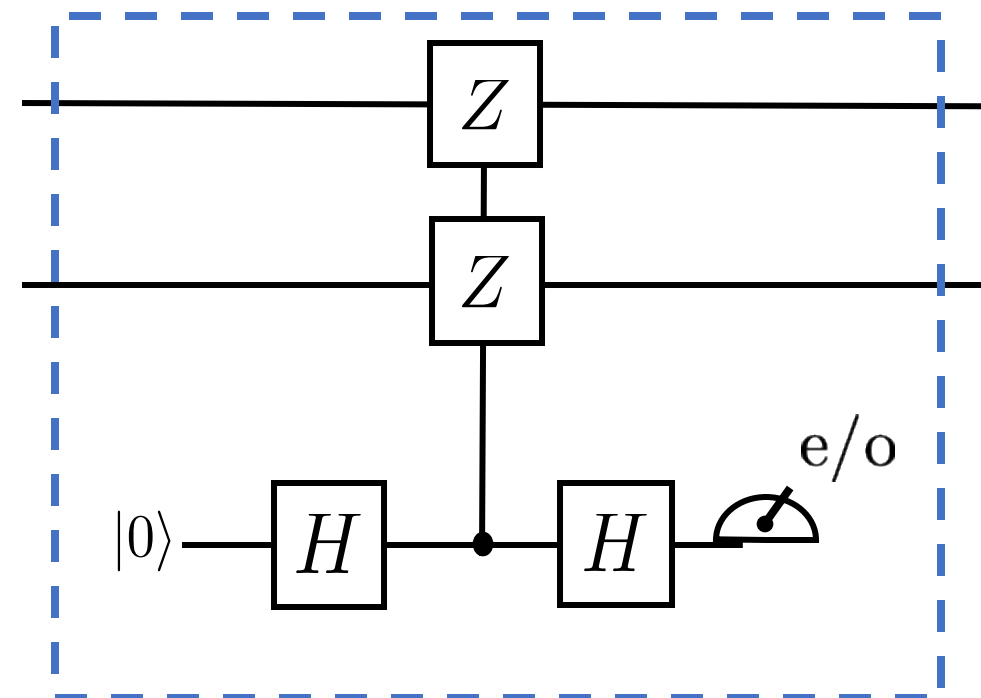
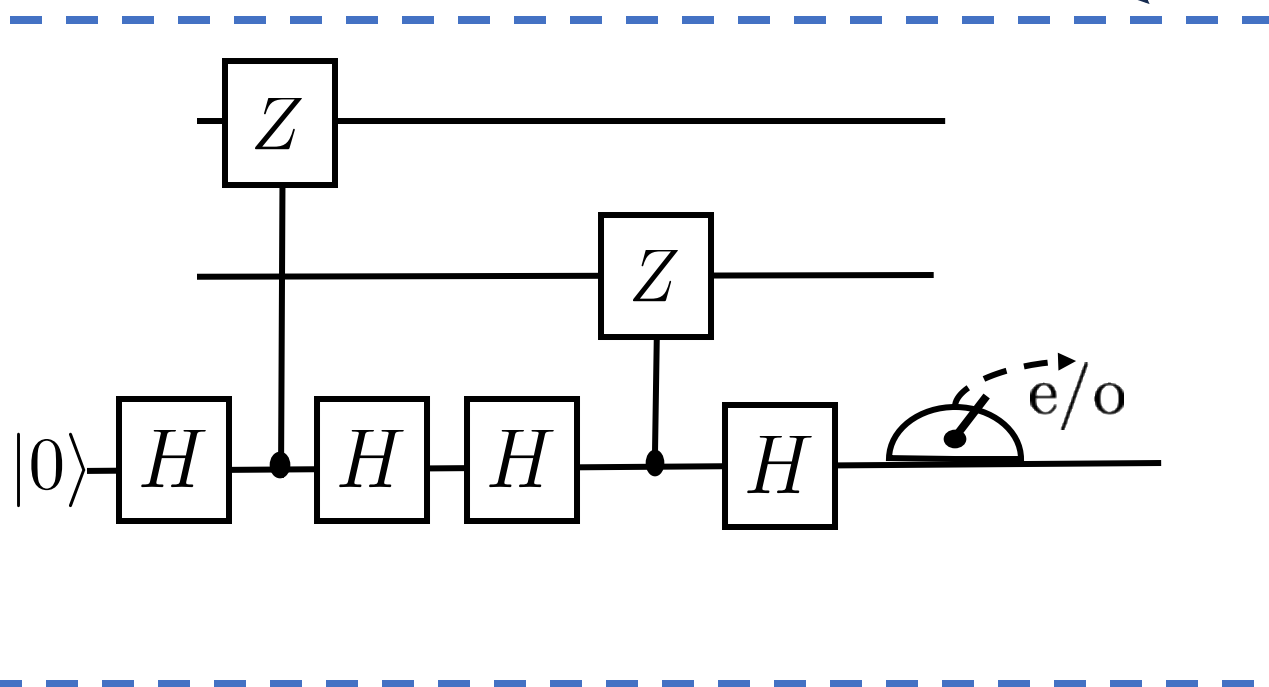
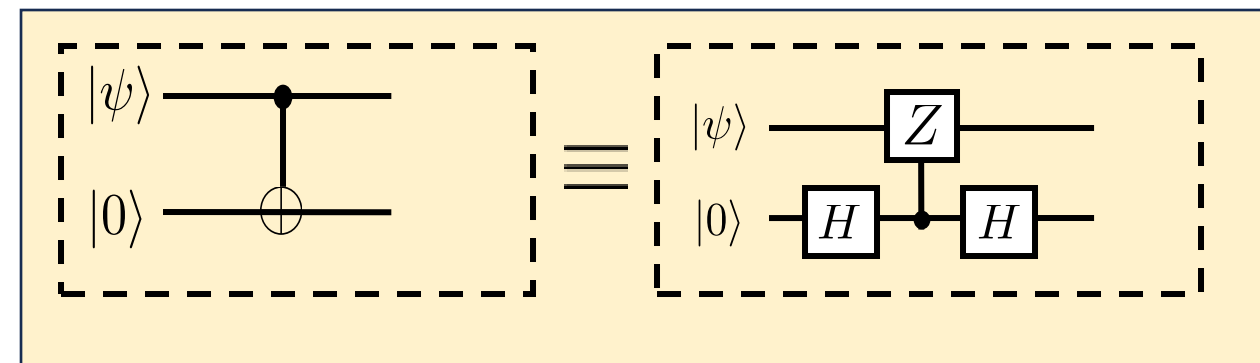
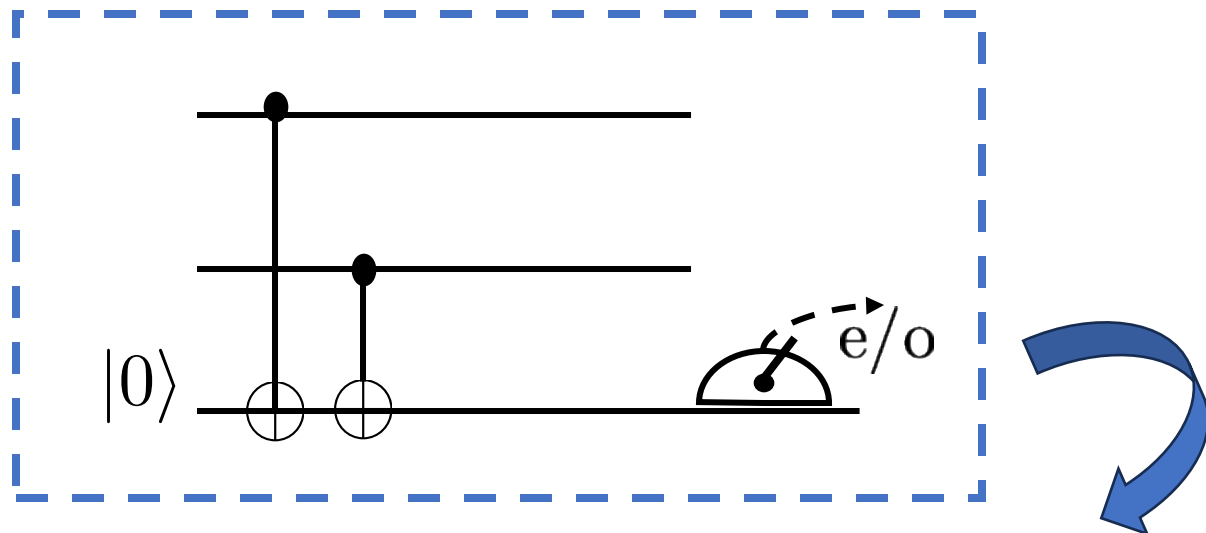
An equivalent circuit



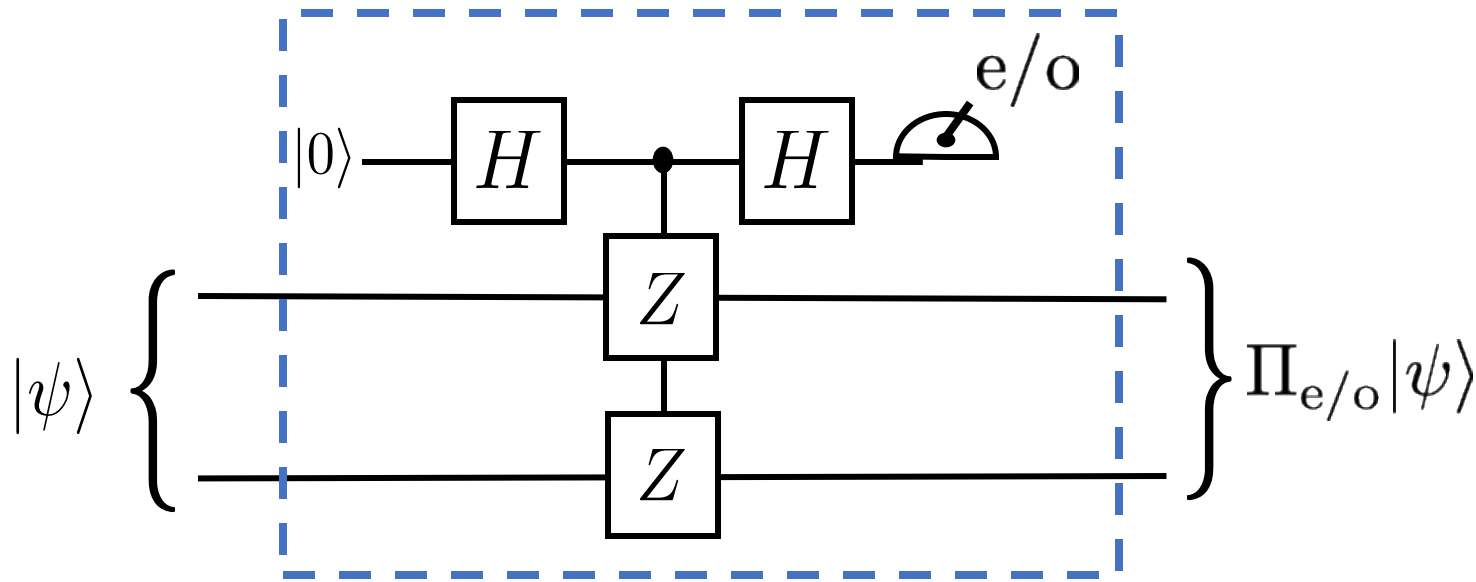
Tutorial 2 problem 2



Hadamard test circuit for parity check



Hadamard test circuit for parity check



$$Z_1 \otimes Z_2 |x_1 x_2\rangle = (-1)^{x_1 \oplus x_2} |x_1 x_2\rangle$$

$$\Pi_e = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\Pi_o = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$(Z \otimes Z) \Pi_{e/o} = \pm \Pi_{e/o}$$

The operator ZZ:

Eigenspace eigenvalue 1 is Even parity

Eigenspace eigenvalue -1 is Odd parity

This will generalize to the
Hadamard Test, see next lecture



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Parity check of phase



The intuition

$$-\boxed{X}- \equiv -\boxed{H}-\boxed{Z}-\boxed{H}-$$

$$H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) = |\pm\rangle$$

Bit flip error

$$-\boxed{X}-$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|x\rangle = |x \oplus 1\rangle$$

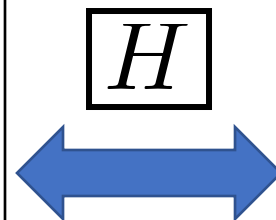
- Basis $Z|x\rangle = (-1)^x|x\rangle$

- Parity check $Z \otimes Z$

$$\Pi_e = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\Pi_o = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$(Z \otimes Z)\Pi_{e/o} = \pm\Pi_{e/o}$$



Phase flip error

$$-\boxed{Z}-$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|\pm\rangle = |\mp\rangle$$

$$X|\pm\rangle = \pm|\pm\rangle$$

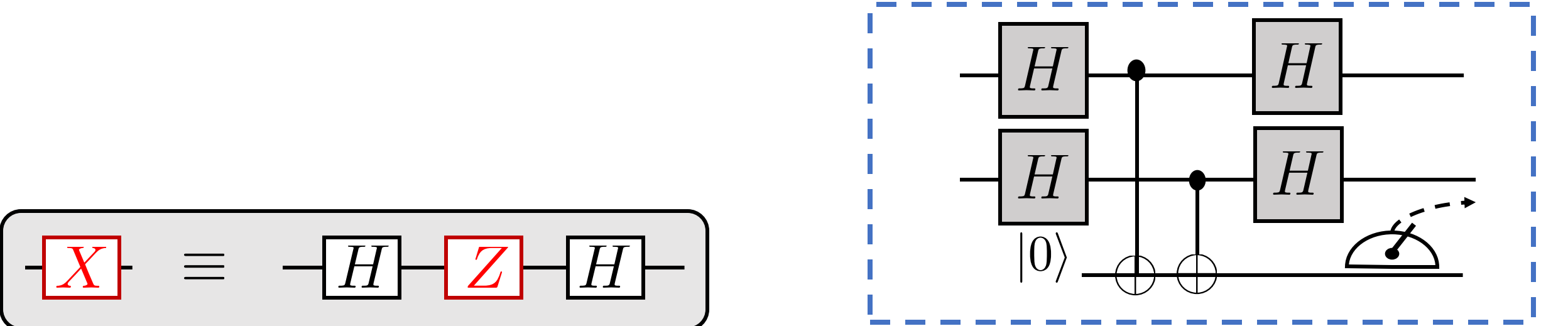
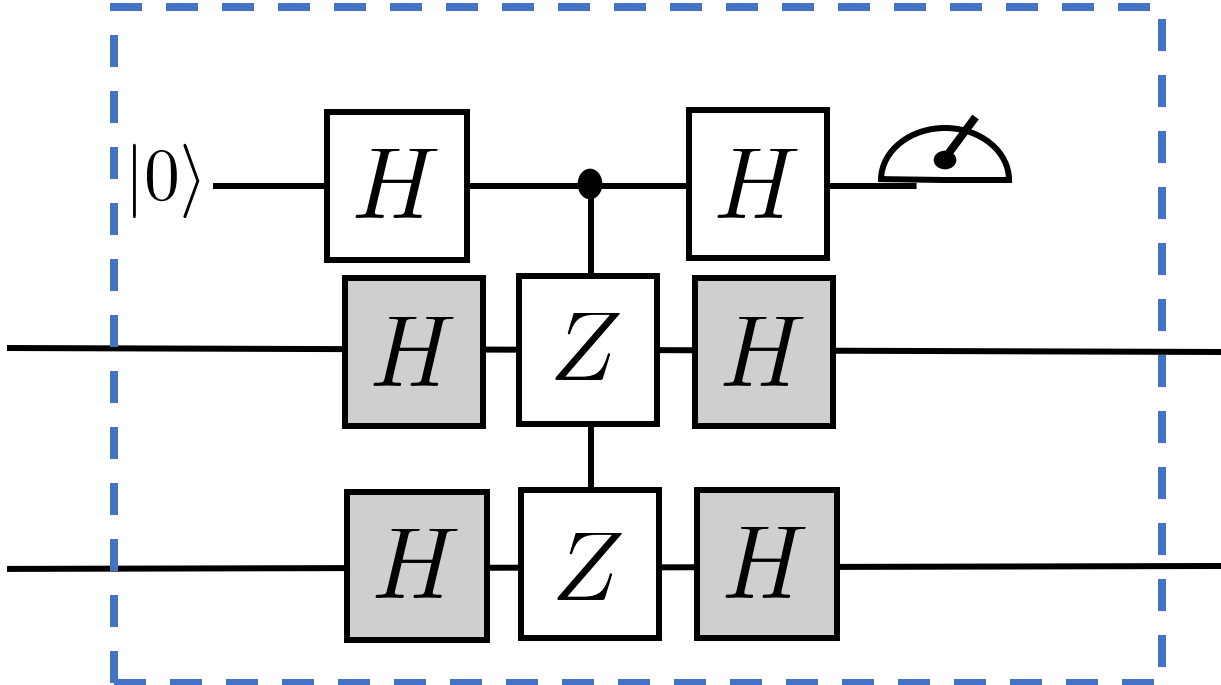
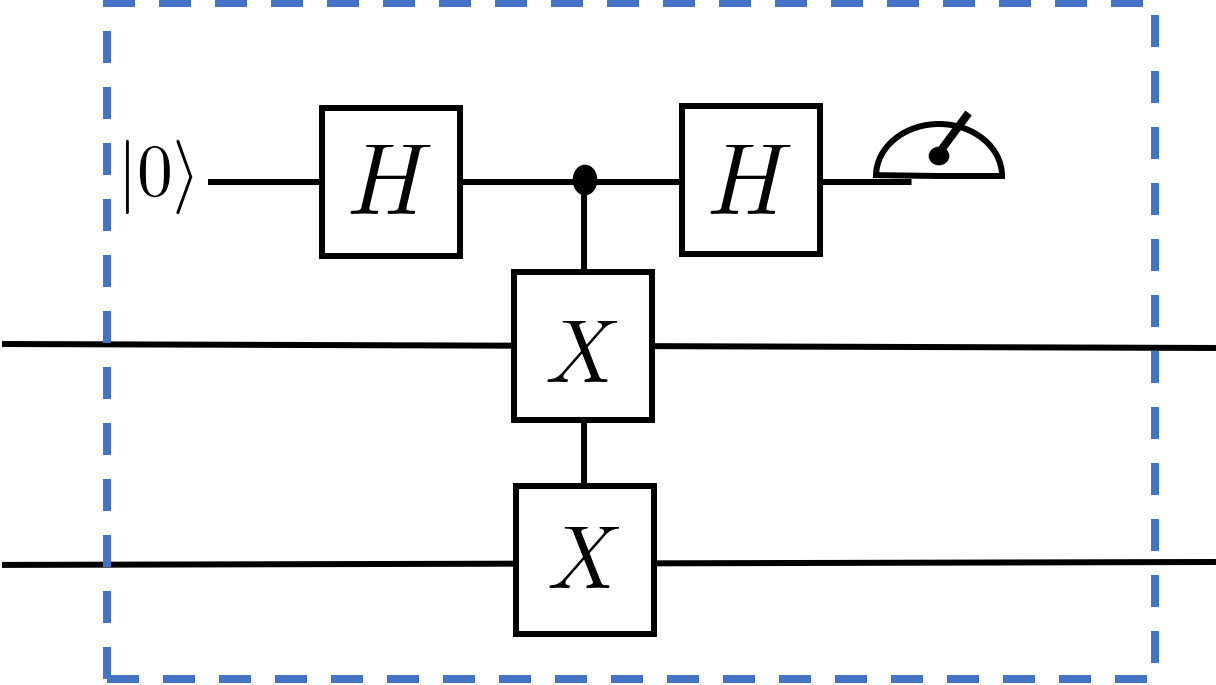
- Parity check $X \otimes X$

$$\tilde{\Pi}_e = |++\rangle\langle ++| + |--\rangle\langle --|$$

$$\tilde{\Pi}_o = |+-\rangle\langle +-| + -+\rangle\langle -+|$$

$$(X \otimes X)\tilde{\Pi}_{e/o} = \pm\tilde{\Pi}_{e/o}$$

Parity on X basis



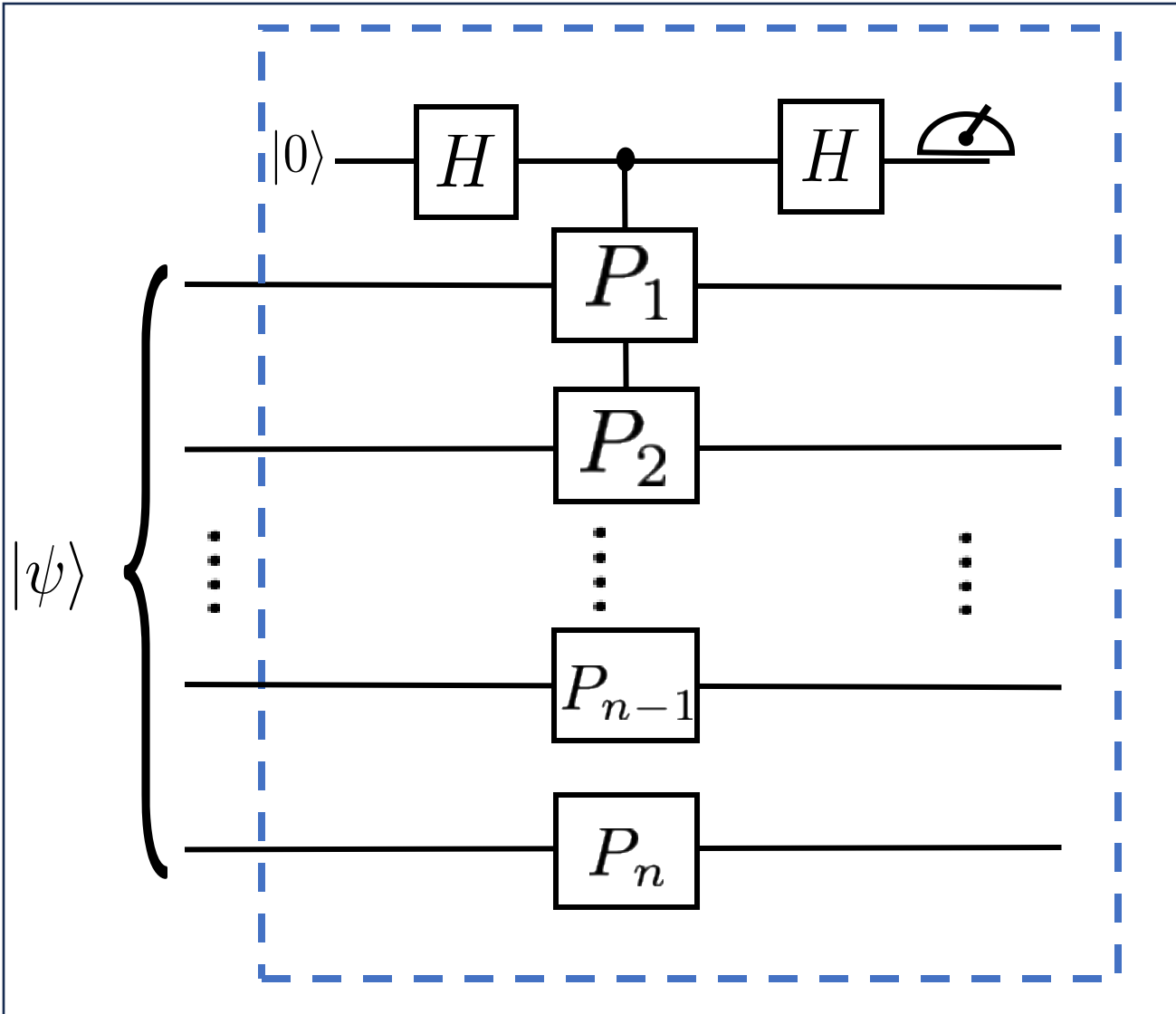


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General Parity Check – QEC lectures



Generalized parity check



Parity-check operator (syndrome)

$$P = P_1 \otimes P_2 \otimes \dots \otimes P_{n-1} \otimes P_n$$

A string of Pauli matrices

$$P_i \in \{I, X, Y, Z\}$$

$$P\Pi_{\pm 1} = \pm\Pi_{\pm 1}$$

$\Pi_{\pm 1}$ eigenspace with eigenvalue ± 1

$$\Pi_{+1} + \Pi_{-1} = I_{2^n}$$

References

Reading references

1. Operators and Observables: NC 2.1.2, 2.15, 2.1.6,
2. Projective measurement: NC 2.2.5
3. Parity measurement: NC 10.5.8

NC $\equiv$ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information
Cambridge University Press (2010)