



THE UNIVERSITY of EDINBURGH
informatics

Lecture 7: Phase-kickback and Deutsch-Jozsa Algorithm

Raul Garcia-Patron Sanchez



Balanced or equal function Game

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

Constant: $f(x) = c \ \forall x$

Balanced: $f(x) = 1$ on half of the bits

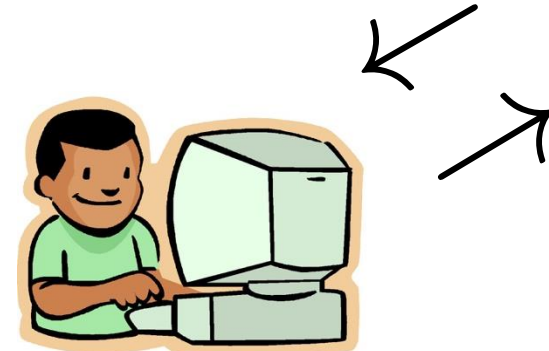
x	000	001	010	011	100	101	110	111
f(x)	0	0	0	0	0	0	0	0

x	000	001	010	011	100	101	110	111
f(x)	1	0	0	0	1	0	1	1

- How many queries to the function we need to answer with full certainty?
- Classical oracle $2^n/2$ queries.

- Quantum:

1 single query!



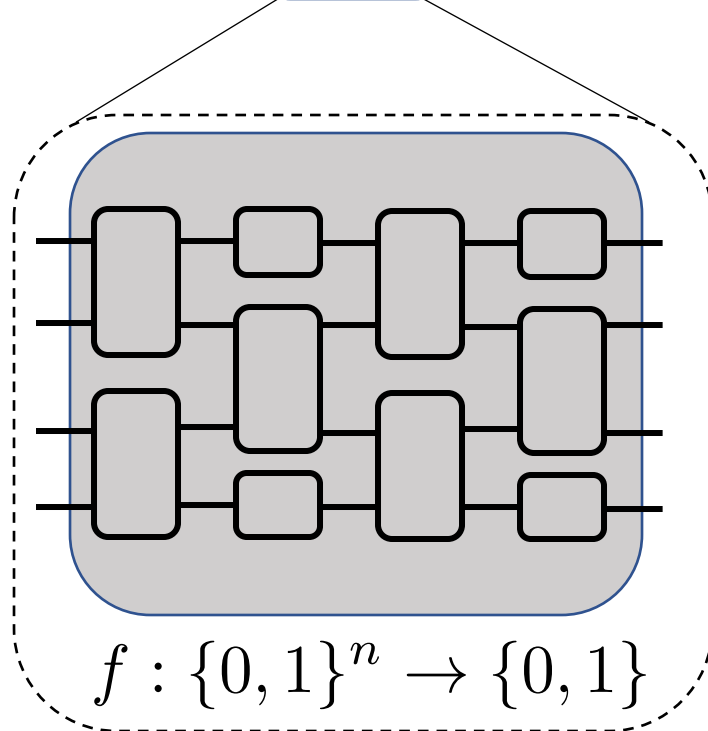
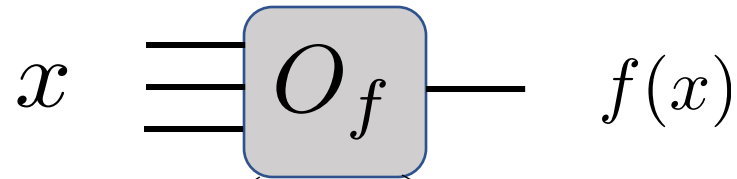
O_f

Oracles

“An **oracle** is a person or agency considered to provide wise and insightful counsel or **prophetic predictions** or precognition of the future, inspired by the gods. As such it is **a form of divination**.” [Wikipedia](#)



Classical oracle



An oracle is a sort of subroutine

In the oracle model we do not care about the complexity of the circuit, only about the existence of a circuit.

In the oracle model the resource is the number of calls to the oracle: *queries*.

Deutsch-Jozsa Algorithm

Parallelization

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

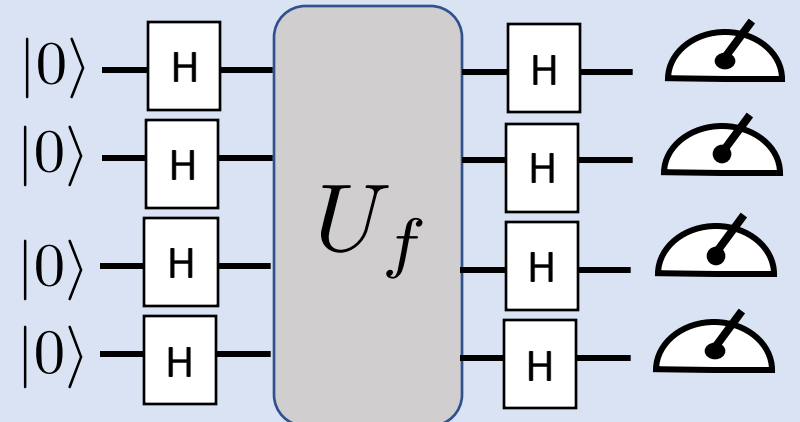
Phase Kickback

$$U_f : |x\rangle \rightarrow (-1)^{f(x)} |x\rangle$$

U_f

1. Unitary
2. Its own inverse
3. Diagonal in computational basis

Interference
with
Walsh–Hadamard transform

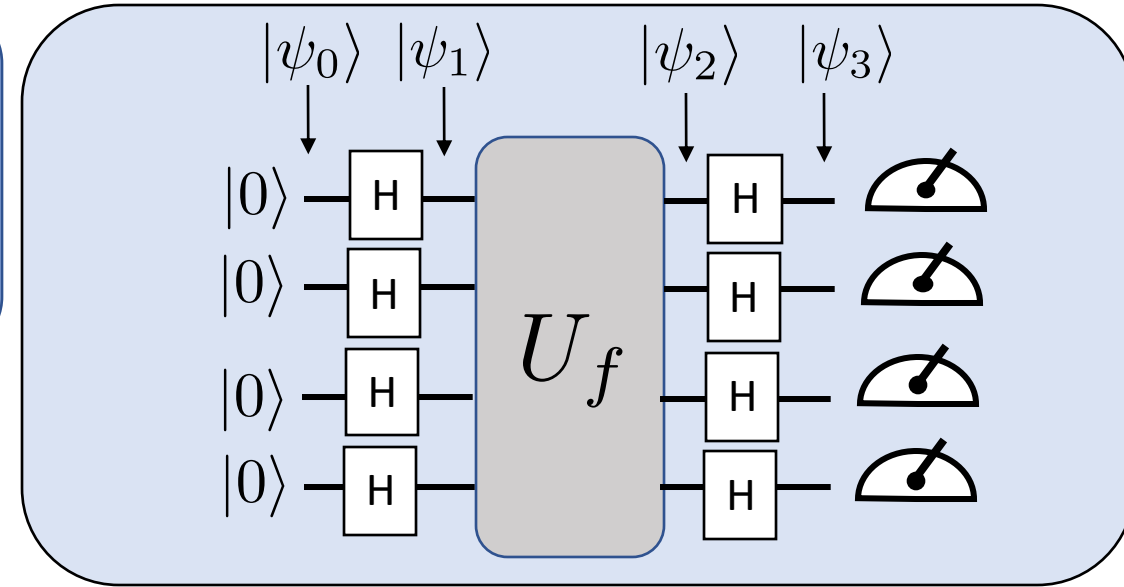


Parallelization

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

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$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} (H|0\rangle)^{\otimes n} = |+\rangle^{\otimes n} = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes n}$$

$$|\psi_1\rangle$$

Recap

$$|0\rangle \longrightarrow \boxed{H} \longrightarrow |+\rangle$$

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$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

$$|u\rangle \otimes (|v_1\rangle + |v_2\rangle) = |u\rangle \otimes |v_1\rangle + |u\rangle \otimes |v_2\rangle$$

$$(H \otimes H)(|0\rangle \otimes |0\rangle) = (H|0\rangle) \otimes (H|0\rangle) = |+\rangle \otimes |+\rangle$$

$$= \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

$$= \frac{1}{2}|0\rangle \otimes (|0\rangle + |1\rangle) + \frac{1}{2}|1\rangle \otimes (|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

Recap

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$$(H \otimes H)(|0\rangle \otimes |0\rangle) = (H|0\rangle) \otimes (H|0\rangle) = |+\rangle \otimes |+\rangle$$

$$= \frac{1}{2} (|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

$$= \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Recap

$$|0\rangle \text{ --- } \boxed{H} \text{ --- } |+\rangle$$

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$$|0\rangle \text{ --- } \boxed{H} \text{ --- } |+\rangle$$

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$$|u\rangle \otimes (|v_1\rangle + |v_2\rangle) = |u\rangle \otimes |v_1\rangle + |u\rangle \otimes |v_2\rangle$$

$$H^{\otimes 3}|0\rangle \otimes |0\rangle \otimes |0\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

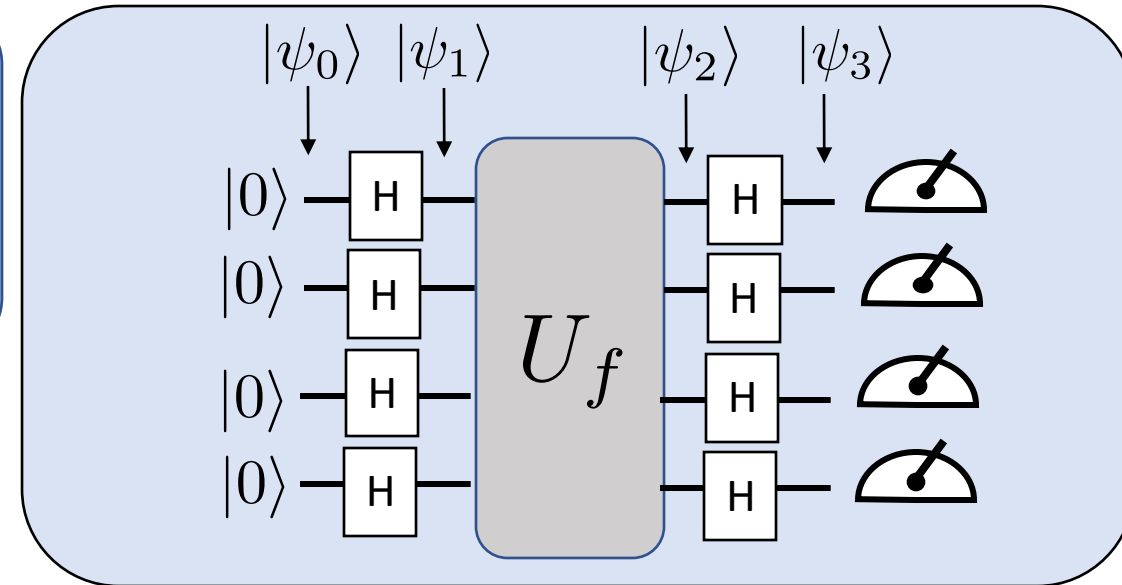
$$\begin{aligned} H^{\otimes 3}|0\rangle \otimes |0\rangle \otimes |0\rangle = \frac{1}{\sqrt{2^3}}(&|000\rangle + |001\rangle + |010\rangle + |011\rangle \\ &+ |100\rangle + |101\rangle + |110\rangle + |111\rangle) \end{aligned}$$

Parallelization

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

Constant: $f(x) = c \ \forall x$

Balanced: $f(x) = 1$ on half of the bits



$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} (H|0\rangle)^{\otimes n} = |+\rangle^{\otimes n} = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes n} \quad \text{(where } |\psi_1\rangle \text{ is indicated in a circle)}$$

$$= \frac{1}{\sqrt{2^n}} (|0\dots 00\rangle + |0\dots 01\rangle + \dots + |1\dots 11\rangle)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

Phase Kickback

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

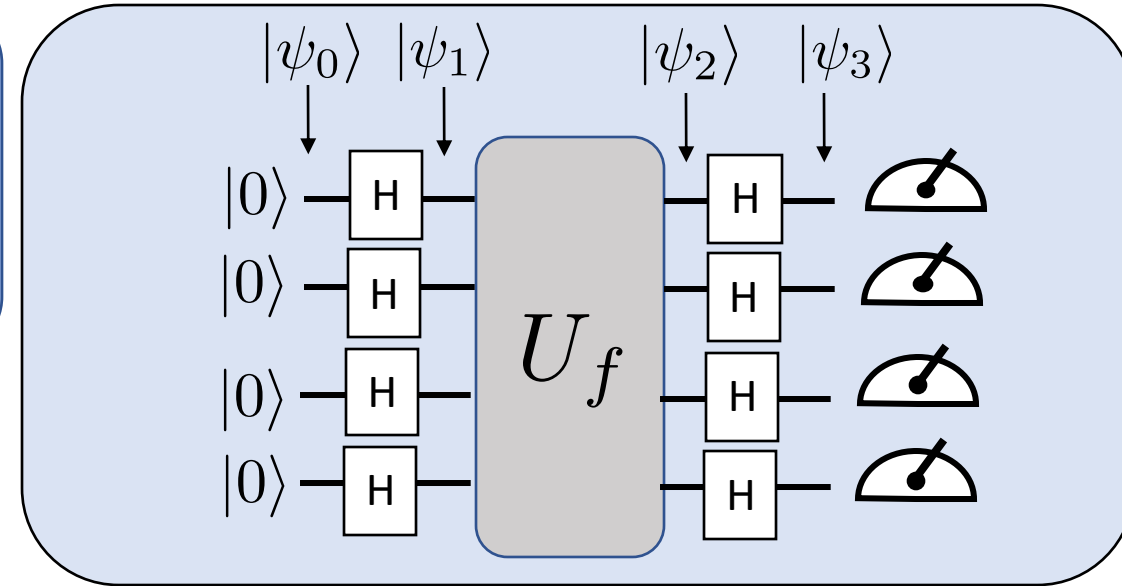
Constant: $f(x) = c \ \forall x$

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$$\begin{aligned} |0\rangle^{\otimes n} &\xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \\ &\xrightarrow{U_f} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \end{aligned}$$

Phase Kickback

$$U_f : |x\rangle \rightarrow (-1)^{f(x)} |x\rangle \quad U_f$$



$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$$

If constant function

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

Constant: $f(x) = c \ \forall x$

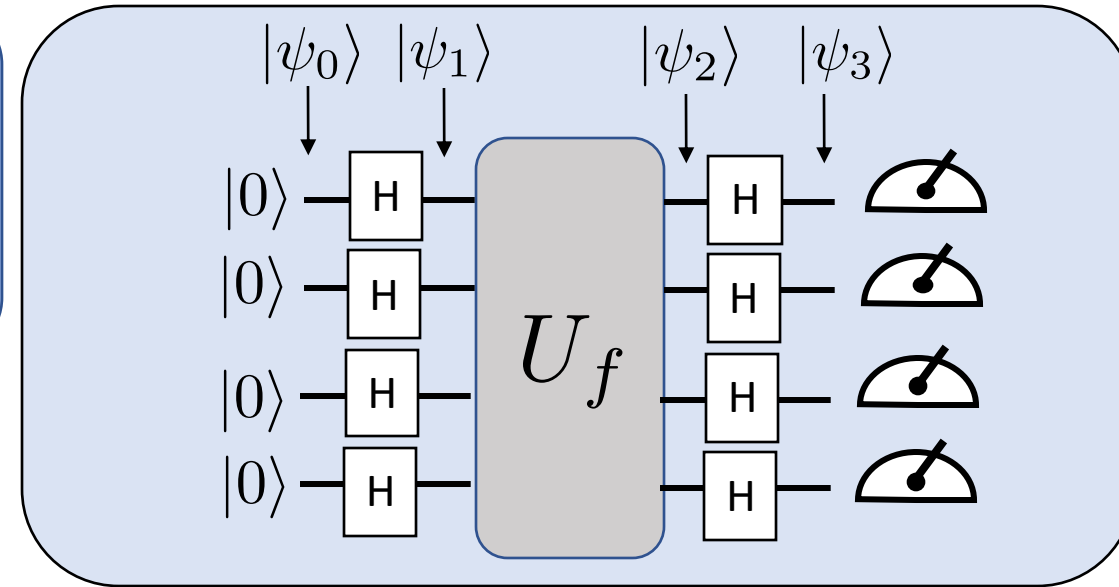
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$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$\xrightarrow{U_f} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$$

$$= (-1)^c \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle, \text{ if } f(x) = c$$

$$\xrightarrow{H^{\otimes n}} (-1)^c (H|+\rangle)^{\otimes n} = (-1)^c |0\rangle^{\otimes n}$$



$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

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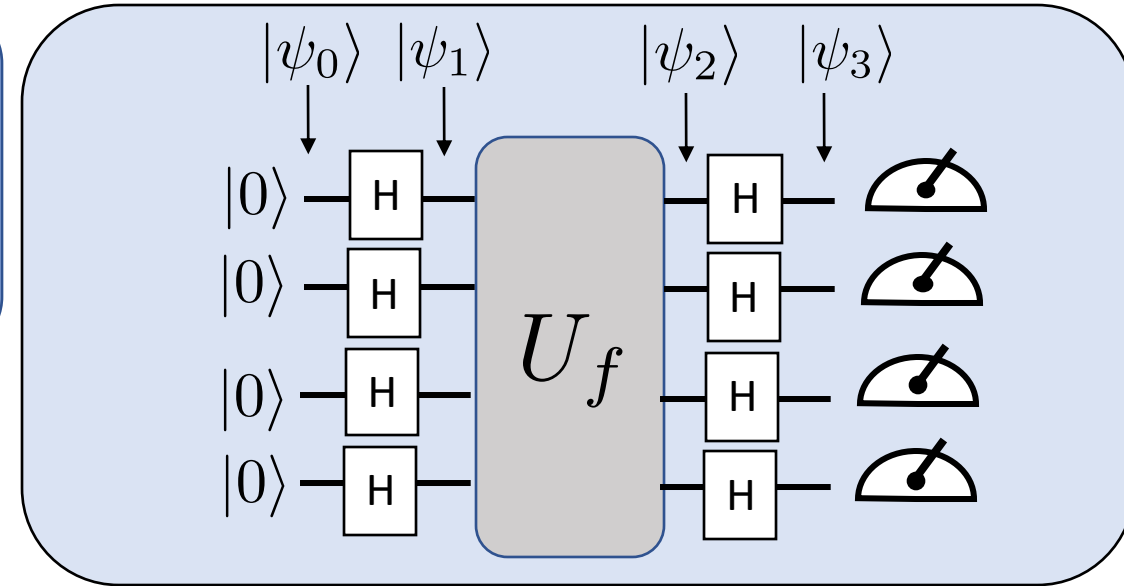
Balanced: $f(x) = 1$ on half of the bits

$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

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$$= (-1)^c \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle, \text{ if } f(x) = c$$

$$\xrightarrow{H^{\otimes n}} (-1)^c |0\rangle^{\otimes n}$$



$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$$

$$P(0) = |\langle 0^n | \psi_3 \rangle|^2 = 1$$

Balanced case

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

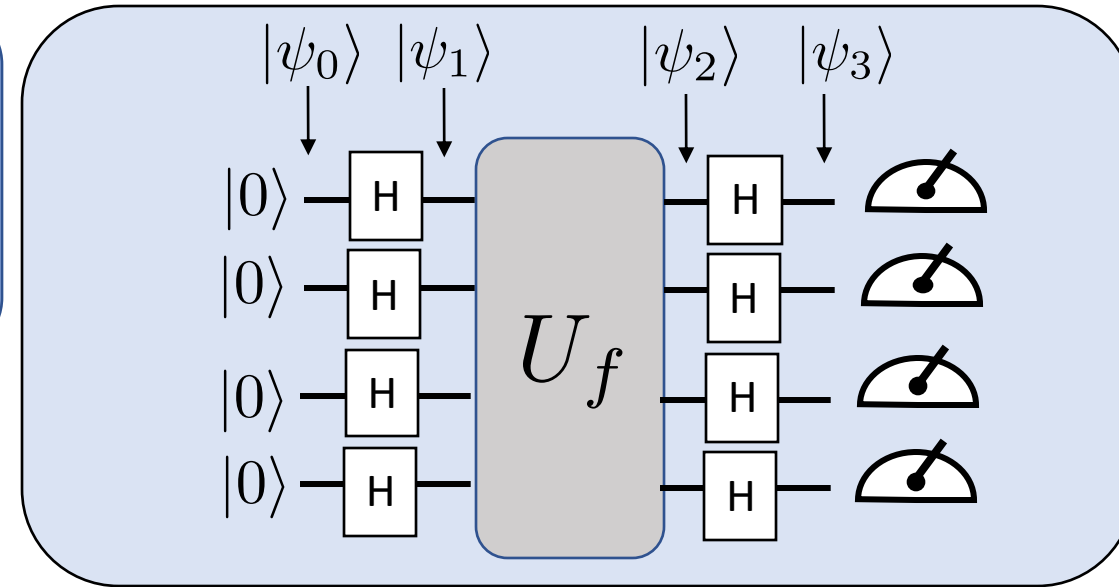
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Phase Kickback

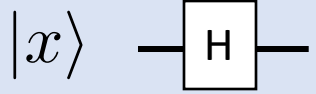
$$U_f : |x\rangle \rightarrow (-1)^{f(x)} |x\rangle \quad U_f$$



$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$$

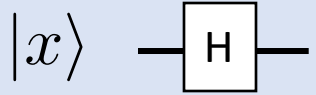
Walsh–Hadamard transform



$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

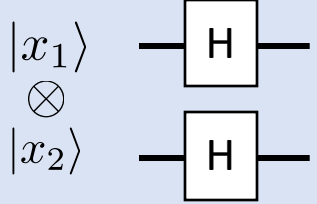
$$H|x_1\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^{x_1}|1\rangle] = \frac{1}{\sqrt{2}} \sum_{y_1 \in \{0,1\}} (-1)^{x_1 y_1} |y_1\rangle$$

Walsh–Hadamard transform



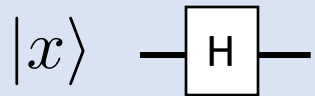
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$$(H \otimes H)|x_1\rangle \otimes |x_2\rangle = \frac{1}{2} \left(\sum_{y_1 \in \{0,1\}} (-1)^{x_1 y_1} |y_1\rangle \right) \otimes \left(\sum_{y_2 \in \{0,1\}} (-1)^{x_2 y_2} |y_2\rangle \right)$$
$$(H \otimes H)|x_1\rangle \otimes |x_2\rangle = \frac{1}{2} \sum_{y_1, y_2 \in \{0,1\}} (-1)^{x_1 y_1 + x_2 y_2} |y_1\rangle \otimes |y_2\rangle$$

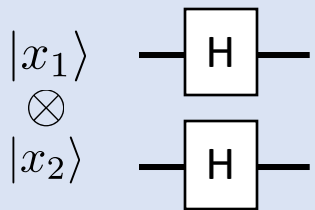
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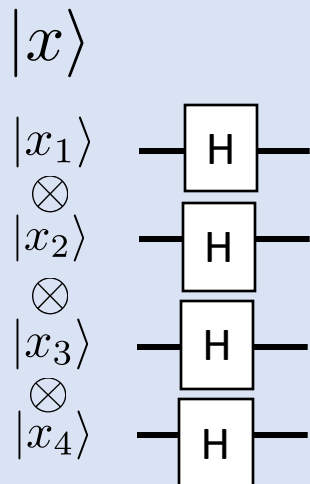
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$$(H \otimes H)|x_1\rangle \otimes |x_2\rangle = \frac{1}{2} \sum_{y_1, y_2 \in \{0,1\}} (-1)^{x_1 y_1 + x_2 y_2} |y_1\rangle \otimes |y_2\rangle$$



$$|x\rangle = |x_1, x_2, \dots, x_n\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \dots \otimes |x_n\rangle$$

$$\text{where } x \cdot y = \sum_{i=0}^n x_i y_i$$

$$|x\rangle \xrightarrow{H^{\otimes n}} \bigotimes_i^n H|x_i\rangle = \frac{1}{\sqrt{2^n}} \bigotimes_i^n \left[\sum_{y_i \in \{0,1\}} (-1)^{x_i y_i} |y_i\rangle \right] = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

Balanced case

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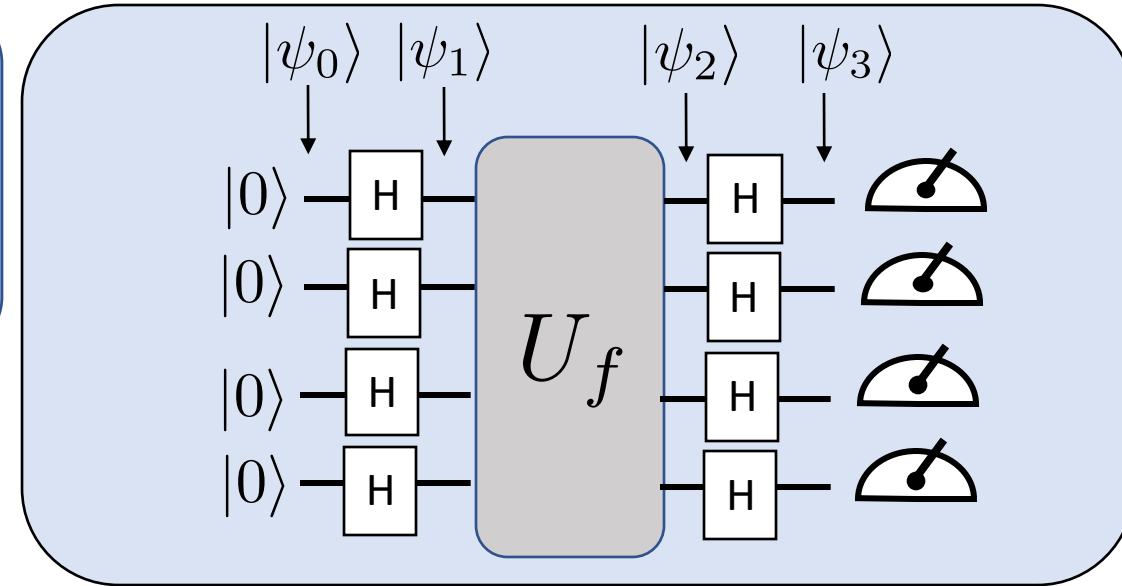
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$$\xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \left(\frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \right)$$

$$= \sum_{y \in \{0,1\}^n} \left(\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + x \cdot y} \right) |y\rangle = \sum_{y \in \{0,1\}^n} \psi_y |y\rangle$$



Walsh-Hadamard transform

$$|x\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

Balanced case

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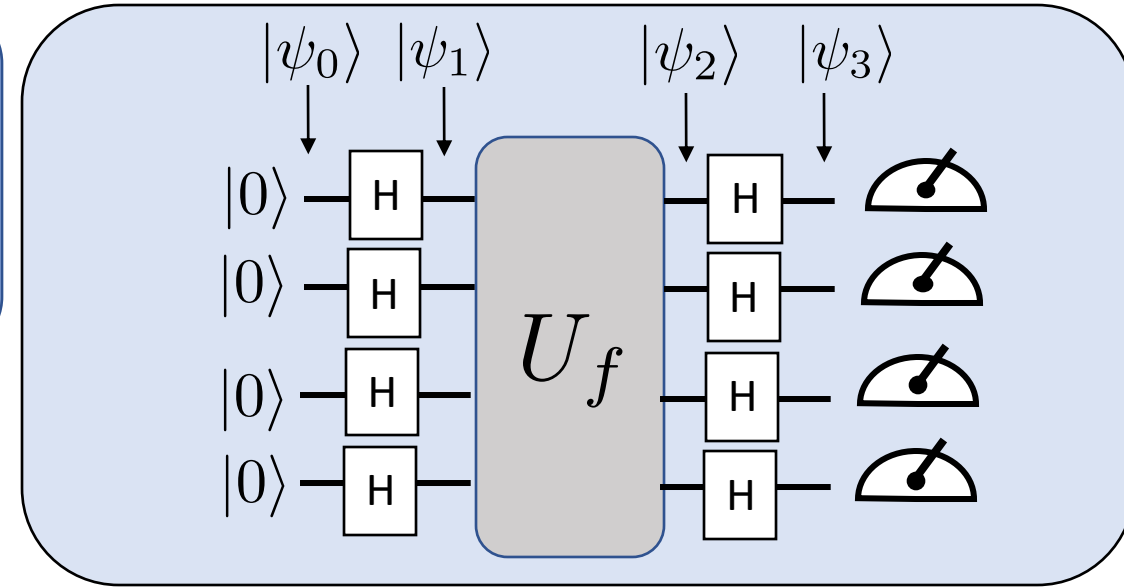
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$$= \sum_{y \in \{0,1\}^n} \left(\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + x \cdot y} \right) |y\rangle = \sum_{y \in \{0,1\}^n} \psi_y |y\rangle$$

$$P(y = 0) = \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \right|^2 = 0, \text{ for balance } f(x)$$



Walsh-Hadamard transform

$$|x\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

Measurement postulate

$$P(0) = |\langle 0^n | \psi_3 \rangle|^2$$

Balanced or equal function Game

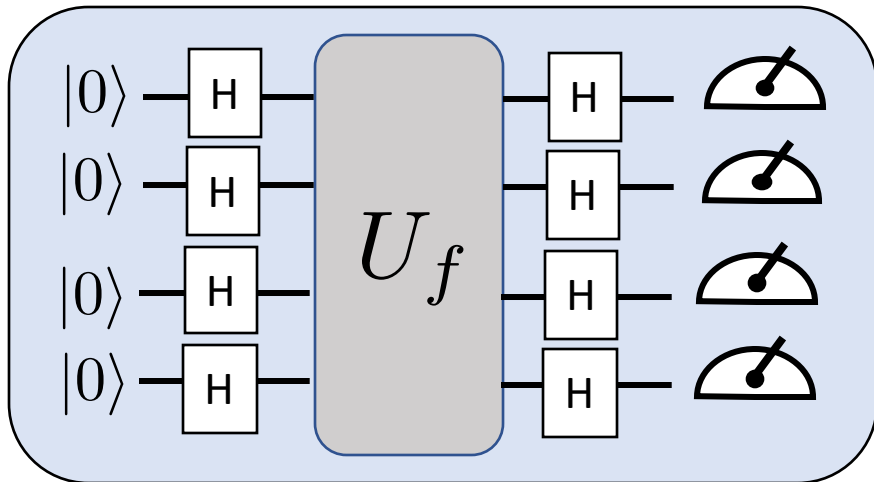
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f(x)	0	0	0	0	0	0	0	0

x	000	001	010	011	100	101	110	111
f(x)	1	0	0	0	1	0	1	1



Constant: $P(0^n) = 1$

Balanced: $P(0^n) = 0$



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Discussion

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Balanced or equal function Game

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

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Balanced: $f(x) = 1$ on half of the bits

x	000	001	010	011	100	101	110	111
f(x)	0	0	0	0	0	0	0	0

x	000	001	010	011	100	101	110	111
f(x)	1	0	0	0	1	0	1	1

- Classical oracle $2^n/2$ queries.
- Quantum:

1 single query!



Classical strategies

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

Constant: $f(x) = c \ \forall x$

Balanced: $f(x) = 1$ on half of the bits



O_f



- Deterministic solution

We need $> 2^n/2$ queries: if balanced we could just always obtain the same value

- Probabilistic solution

We need only 2 queries to solve it with probability $2/3$

Protocol

- 1 – Query two values x and x'
- 2 – If $f(x) \neq f(x')$ output balanced
- 3 – If $f(x) = f(x')$ say constant with probability $2/3$
and balanced with probability $1/3$

References

Reading references

1. Deutsch-Jozsa NC 1.4.3 and 1.4.4
2. Phase kick-back RdW 2.4.1 and G 7.1-7.4

NC $\equiv$ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information
Cambridge University Press (2010)

RdW $\equiv$ Quantum Computing Lecture Notes, Ronald de Wolf, <https://arxiv.org/abs/1907.09415>

G $\equiv$ Introduction to Quantum Computation, Sevag Gharibian, [Lectures notes](#)