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informatics

Introduction to Quantum Computing

Lecture 11: Projective measurements

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Projectors



Projectors on computational basis

● Single-qubit projector on computational basis state

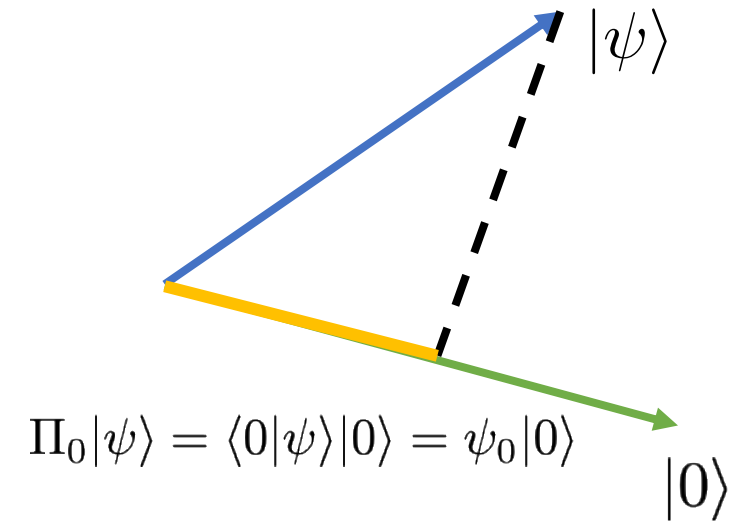
$$\circ \Pi_0 = |0\rangle\langle 0| \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\circ \Pi_1 = |1\rangle\langle 1| \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Pi_0|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} \psi_0 \\ 0 \end{bmatrix} = \psi_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \psi_0|0\rangle$$

Definition of projectors

$$\Pi_S^2 = \Pi_S$$



● Multi-qubit projector on computational basis state $|j\rangle$

$$\Pi_j = |j\rangle\langle j| \quad \Pi_j|\psi\rangle = \psi_j|j\rangle$$

$$\sum_j \Pi_j = I_d$$

Properties of projectors

- $\Pi_{\mathcal{S}}^2 = \Pi_{\mathcal{S}}$

- $\Pi_{\mathcal{S}}^{\dagger} = \Pi_{\mathcal{S}}$

In this course



Self-adjoint projectors

$$\Pi_{\mathcal{S}}^2 = \Pi_{\mathcal{S}} = \Pi_{\mathcal{S}}^{\dagger}$$



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Revisiting basis measurements



Projective measurement

A projective measurement consists of a set of projectors Π_i

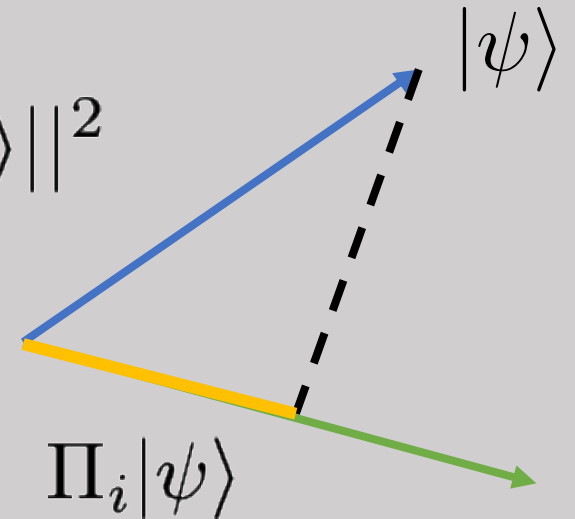
Completeness relation: $\sum_{i=0}^l \Pi_i = I_d$

Orthogonality relation: $\Pi_m \Pi_n = \delta_{n,m} \Pi_m$

Probability of outcome i reads: $P(i) = ||\Pi_i |\psi\rangle||^2$

The quantum state is updated to

$$\frac{\Pi_i |\psi\rangle}{||\Pi_i |\psi\rangle||}$$



Projective measurement: o.n. basis

Any orthonormal basis $\{|v_i\rangle\}$: $\Pi_i = |v_i\rangle\langle v_i|$

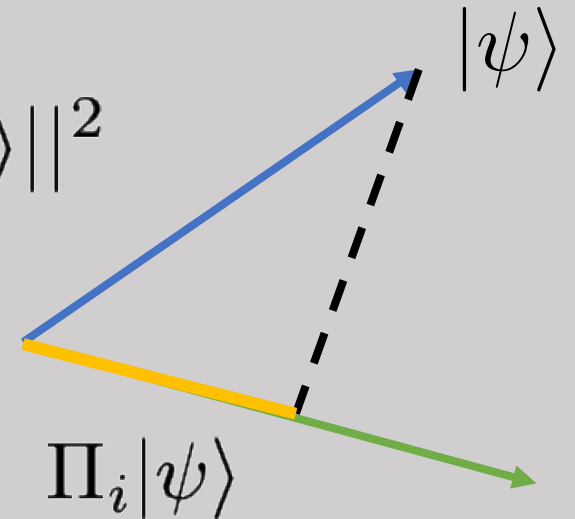
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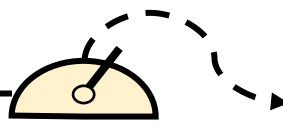
$$\frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$

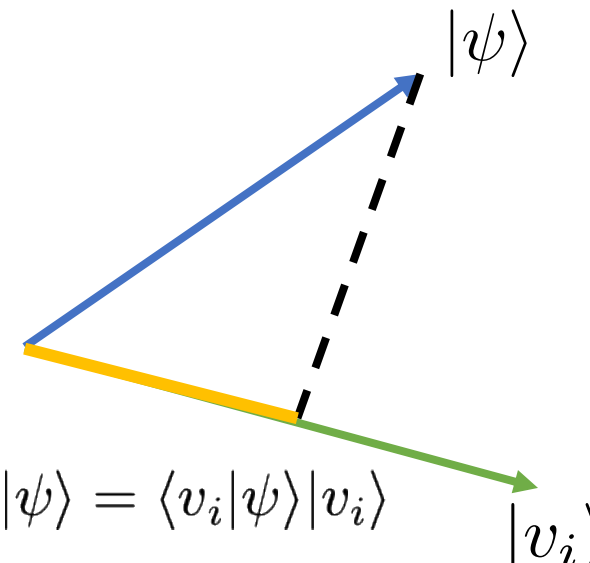


Example: +/- basis of a single qubit

$$\mathcal{H}_Q = \text{Span}\{|+\rangle, |-\rangle\}$$

● Completeness: $|+\rangle\langle+| + |-\rangle\langle-| \equiv \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv I_2$

● $|\psi\rangle$ ——— 
 $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$



$\Pi_i|\psi\rangle = \langle v_i|\psi\rangle|v_i\rangle$

$$P(-) = \|\Pi_{|-\rangle}|\psi\rangle\|^2 \quad \text{Updated state: } \frac{\Pi_{|-\rangle}|\psi\rangle}{\|\Pi_{|-\rangle}|\psi\rangle\|}$$

$$\Pi_{|-\rangle} = |-\rangle\langle-| \equiv \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \times [1/\sqrt{2} \quad -1/\sqrt{2}] = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Pi_{|-\rangle}|\psi\rangle = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \frac{\psi_0 - \psi_1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{\psi_0 - \psi_1}{\sqrt{2}} |-\rangle$$

$$P(-) = |\psi_0 - \psi_1|^2/2$$

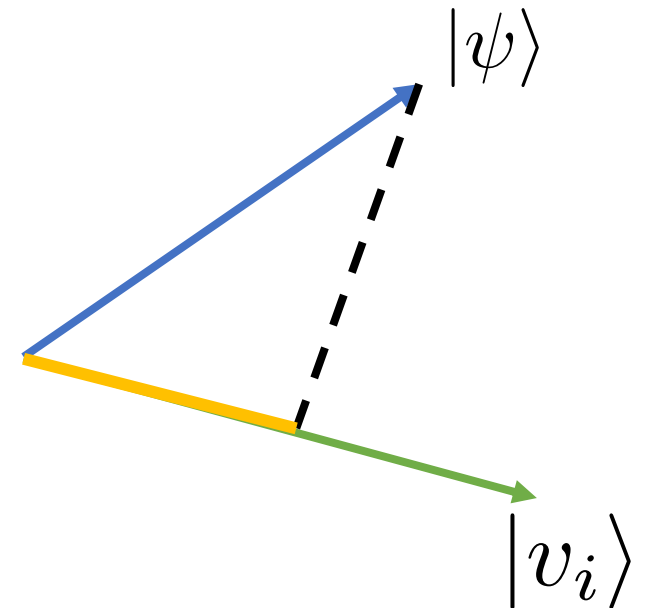
Updated state: $|-\rangle$
up to a global phase (see slide 12)

Decomposition in an arbitrary basis

$$\forall \text{ o.n. basis } \{|v_i\rangle\} : |\psi\rangle = \sum_i \langle v_i | \psi \rangle |v_i\rangle$$

$$\text{Completeness relation: } \sum_{i=0}^l \Pi_i = I_d$$

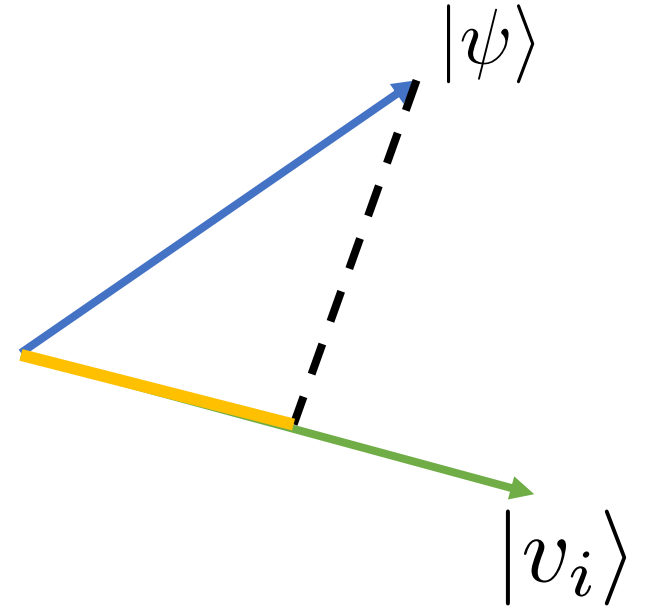
$$\begin{aligned} |\psi\rangle &= \left(\sum_i \Pi_{v_i} \right) |\psi\rangle \\ &= \left(\sum_i |v_i\rangle \langle v_i| \right) |\psi\rangle = \sum_i |v_i\rangle \langle v_i | \psi \rangle \\ &= \sum_i \langle v_i | \psi \rangle |v_i\rangle = \sum_i \psi_{v_i} |v_i\rangle \end{aligned}$$



Global phase

The quantum state is updated to

$$\frac{\Pi_i |\psi\rangle}{||\Pi_i |\psi\rangle||} = \frac{\langle v_i | \psi \rangle}{||\Pi_i |\psi\rangle||} |v_i\rangle = e^{i\varphi} |v_i\rangle$$



- Define a state up to a global phase: $|\tilde{\psi}\rangle \equiv e^{i\varphi} |\psi\rangle$

Output probability: $P(i) = ||\Pi_i |\tilde{\psi}\rangle||^2 = |e^{i\varphi} \langle v_i | \psi \rangle|^2 = |\langle v_i | \psi \rangle|^2$

Projectors on basis states are rank-1 projectors

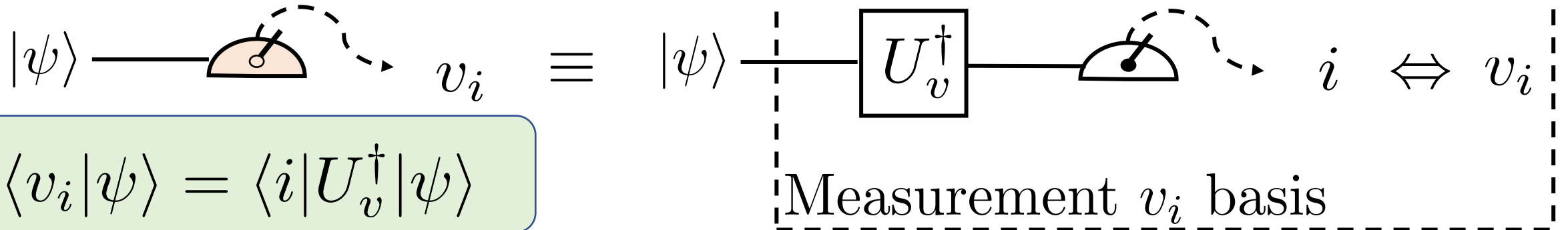
$\forall$ o.n. basis $\{|v_i\rangle\}, \exists U_v$ s.t. $|v_i\rangle = U_v|i\rangle$

$$\Pi_i = |v_i\rangle\langle v_i| = U_v|i\rangle\langle i|U_v^\dagger = U_v$$

$$\begin{bmatrix} \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix} U_v^\dagger$$

Rank-1 matrix

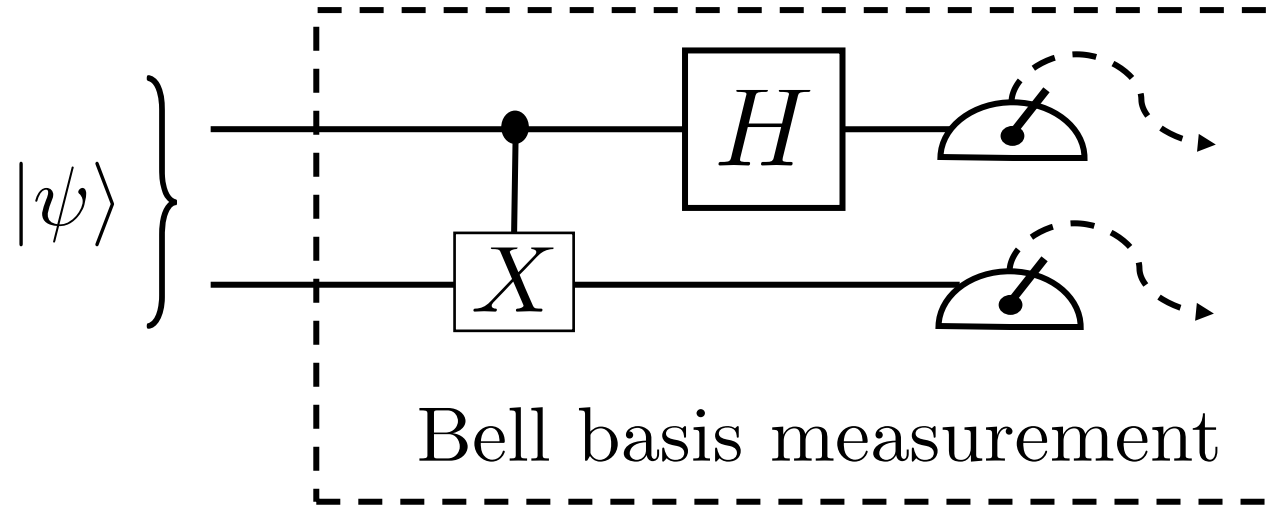
- Measurement basis $\{|v_i\rangle\}$:



$$\langle v_i|\psi\rangle = \langle i|U_v^\dagger|\psi\rangle$$

General multi-qubit basis measurement

- Bell basis measurement:



Bell basis

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

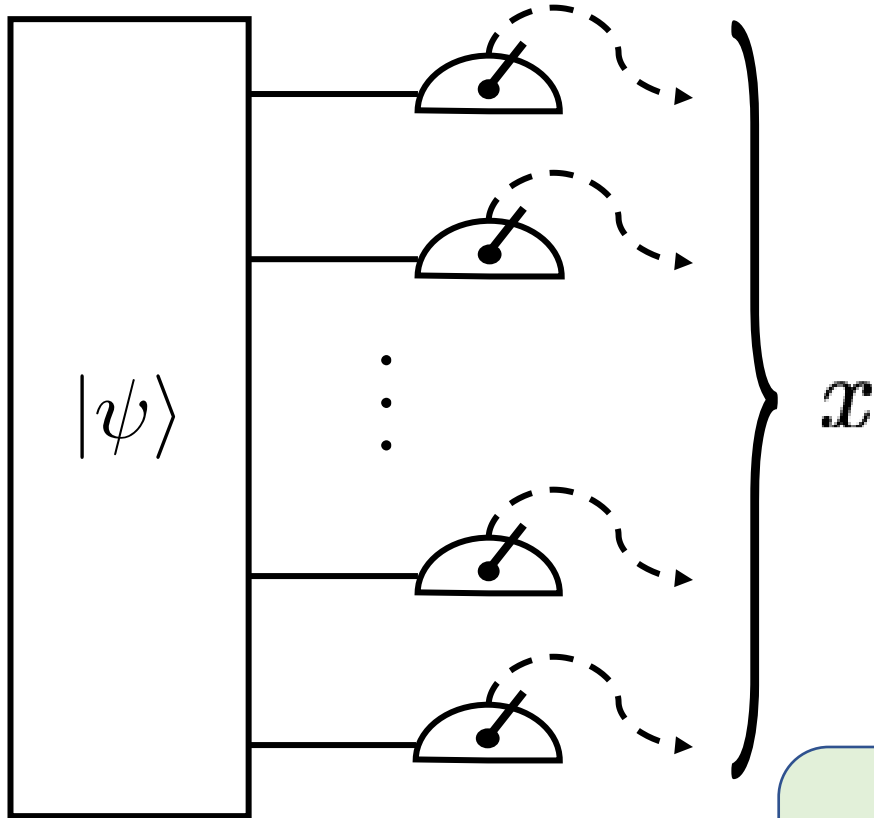
$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

$$|\Phi^\pm\rangle\langle\Phi^\pm| = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix}$$

$$|\Psi^\pm\rangle\langle\Psi^\pm| = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Measurement: computational basis

- Let x encode the outcome of n bits



$$|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$

$$\Pi_x = |x\rangle\langle x| = |x_1\rangle\langle x_1| \otimes \cdots \otimes |x_n\rangle\langle x_n|$$

$$P(x) = \|\Pi_x|\psi\rangle\|^2 = |\langle x|\psi\rangle|^2$$

$$\text{Update: } \frac{\Pi_x|\psi\rangle}{\|\Pi_x|\psi\rangle\|} = \frac{\psi_x}{|\psi_x|}|x\rangle = e^{i\phi}|x\rangle$$

$$|\langle x|\psi\rangle|^2 = |\langle x| \sum_{y \in \{0,1\}^n} \psi_y |y\rangle|^2 = |\psi_x|^2$$



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Projectors on subspaces



Projective measurement

A projective measurement consists of a set of projectors Π_i

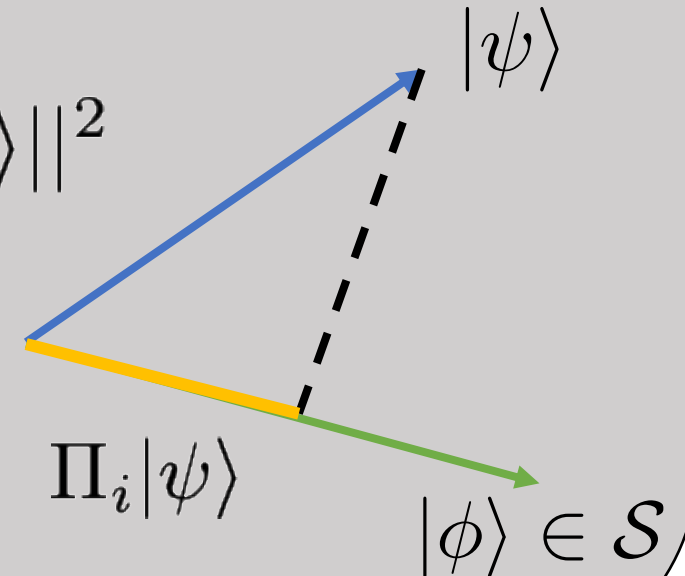
Completeness relation: $\sum_{i=0}^l \Pi_i = I_d$

Orthogonality relation: $\Pi_m \Pi_n = \delta_{n,m} \Pi_m$

Probability of outcome i reads: $P(i) = ||\Pi_i |\psi\rangle||^2$

The quantum state is updated to

$$\frac{\Pi_i |\psi\rangle}{||\Pi_i |\psi\rangle||}$$



A 3-dimensional quantum system

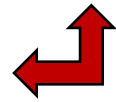
- Orthonormal basis:

- Completeness:

- $\Pi_3 + \Pi_{12} = I$

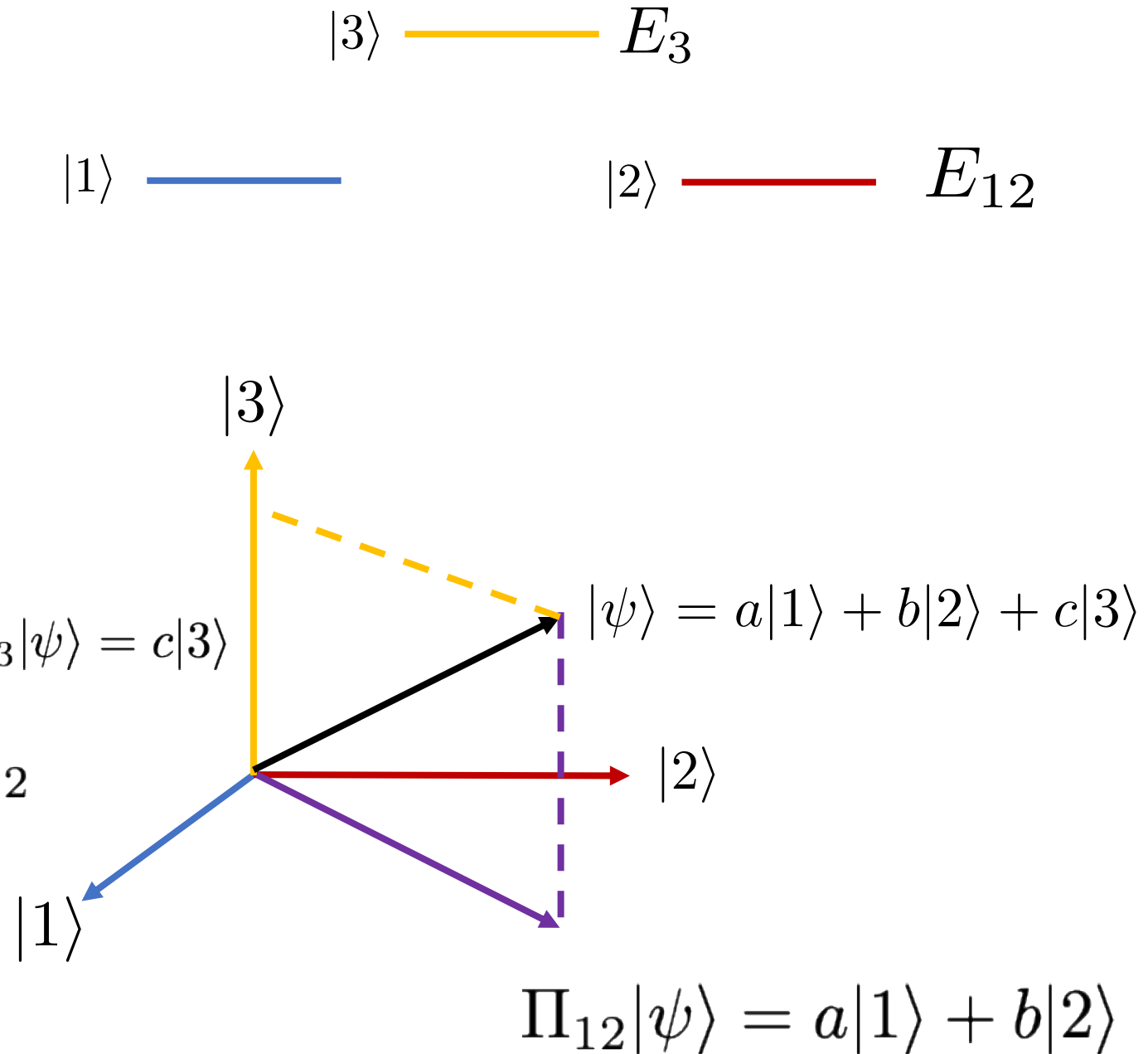
- $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rank 2 projector



- $P(12) = \|\Pi_{12}|\psi\rangle\|^2 = |a|^2 + |b|^2$

Update: $|\psi'\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}}(a|1\rangle + b|2\rangle)$



Projectors on vector subspaces

- Projectors on a 1-dim vector subspace: $\Pi_i = |v_i\rangle\langle v_i|$
- Projector on vector subspace $\mathcal{S}$ of dim k ($\mathcal{S} \subset \mathcal{H}$) :
 - Being a vector space, $\mathcal{S}$ has an orthonormal basis $\{|u_i\rangle\}_{i=0}^{k-1}$
 - $\Pi_{\mathcal{S}} = \sum_{i=0}^{k-1} |u_i\rangle\langle u_i|$
 - $\Pi_{\mathcal{S}}^2 = \Pi_{\mathcal{S}}$
 - $\Pi_{\mathcal{S}}^\dagger = \Pi_{\mathcal{S}}$

$$\Pi_{\mathcal{S}}^2 = \Pi_{\mathcal{S}} = \Pi_{\mathcal{S}}^\dagger$$

Projective measurement

Completeness relation

$$\sum_{i=0}^l \Pi_i = I_d$$

Completeness implies probabilities add to 1:

$$\begin{aligned}\sum_i P(i) &= \sum_i ||\Pi_i|\psi\rangle||^2 \\ &= \sum_i \langle\psi|\Pi_i^\dagger\Pi_i|\psi\rangle = \langle\psi|\sum_i \Pi_i^\dagger\Pi_i|\psi\rangle \\ &= \langle\psi|\sum_i \Pi_i|\psi\rangle = \langle\psi|\psi\rangle = 1\end{aligned}$$

We use:

- $P(i) = ||\Pi_i|\psi\rangle||^2$
- Linearity
- $\Pi_{\mathcal{S}}^2 = \Pi_{\mathcal{S}} = \Pi_{\mathcal{S}}^\dagger$

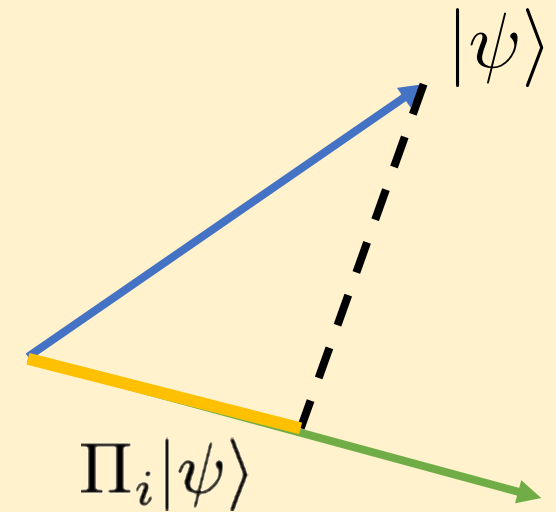
Reproducibility of measurement

Repeating the same measurement immediately after gives the same answer.

Results from: $\Pi_i^2 = \Pi_i$ and $\Pi_i \Pi_j = \delta_{i,j} \Pi_i$

$$\Pi_i \frac{\Pi_i |\psi\rangle}{||\Pi_i |\psi\rangle||} = \frac{\Pi_i^2 |\psi\rangle}{||\Pi_i |\psi\rangle||} = \frac{\Pi_i |\psi\rangle}{||\Pi_i |\psi\rangle||}$$

$$\Pi_j \frac{\Pi_i |\psi\rangle}{||\Pi_i |\psi\rangle||} = 0 \quad (\forall j \neq i)$$



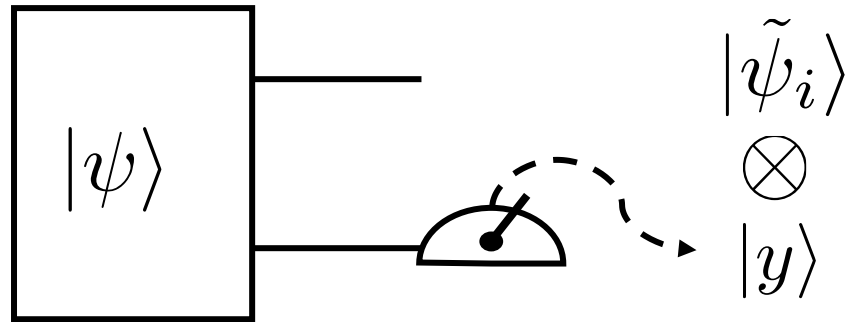


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Partial measurement



Two-qubit example



$$\begin{aligned}\tilde{\Pi}_0 &= I \otimes |0\rangle\langle 0| \\ \tilde{\Pi}_1 &= I \otimes |1\rangle\langle 1| \\ \tilde{\Pi}_0 + \tilde{\Pi}_1 &= I\end{aligned}$$

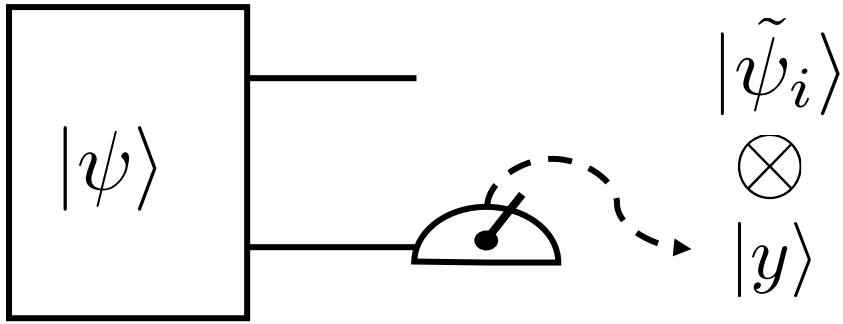
$$|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{01}|0\rangle \otimes |1\rangle + \psi_{10}|1\rangle \otimes |0\rangle + \psi_{11}|1\rangle \otimes |1\rangle$$

$$\tilde{\Pi}_0|i\rangle \otimes |0\rangle = (I \otimes |0\rangle\langle 0|)(|i\rangle \otimes |0\rangle) = I|i\rangle \otimes |0\rangle \underbrace{\langle 0|0\rangle}_{=1} = |i\rangle \otimes |0\rangle$$

$$\tilde{\Pi}_0|i\rangle \otimes |1\rangle = (I \otimes |0\rangle\langle 0|)(|i\rangle \otimes |1\rangle) = I|i\rangle \otimes |0\rangle \underbrace{\langle 0|1\rangle}_{=0} = 0$$

$$\tilde{\Pi}_0|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{10}|1\rangle \otimes |0\rangle = (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle$$

Two-qubit example



$$\begin{aligned}\tilde{\Pi}_0 &= I \otimes |0\rangle\langle 0| \\ \tilde{\Pi}_1 &= I \otimes |1\rangle\langle 1| \\ \tilde{\Pi}_0 + \tilde{\Pi}_1 &= I\end{aligned}$$

$$|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{01}|0\rangle \otimes |1\rangle + \psi_{10}|1\rangle \otimes |0\rangle + \psi_{11}|1\rangle \otimes |1\rangle$$

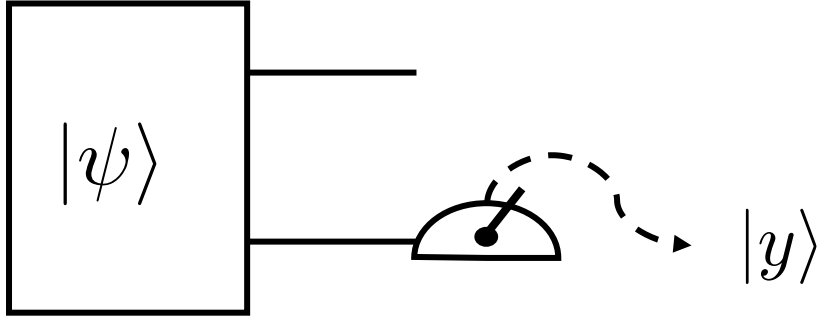
$$\tilde{\Pi}_0|\psi\rangle = \psi_{00}|0\rangle \otimes |0\rangle + \psi_{10}|1\rangle \otimes |0\rangle = (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle$$

$$\|\tilde{\Pi}_0|\psi\rangle\|^2 = |\psi_{00}|^2 + |\psi_{10}|^2$$

$$\frac{\tilde{\Pi}_0|\psi\rangle}{\|\tilde{\Pi}_0|\psi\rangle\|} = \frac{1}{\sqrt{|\psi_{00}|^2 + |\psi_{10}|^2}} (\psi_{00}|0\rangle + \psi_{10}|1\rangle) \otimes |0\rangle = |\tilde{\psi}_0\rangle \otimes |0\rangle$$

Two-register example

Two registers of n and m qubits respectively :



$$|\psi\rangle = \sum_{x,w} \psi_{x,w} |x\rangle \otimes |w\rangle$$

$$\tilde{\Pi}_y = I_{\mathcal{H}^{\otimes n}} \otimes |y\rangle\langle y|$$

$$\sum_{y \in \{0,1\}^m} \Pi_y = I_{\mathcal{H}^{\otimes m}}$$

$$\tilde{\Pi}_y |\psi\rangle = (I_{\mathcal{H}^{\otimes n}} \otimes |y\rangle\langle y|) |\psi\rangle = \left(\sum_x \psi_{x,y} |x\rangle \right) \otimes |y\rangle \quad ||\tilde{\Pi}_y |\psi\rangle||^2 = \sum_x |\psi_{x,y}|^2$$

$$\frac{\tilde{\Pi}_y |\psi\rangle}{||\tilde{\Pi}_y |\psi\rangle||} = \frac{1}{\sqrt{\sum_x |\psi_{x,y}|^2}} \left(\sum_x \psi_{x,y} |x\rangle \right) \otimes |y\rangle = |\tilde{\psi}_y\rangle \otimes |y\rangle$$

References

Reading references

1. Adjoints and Hermitian operators NC 2.1.6
2. Projective measurement NC 2.2.5 (notation different from the course)
3. What is a phase? NC 2.2.7
4. Composite system and measurement NC 2.2.8

NC $\equiv$ Michael Nielsen and Isaac Chuang, Quantum Computation and Quantum Information
Cambridge University Press (2010)