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informatics

Introduction to Quantum Computing

Lecture 18: Quantum Fourier Transform

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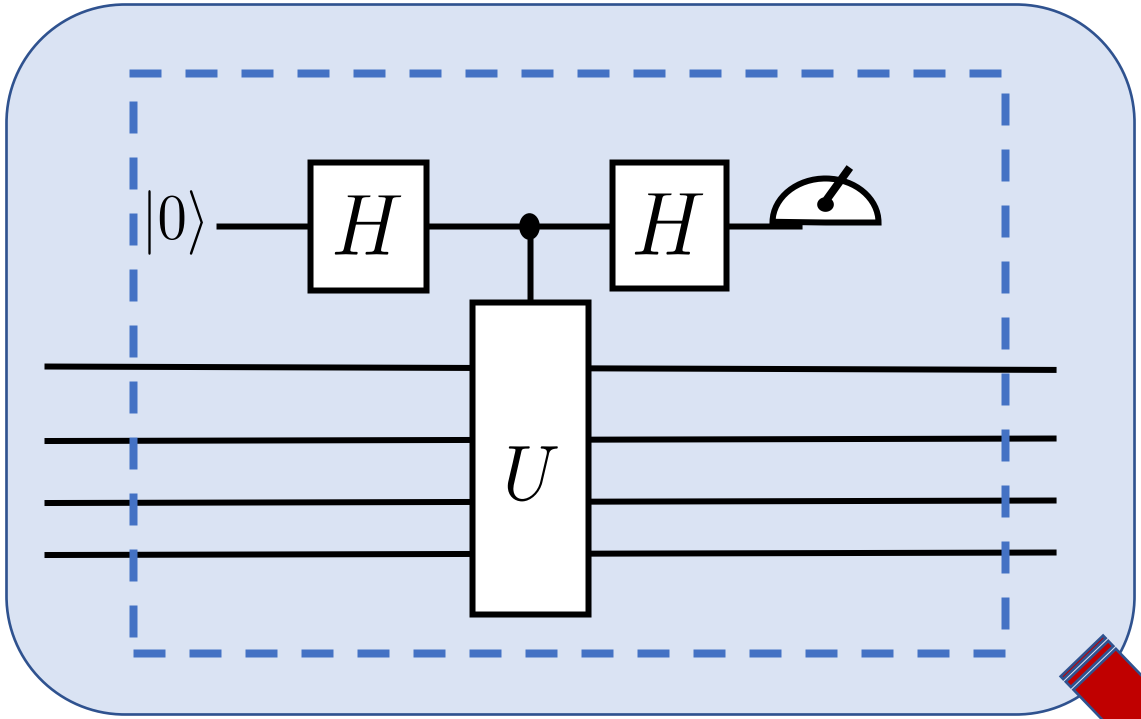
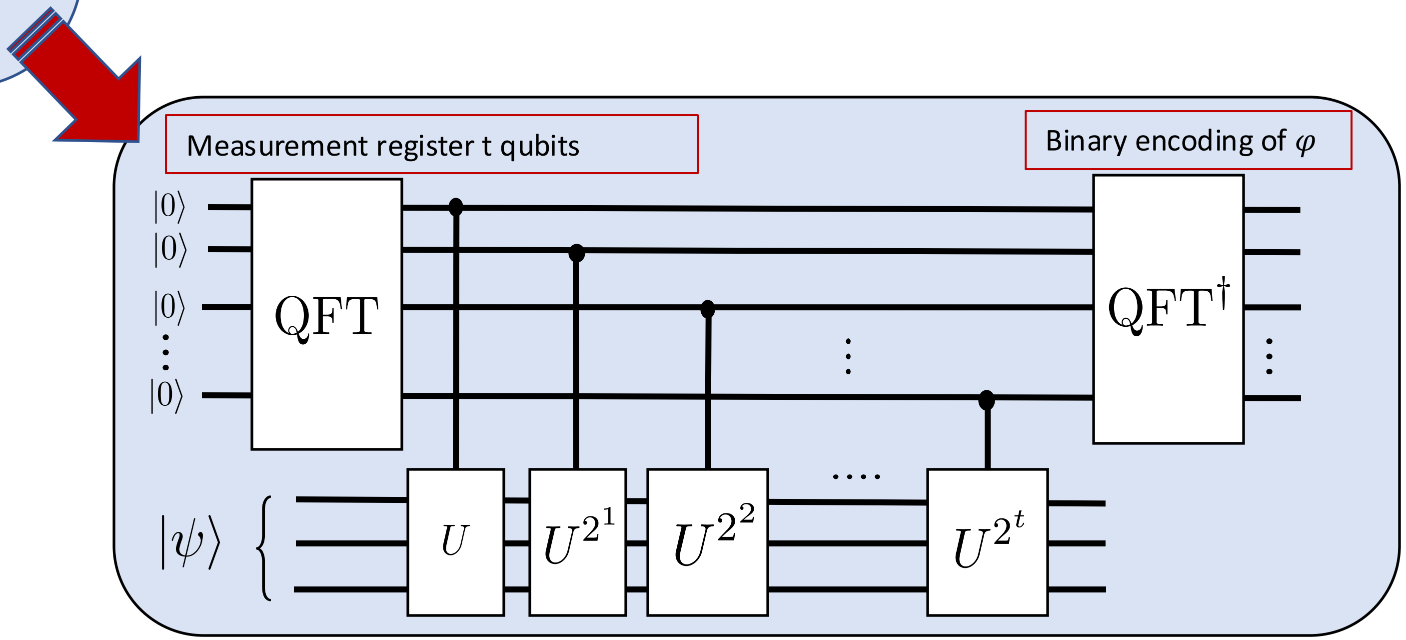
Generalization to N outcomes measurement





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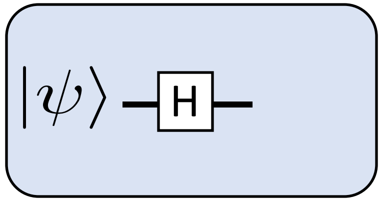
Quantum Fourier Transform

From Hadamard to QFT

Hadamard Gate

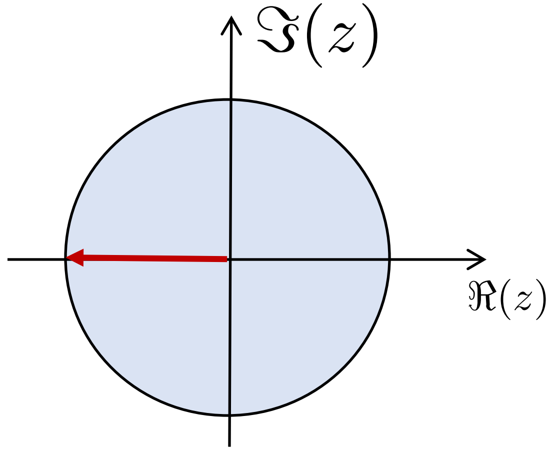
$$\boxed{-1 = e^{i\pi}}$$

$$\boxed{(-1)^2 = 1}$$



$$\boxed{|x\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} (-1)^{xy}|y\rangle}$$

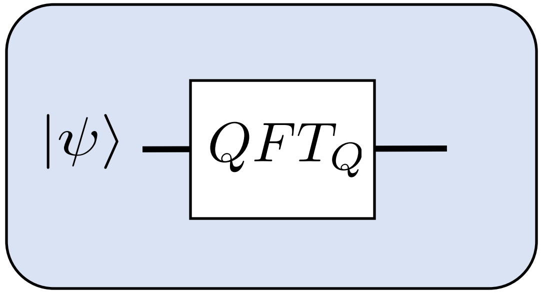
$$(-1)^x(-1)^y = (-1)^{x\oplus y}$$



Fourier Transform over Z_Q

$$\boxed{\omega = e^{i2\pi/Q}}$$

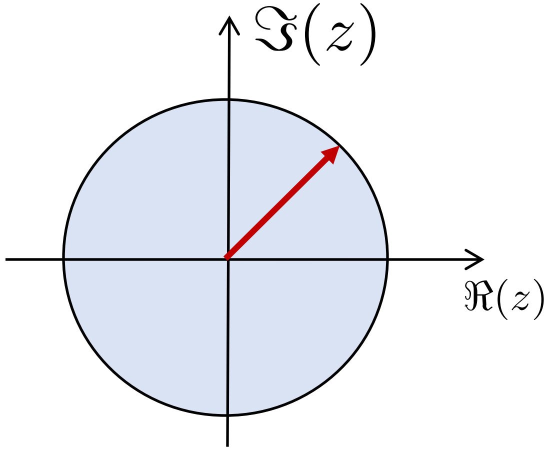
$$\boxed{\omega^Q = 1}$$



$$\boxed{|x\rangle \xrightarrow{QFT_Q} \frac{1}{\sqrt{Q}} \sum_{y\in\mathbb{Z}_Q} \omega^{xy}|y\rangle}$$

$$\text{Basis of size } N : |0\rangle, |1\rangle, \ldots, |Q-1\rangle$$

$$\omega^x\omega^y = e^{ix2\pi/Q}e^{iy2\pi/Q} = \omega^{(x+y \text{ mod} Q)}$$



Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

Binary representation of integers:

$$z \equiv z_1 z_2 .... z_n$$

$$z = z_1 2^{n-1} + z_2 2^{n-2} + ... + z_{n-1} 2 + z_n$$

Binary fraction:

$$0.w_l w_{l+1} .... w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + ... + \frac{w_m}{2^{m-l+1}}$$

Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x\rangle \xrightarrow{QFT_{2^n}} \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

$$H|x_1\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^{x_1}|1\rangle]$$

$$H|x_1\rangle = \frac{1}{2^{1/2}}[|0\rangle + e^{2\pi i 0.x_1}|1\rangle]$$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + ... + z_{n-1}2 + z_n$

Binary fraction: $0.w_l w_{l+1} .... w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + ... + \frac{w_m}{2^{m-l+1}}$

We will now prove

$$|x_1, x_2, ..., x_{n-1}, x_n\rangle \rightarrow \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle)...(|0\rangle + e^{2\pi i 0.x_1 x_2 ... x_n}|1\rangle)$$

Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x\rangle \to \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + \ldots + z_{n-1} 2 + z_n$

Binary fraction: $0.w_l w_{l+1} \ldots . w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \ldots + \frac{w_m}{2^{m-l+1}}$

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} \ldots \sum_{y_n=0}^{1} e^{2\pi i x (\sum_{l=1}^{n} y_l 2^{-l})} |y_1 \ldots y_n\rangle$$

re-writing y in binary notation

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} \ldots \sum_{y_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i x y_l 2^{-l}} |y_l\rangle$$

exp. of sum = product of exp.

Property: $c(|u\rangle \otimes |v\rangle) = (c|u\rangle) \otimes |v\rangle = |u\rangle \otimes (c|v\rangle)$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{y_l=0}^{1} e^{2\pi i x y_l 2^{-l}} |y_l\rangle \right]$$

Property: $|u\rangle \otimes (|v_1\rangle + |v_2\rangle) = |u\rangle \otimes |v_1\rangle + |u\rangle \otimes |v_2\rangle$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi i x 2^{-l}} |1\rangle \right]$$

Expanding each sum

Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x\rangle \to \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + \dots + z_{n-1} 2 + z_n$

Binary fraction: $0.w_l w_{l+1} \dots . w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \dots + \frac{w_m}{2^{m-l+1}}$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{y_l=0}^{1} e^{2\pi i x y_l 2^{-l}} |y_l\rangle \right]$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi i x 2^{-l}} |1\rangle \right] = \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi i 0.x_{n-l+1} \dots x_n} |1\rangle \right]$$

$$z = z_1 2^{n-1} + z_2 2^{n-2} + \dots + z_{n-1} 2 + z_n \qquad e^{2\pi i z} = e^{2\pi i (z_1 2^{n-1} + z_2 2^{n-2} + \dots + z_{n-1} 2 + z_n)} = 1 (z \in \mathbb{Z})$$

$$\frac{z}{2} = z_1 2^{n-2} + z_2 2^{n-3} + \dots + z_{n-2} 2 + z_{n-1} + \frac{z_n}{2} \Rightarrow e^{2\pi i \frac{z}{2}} = e^{2\pi i (z_1 2^{n-2} + z_2 2^{n-3} + \dots + z_{n-2} 2 + z_{n-1})} e^{2\pi i \frac{z_n}{2}} = e^{2\pi i 0.z_n}$$

$$\frac{z}{4} = z_1 2^{n-3} + z_2 2^{n-4} + \dots + z_{n-2} + \frac{z_{n-1}}{2} + \frac{z_n}{4} \Rightarrow e^{2\pi i \frac{z}{4}} = e^{2\pi i (z_1 2^{n-3} + z_2 2^{n-4} + \dots + z_{n-2})} e^{2\pi i (\frac{z_{n-1}}{2} + \frac{z_n}{4})} = e^{2\pi i 0.z_{n-1} z_n}$$

$$\vdots$$

$$\frac{z}{2^l} = z_1 2^{n-1-l} + \dots . + z_{n-l} + \frac{z_{n-l+1}}{2} + \dots + \frac{z_n}{2^l}$$

$$e^{2\pi i z 2^{-l}} = e^{2\pi i (z_1 2^{n-l-1} + z_2 2^{n-l-2} + \dots + z_{n-l})} e^{2\pi i 0.z_{n-l+1} z_{n-l+2} \dots z_n} = e^{2\pi i 0.z_{n-l+1} z_{n-l+2} \dots x_n}$$

Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + ... + z_{n-1}2 + z_n$

Binary fraction: $0.w_l w_{l+1} .... w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + ... + \frac{w_m}{2^{m-l+1}}$

$$|x\rangle \to \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y / 2^n} |y\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} ... \sum_{y_n=0}^{1} e^{2\pi i x (\sum_{l=1}^{n} y_l 2^{-l})} |y_1 ... y_n\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} ... \sum_{y_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i x y_l 2^{-l}} |y_l\rangle$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{y_l=0}^{1} e^{2\pi i x y_l 2^{-l}} |y_l\rangle \right]$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi i x 2^{-l}} |1\rangle \right] = \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi i 0.x_{n-l+1} ... x_n} |1\rangle \right]$$

$$|x_1, x_2, ..., x_{n-1}, x_n\rangle \to \frac{1}{2^{n/2}} (|0\rangle + e^{2\pi i 0.x_n} |1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1} x_n} |1\rangle) ... (|0\rangle + e^{2\pi i 0.x_1 x_2 ... x_n} |1\rangle)$$

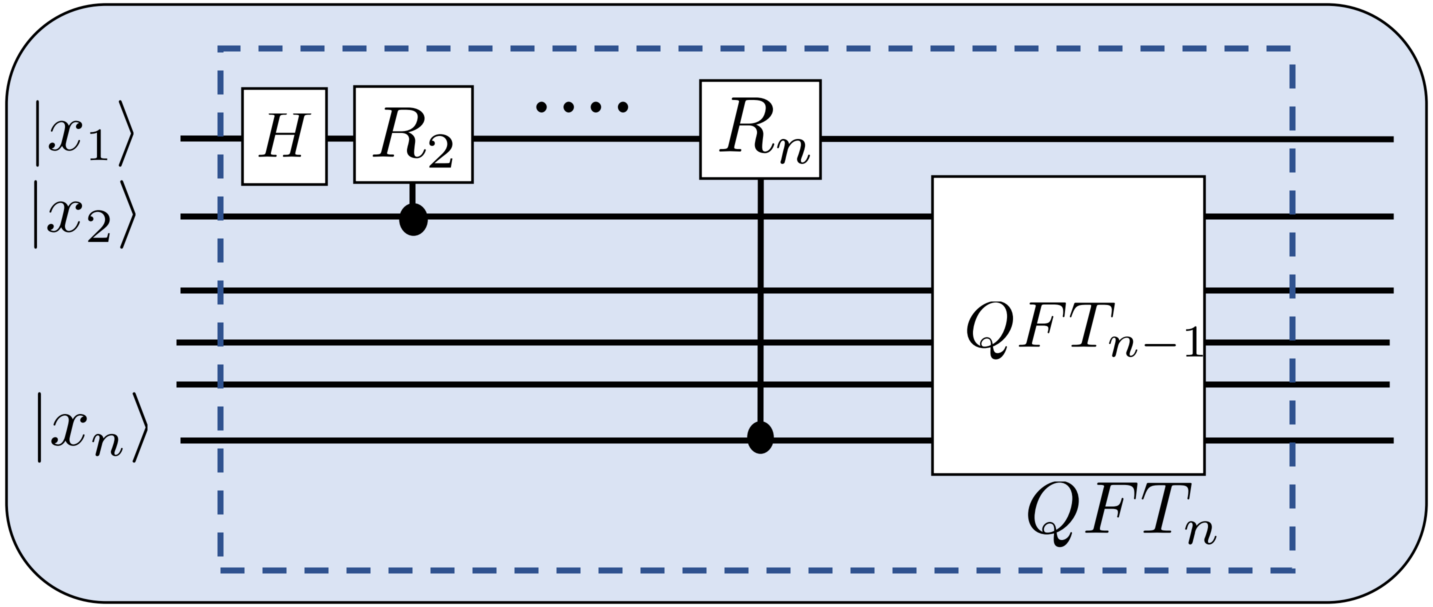
Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x_1, x_2, ..., x_{n-1}, x_n\rangle \rightarrow \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle)...(|0\rangle + e^{2\pi i 0.x_1 x_2 ... x_n}|1\rangle)$$

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{bmatrix}$$

Binary fraction:

$$0.w_l w_{l+1} .... w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + ... + \frac{w_m}{2^{m-l+1}}$$



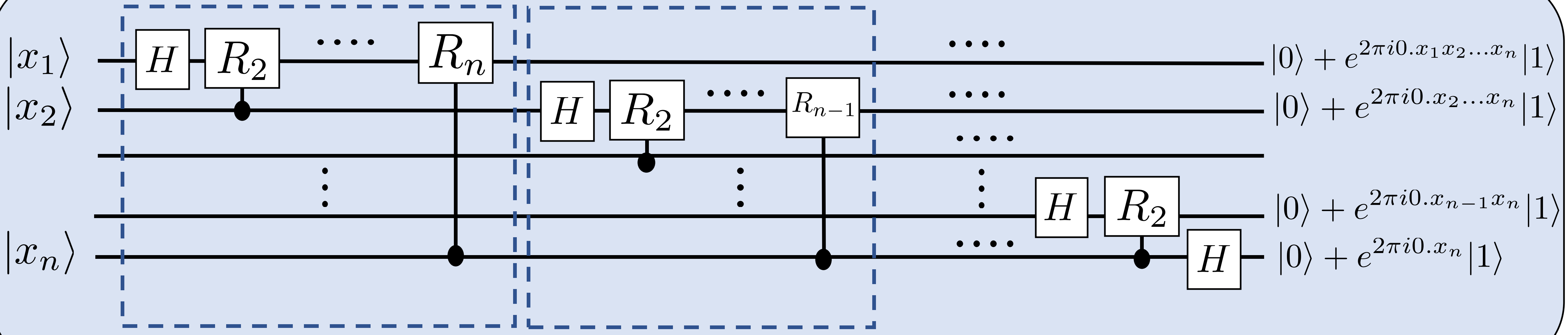
$$|x_1\rangle \otimes |x_2 ... x_n\rangle \xrightarrow{H} \frac{1}{2^{1/2}} \left[|0\rangle + e^{2\pi i 0.x_1}|1\rangle\right] \otimes |x_2 ... x_n\rangle$$

$$\xrightarrow{cR_2} \frac{1}{2^{1/2}} \left[|0\rangle + e^{2\pi i 0.x_1 x_2}|1\rangle\right] \otimes |x_2 ... x_n\rangle$$

$$\xrightarrow{cR_n} \frac{1}{2^{1/2}} \left[|0\rangle + e^{2\pi i 0.x_1 x_2 .. x_n}|1\rangle\right] \otimes |x_2 ... x_n\rangle$$

Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x_1, x_2, ..., x_{n-1}, x_n\rangle \to \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)...(|0\rangle + e^{2\pi i 0.x_2 \dots x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_1 x_2 \dots x_n}|1\rangle)$$



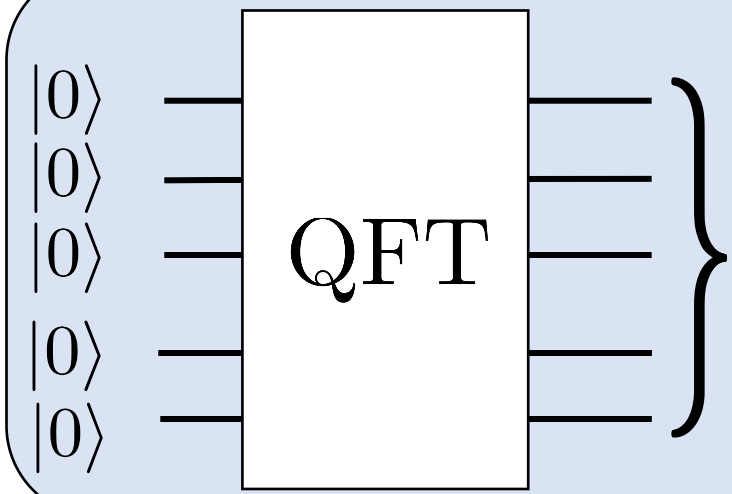
$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{bmatrix}$$

- QFT up to a reverse of the order of qubits.
- Unitary as composed of unitary gates.
- $n + (n-1) + ... + 1 = n(n+1)/2$ gates are required
  $+n/2$ SWAP gates (3 CNOT)

$\Theta(n^2)$ gates

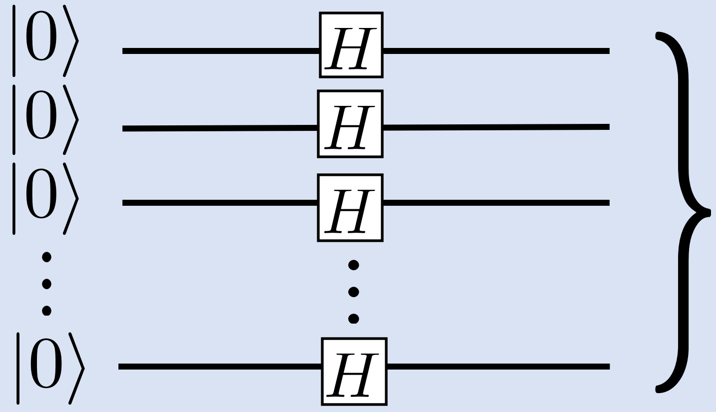
Parallelization

$$|x_1, x_2, \ldots, x_{n-1}, x_n\rangle \rightarrow \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)\ldots(|0\rangle + e^{2\pi i 0.x_2\ldots x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_1 x_2 \ldots x_n}|1\rangle)$$



$$\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$|x\rangle \xrightarrow{QFT_{2^n}} \frac{1}{2^{n/2}} \sum_{y=0}^{2^n - 1} e^{2\pi i x y / 2^n} |y\rangle$$



$$\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$$

Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

- Quantum circuit: $\Theta(n^2)$ gates.
- Classical FFT: $\Theta(n2^n)$ operations on a 2^n vector.

> the first step in phoneme recognition is to Fourier transform the digitized sound. Can we use the quantum Fourier transform to speed up the computation of these Fourier transforms? Unfortunately, the answer is that there is no known way to do this. The problem is that the amplitudes in a quantum computer cannot be directly accessed by measurement. Thus, there is no way of determining the Fourier transformed amplitudes of the original state. Worse still, there is in general no way to efficiently prepare the original state to be Fourier transformed. Thus, finding uses for the quantum Fourier transform is more subtle than we might have hoped. In this and the next chapter we develop several algorithms based upon a more subtle application of the quantum Fourier transform.

Extract from NC page 220

- A quantum computer does not output the wave-function amplitudes!
  - A quantum computer generates outputs (samples) from a probability distribution $|\psi(x)|^2$
- Quantum mechanics allows you to FFT an exponentially big vector but….

  … encoding an exponentially big classical dataset in a quantum state is highly non-trivial
- Quantum Machine Learning to be practical needs to deal with a similar issue.

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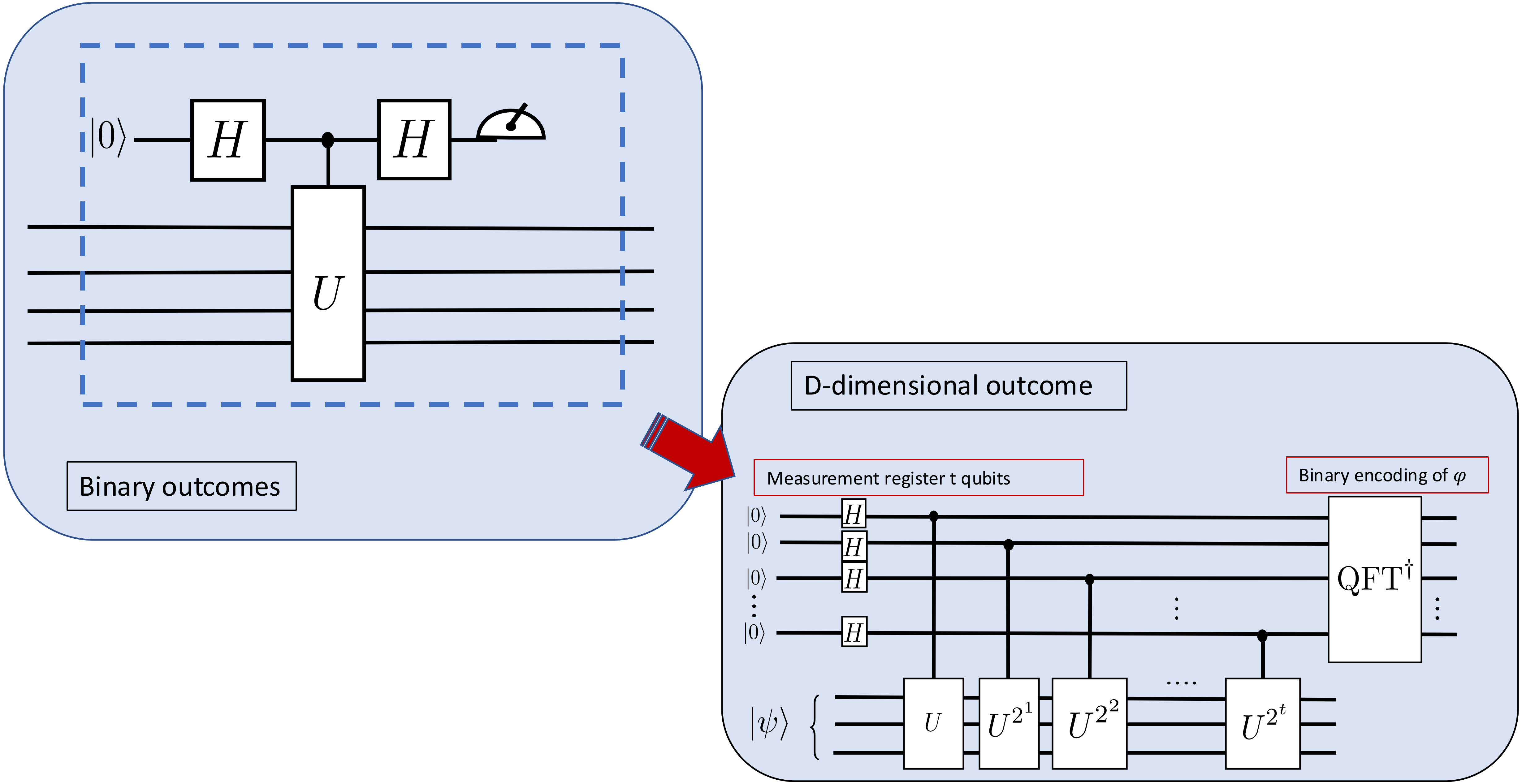
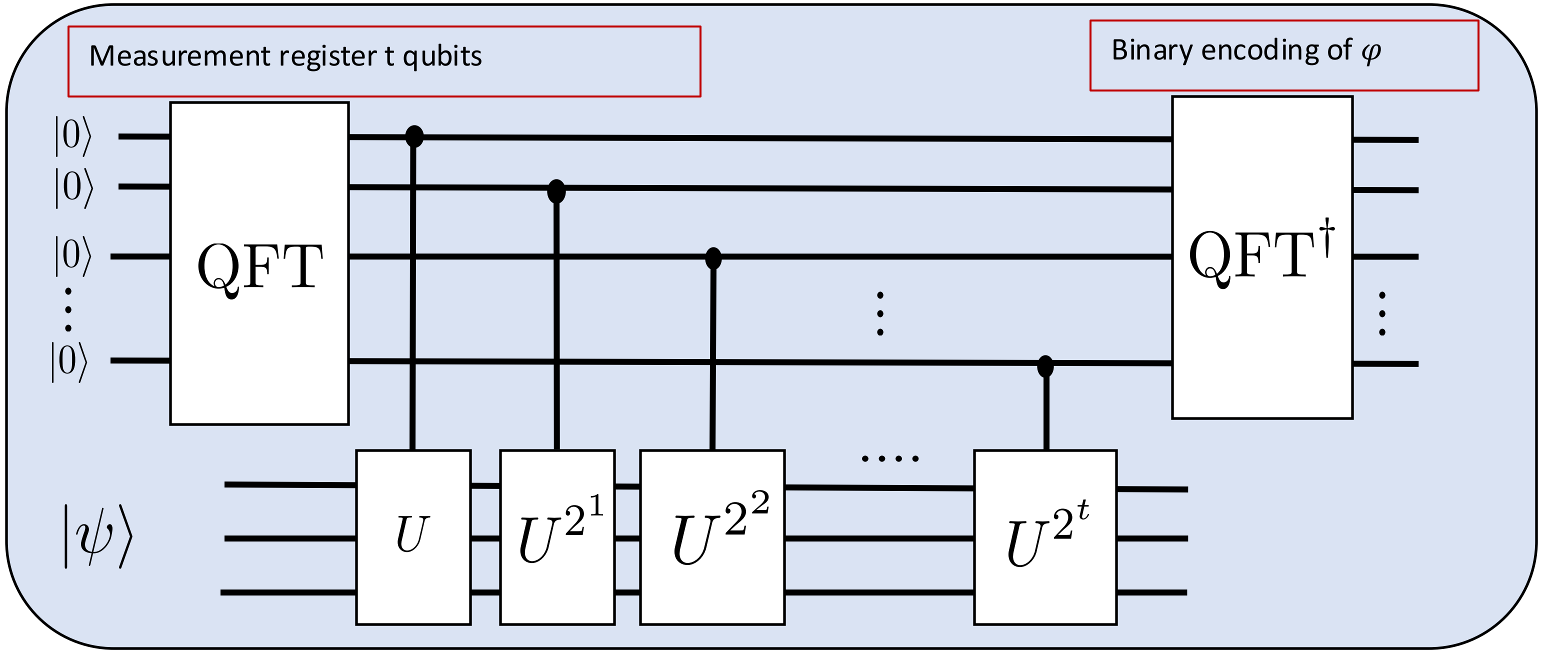
Quantum phase estimation

Generalization to N outcomes measurement

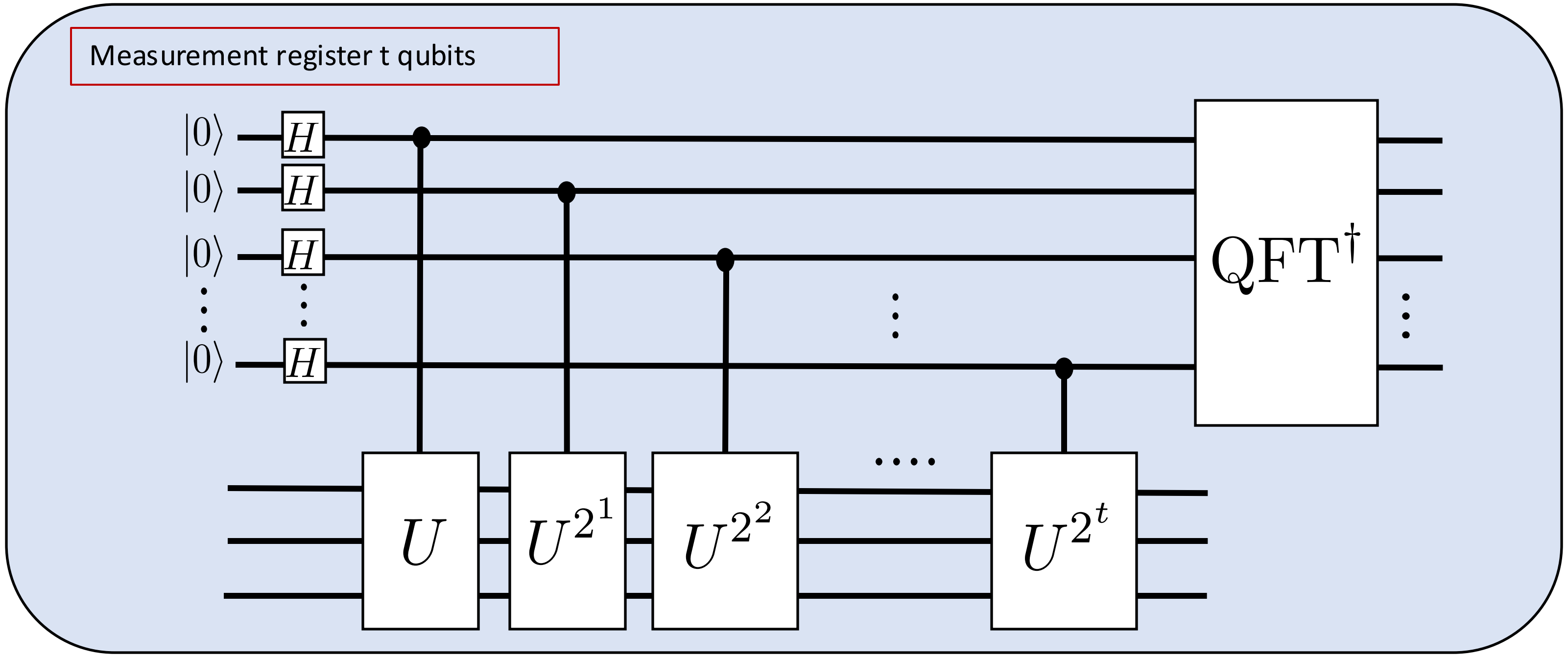


Quantum phase estimation: applications



- Factoring: Shor's algorithm
- Quantum counting = Grover + Quantum phase estimation
- Quantum chemistry and many-body physics: energies estimation
- Quantum Metropolis Sampling

Quantum phase estimation: applications



- Factoring: Shor's algorithm
- Quantum counting = Grover + Quantum phase estimation
- Quantum chemistry and many-body physics: energies estimation
- Quantum Metropolis Sampling

The eigenspace and eigenvalues of unitary matrices

$\boxed{\text{Spectral theorem}}$ Unitary $U : UU^\dagger = U^\dagger U = I$

$$\exists V : VUV^\dagger = D = \mathrm{diag}(e^{2\pi i\varphi_1}, ..., e^{2\pi i\varphi_N})$$

- N eigenvalues in the unit circle: $\lambda_i = e^{2\pi i\varphi_i}$
- $U = \sum_{i=1}^{N} e^{2\pi i\varphi_i} |u_i\rangle\langle u_i|,$ and $V = \sum_{i=1}^{N} |i\rangle\langle u_i|$

$$\boxed{X} \equiv -\boxed{H}-\boxed{Z}-\boxed{H}-$$

The eigenspace and eigenvalues of unitary matrices

$\boxed{\text{Spectral theorem}}$ Unitary $U : UU^\dagger = U^\dagger U = I$

$$\exists V : VUV^\dagger = D = \mathrm{diag}(e^{2\pi i\varphi_1}, \ldots, e^{2\pi i\varphi_N})$$

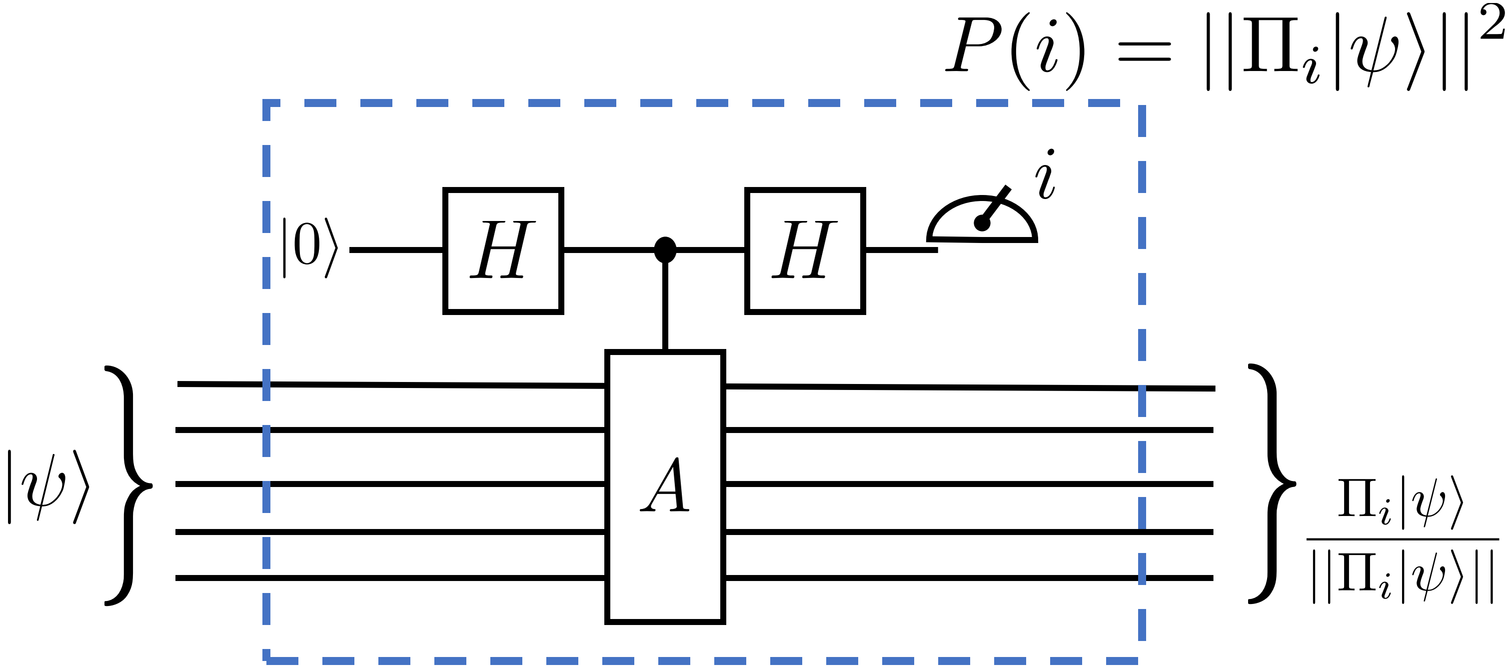
- N eigenvalues in the unit circle: $\lambda_i = e^{2\pi i\varphi_i}$
- $U = \sum_{i=1}^{N} e^{2\pi i\varphi_i} |u_i\rangle\langle u_i|$, and $V = \sum_{i=1}^{N} |i\rangle\langle u_i|$

Degeneracy

- $U = \sum_{i=1}^{L} e^{2\pi i\varphi_i} P_i$
- $\sum_{i=1}^{L} P_i = I$
- $r_i = \mathrm{rank}(P_i) : \sum_{i=1}^{L} r_i = N$

General ± 1 eigenvalues measurement

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$
$$\Pi_0 + \Pi_1 = I$$
$$A = \Pi_0 - \Pi_1$$



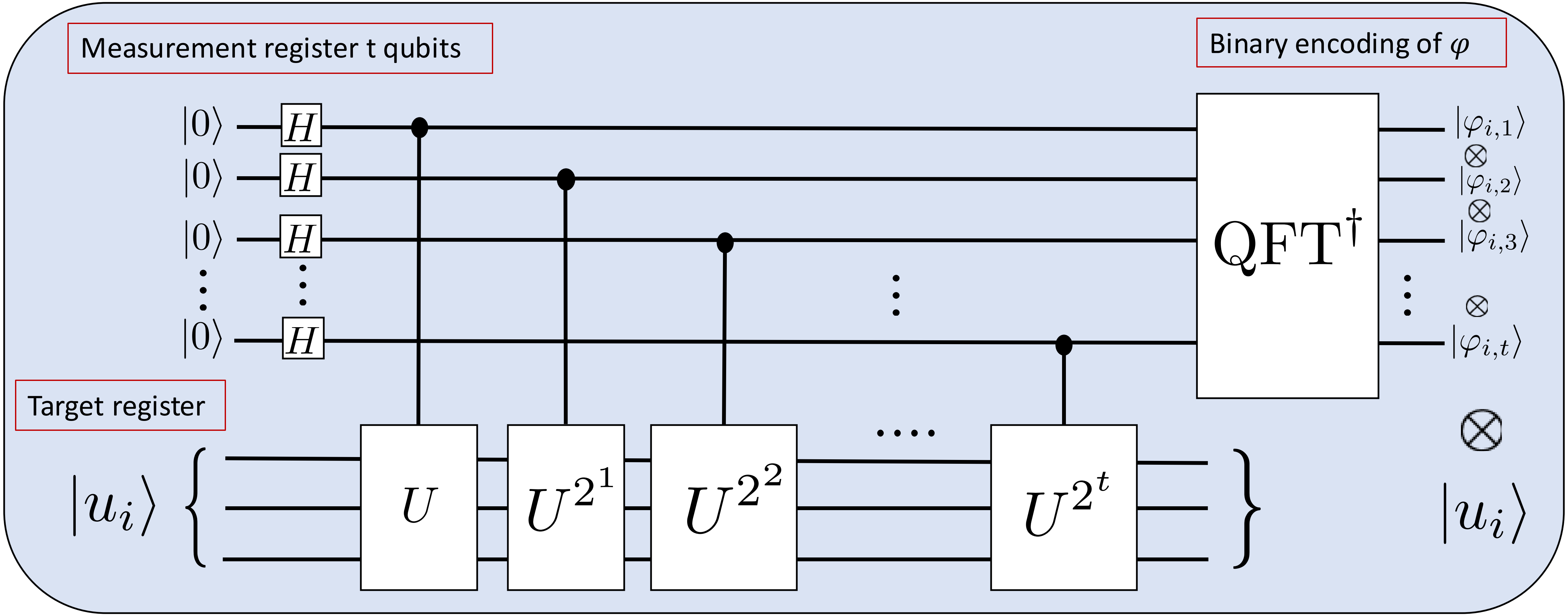
$$|\tilde{\phi}\rangle = |0\rangle \otimes \Pi_0|\psi\rangle + |1\rangle \otimes \Pi_1|\psi\rangle$$

$$(-1)^z = e^{2\pi i 0.z} \text{ where } z \in \{0, 1\}$$

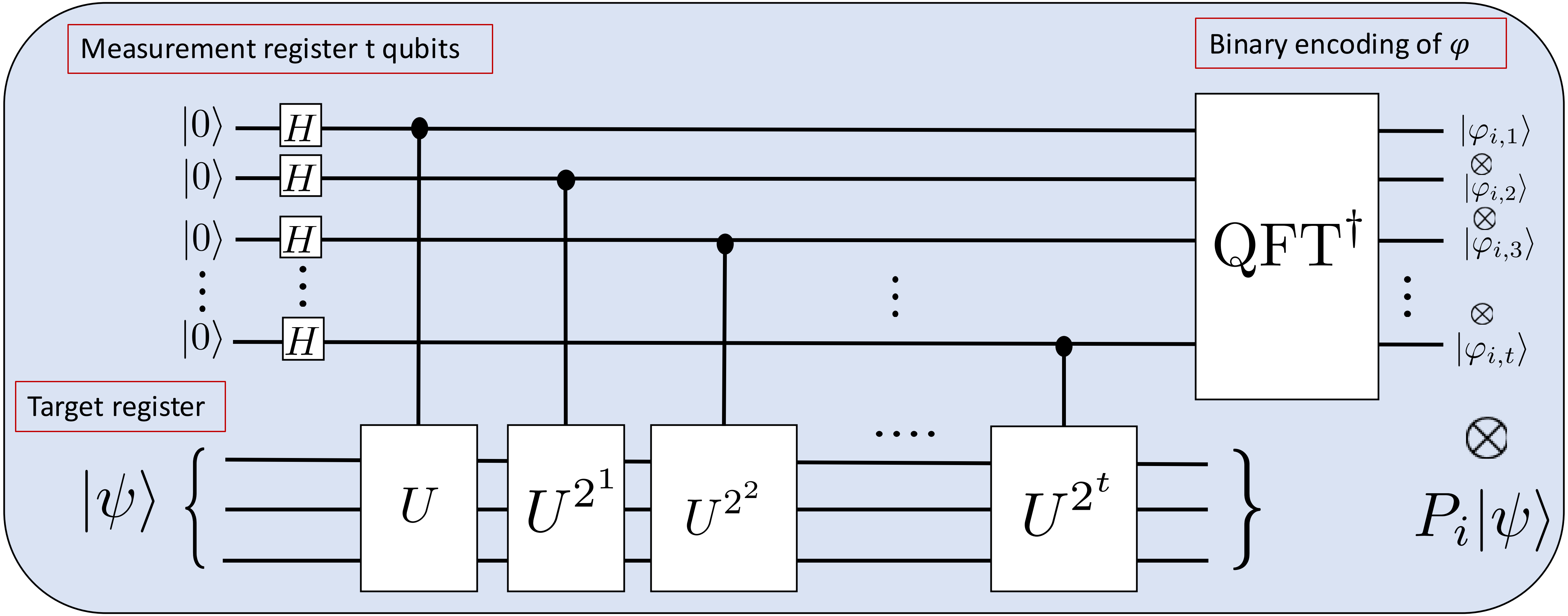
Quantum phase estimation

Projective measurement on the basis $|u_i\rangle$ with outcome $\varphi_i \in [0,1]$

Assumptions:
- we can perform the controled-U^{2^x} operation
- All φ_i have at most t digits in binary encoding



Quantum phase estimation



$$|0^n\rangle \otimes |\psi\rangle = |0^n\rangle \otimes (\sum_i P_i|\psi\rangle) = \sum_i |0^n\rangle \otimes P_i|\psi\rangle \to \sum_i |\varphi_i\rangle \otimes P_i|\psi\rangle$$

- P_i is projector on subspace of eigenvalue $e^{i\varphi_i}$
- With probability $||P_i|\psi\rangle||^2$ the measurement outputs $|\varphi_i\rangle$

References

Reading references

1. Quantum Fourier Transform: NC 5.1,
2. Additional references RdW Ch4, G Ch9

NC ≣ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information Cambridge University Press (2010)
RdW ≣ Quantum Computing Lecture Notes, Ronald de Wolf, https://arxiv.org/abs/1907.09415
G ≣ Introduction to Quantum Computation, Sevag Gharibian, Lectures notes link