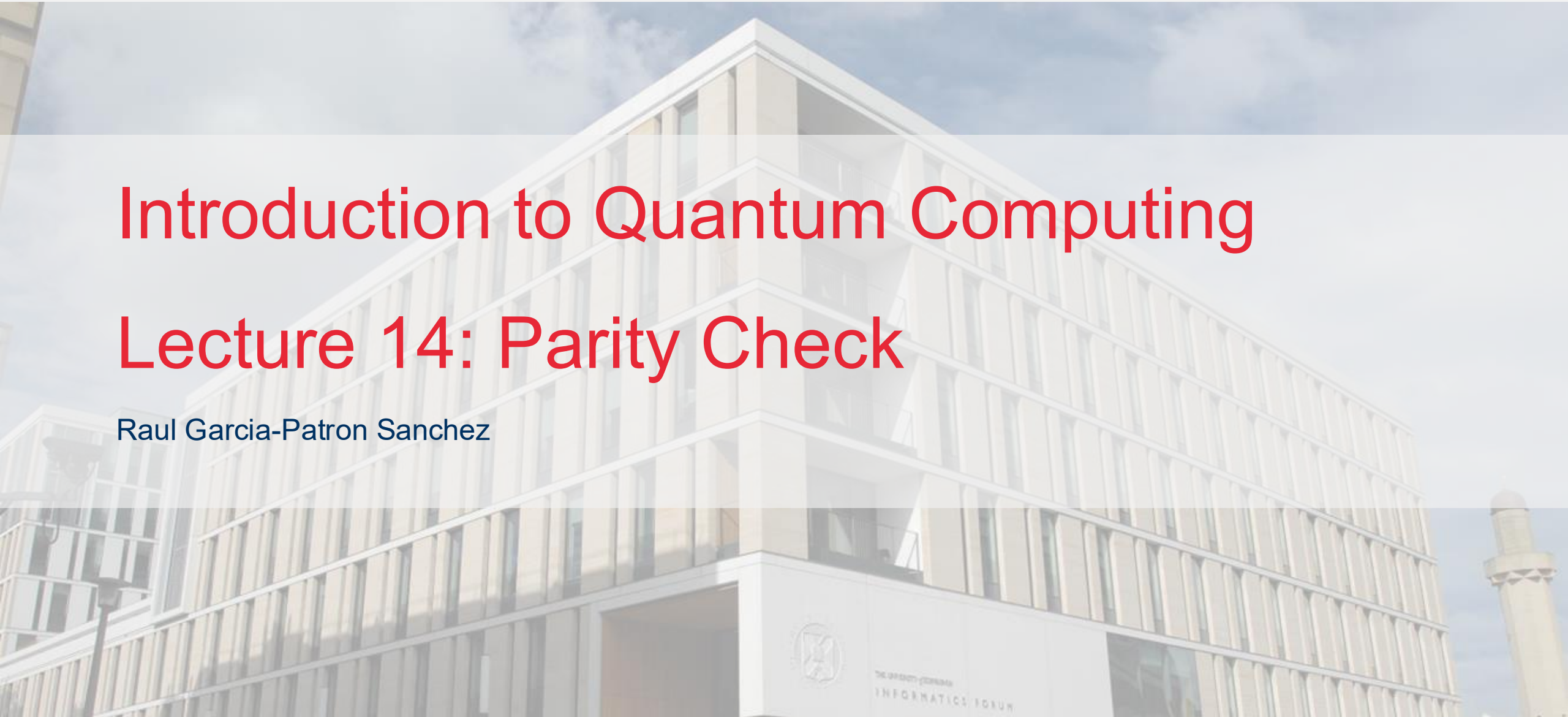


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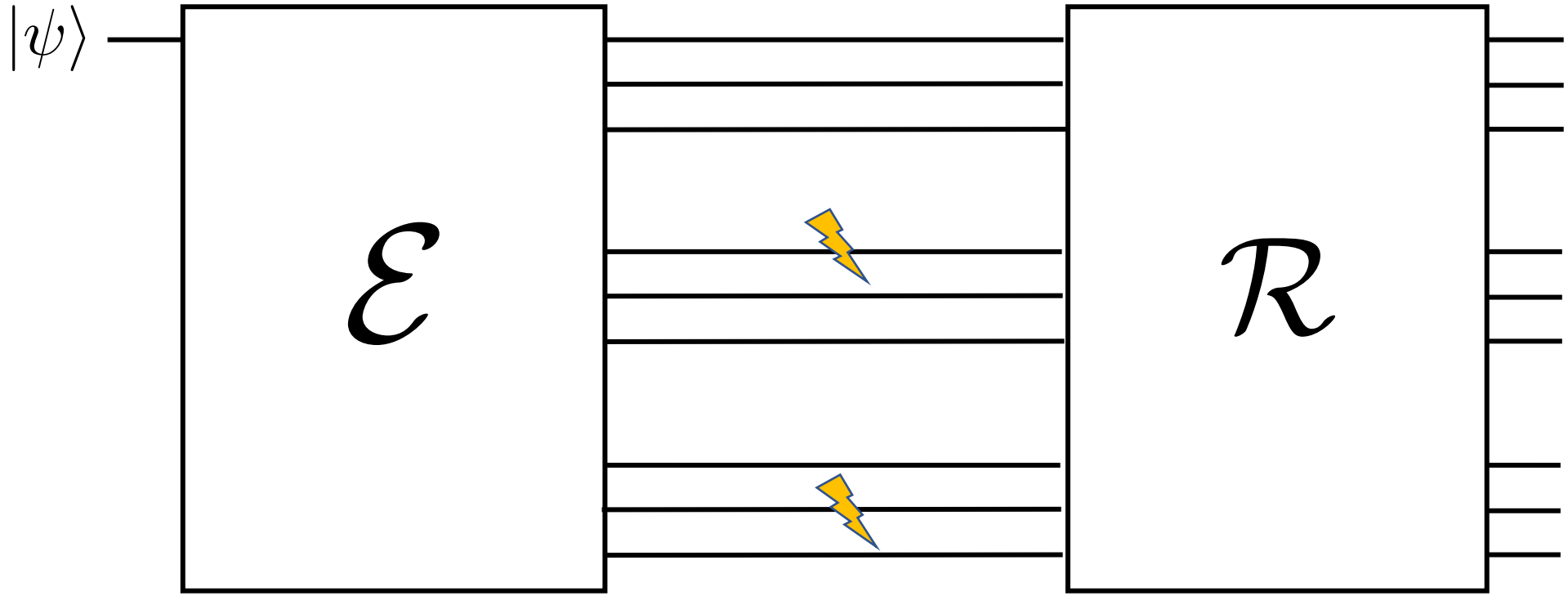
Introduction to Quantum Computing

Lecture 14: Parity Check

Raul Garcia-Patron Sanchez

Quantum Error Correction in one slide

- Noise is local: independent for every qubit
- Our operations are perfect: error due to interaction with local environment.



- In reality we also have correlated errors
- Our operations have errors $\Rightarrow$ Theory of fault-tolerance

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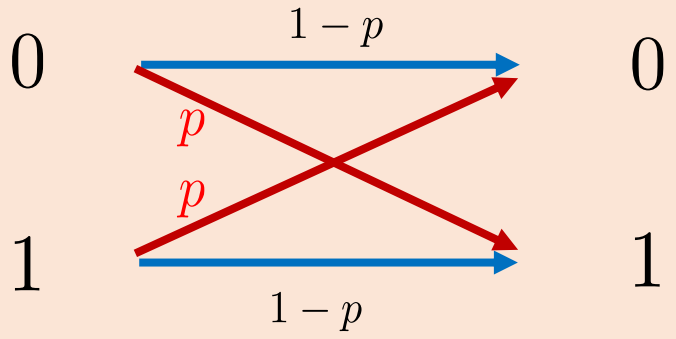
Classical repetition code and parity check

The repetition code

Encoding $0_L \rightarrow 000$ Codeword

$1_L \rightarrow 111$

Binary symmetric channel



$000 \rightarrow 010$

Error correction: majority vote

$\{000, 100, 010, 001\} \rightarrow 0_L$

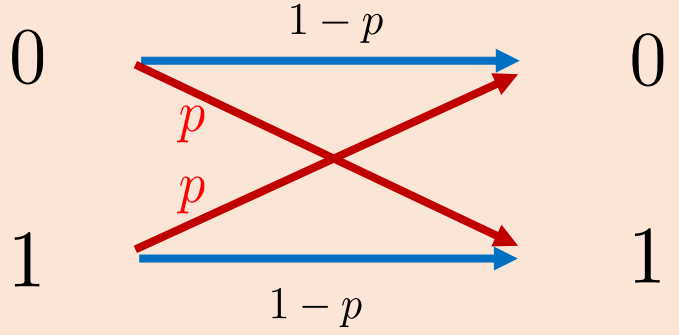
$\{111, 110, 011, 101\} \rightarrow 1_L$

The repetition code

Encoding $0_L \to 000$ Codeword

$1_L \to 111$

Binary symmetric channel



$000 \to 010$

$[n, k, d]$ code

n : codeword length

k : # logical bits

d : min Hamming distance

e : # correctable errors

$e = \lfloor (d-1)/2 \rfloor$

Error correction: majority vote

$\{000, 100, 010, 001\} \to 0_L$

$\{111, 110, 011, 101\} \to 1_L$

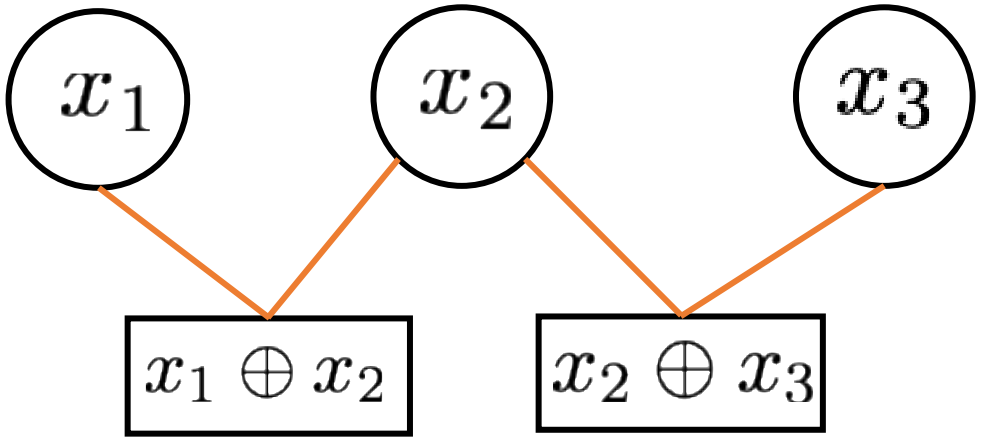
Repetition code

$[3, 1, 3]$ code

Parity checks

Encoding $0_L \rightarrow 000$ Codeword

$1_L \rightarrow 111$



Received Word	Parity Check 1	Parity Check 2	Error location	Output
000	0	0	no	000
001	0	1	3	000
010	1	1	2	000
100	1	0	1	000
011	1	0	1	111
101	1	1	2	111
110	0	1	3	111
111	0	0	no	111



Quantum parity check

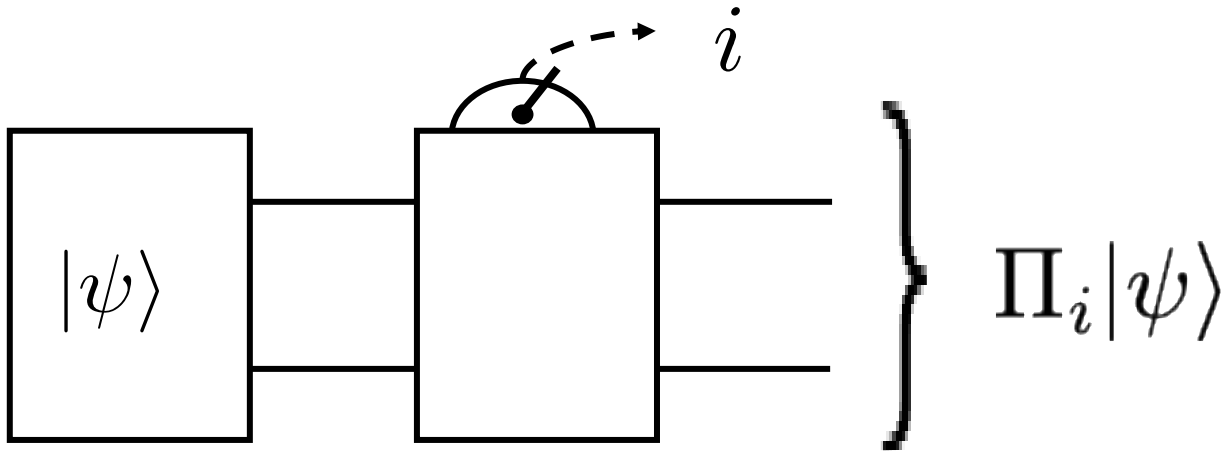
Example: Parity of 2 qubits

- $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o = \text{span}\{|00\rangle, |11\rangle\} \oplus \text{span}\{|01\rangle, |10\rangle\}$

$\oplus \equiv$ Direct sum

- $\Pi_e = |00\rangle\langle 00| + |11\rangle\langle 11|$
  $\Pi_o = |01\rangle\langle 01| + |10\rangle\langle 10|$

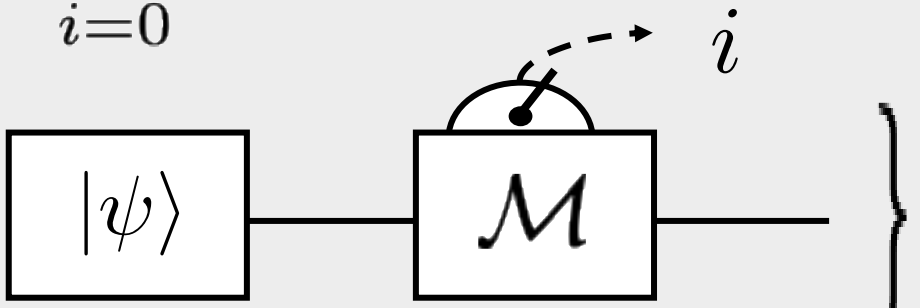
$\Pi_e + \Pi_o = I_4$



$\Pi_i|\psi\rangle$

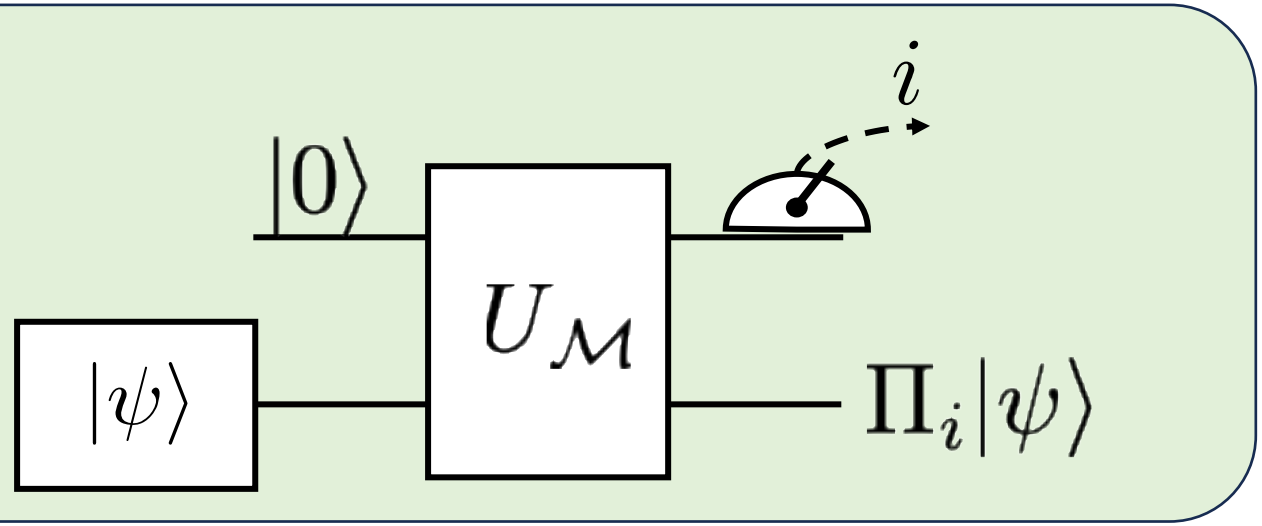
From the math definition to an actual circuit

$$\sum_{i=0}^{l-1} \Pi_i = I_{2^n} \qquad P(i) = ||\Pi_i|\psi\rangle||^2$$

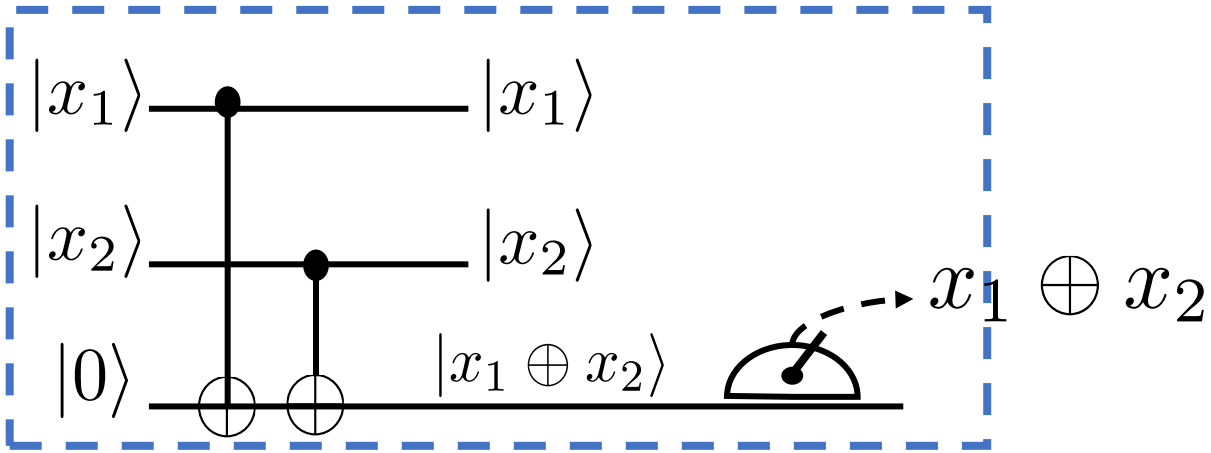


$$\text{Update: } |\psi_i\rangle = \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||}$$

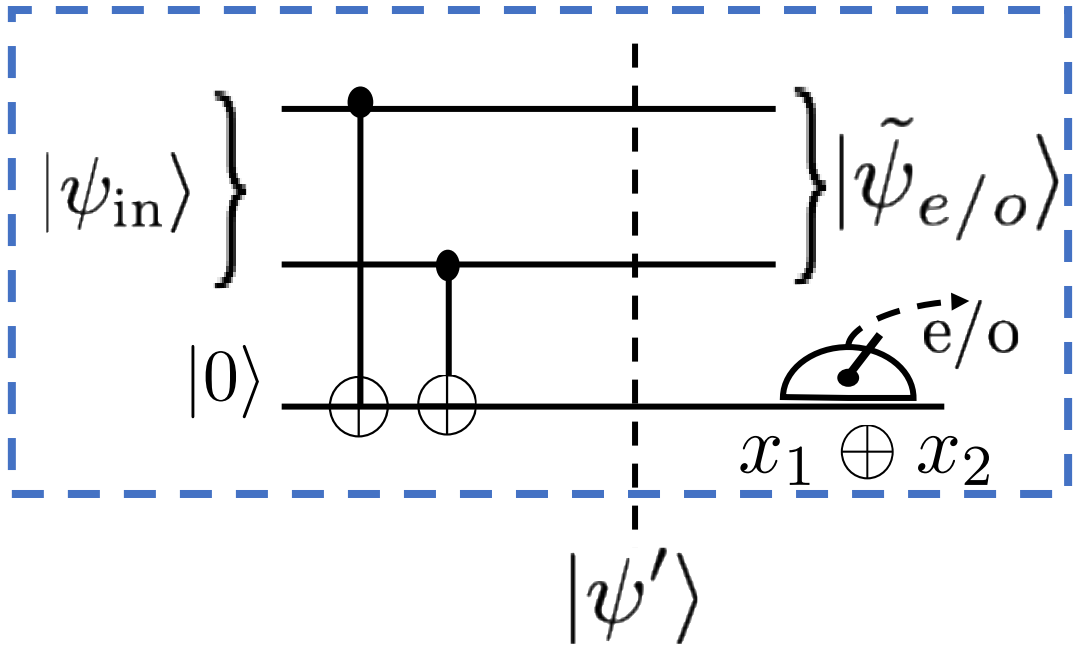
We need to find the circuit $U_{\mathcal{M}}$



Parity check circuit



Parity check circuit – general input

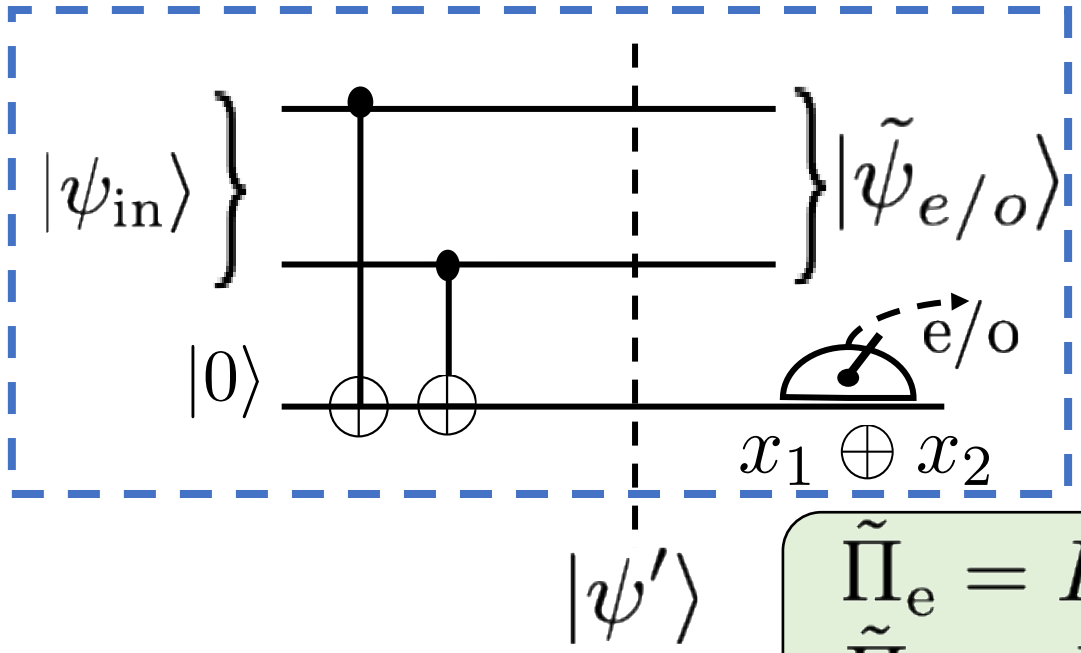


$$|\psi_{\text{in}}\rangle = \left(\sum_{x_1,x_2} \psi_{x_1,x_2}|x_1,x_2\rangle\right) \otimes |0\rangle$$

$$|\psi_{\text{in}}\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2}|x_1,x_2\rangle \otimes |0\rangle$$

$$|\psi'\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2}|x_1,x_2\rangle \otimes |x_1 \oplus x_2\rangle$$

Parity check circuit – general input



$$|\psi_{\text{in}}\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2}|x_1,x_2\rangle \otimes |0\rangle$$

$$|\psi'\rangle = \sum_{x_1,x_2} \psi_{x_1,x_2}|x_1,x_2\rangle \otimes |x_1 \oplus x_2\rangle$$

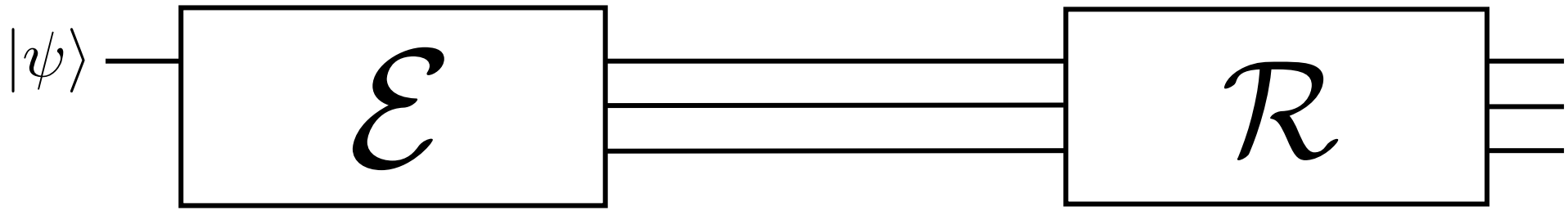
$$\tilde{\Pi}_{\text{e}} = I \otimes I \otimes |0\rangle\langle 0|$$
$$\tilde{\Pi}_{\text{o}} = I \otimes I \otimes |1\rangle\langle 1|$$

$$\tilde{\Pi}_{\text{e}} + \tilde{\Pi}_{\text{o}} = I_8$$

$$\tilde{\Pi}_o|\psi'\rangle = (\psi_{0,1}|0,1\rangle + \psi_{1,0}|1,0\rangle) \otimes |1\rangle \qquad ||\tilde{\Pi}_{\text{o}}|\psi'\rangle||^2 = |\psi_{0,1}|^2 + |\psi_{1,0}|^2$$

$$|\psi'_o\rangle = \frac{\tilde{\Pi}_{\text{o}}|\psi'\rangle}{||\tilde{\Pi}_{\text{o}}|\psi'\rangle||} = \frac{1}{\sqrt{|\psi_{0,1}|^2 + |\psi_{1,0}|^2}} (\psi_{0,1}|0,1\rangle + \psi_{1,0}|1,0\rangle) \otimes |1\rangle = |\tilde{\psi}'_{\text{o}}\rangle \otimes |1\rangle$$

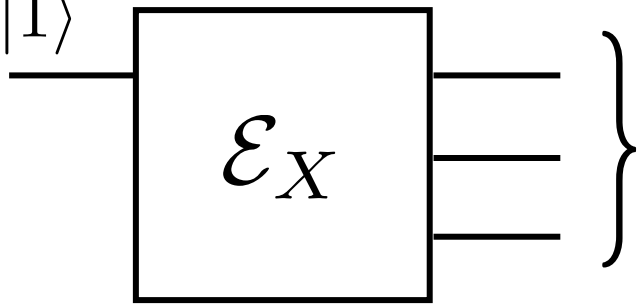
The bit flip repetition code



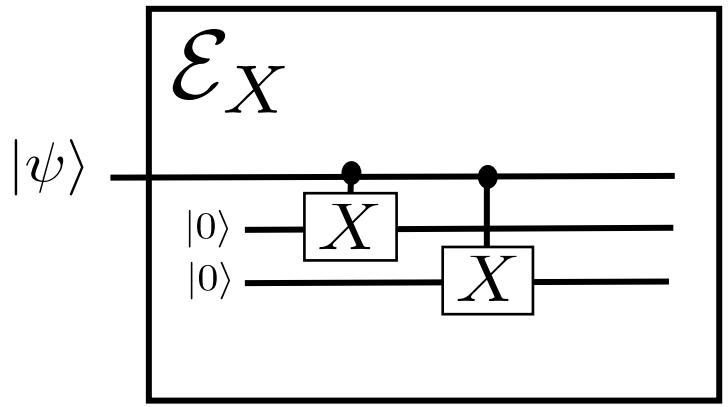
The bit flip repetition code: encoding

$$|0\rangle_L \rightarrow |000\rangle \qquad\qquad |1\rangle_L \rightarrow |111\rangle$$

$|\psi\rangle = a|0\rangle + b|1\rangle$

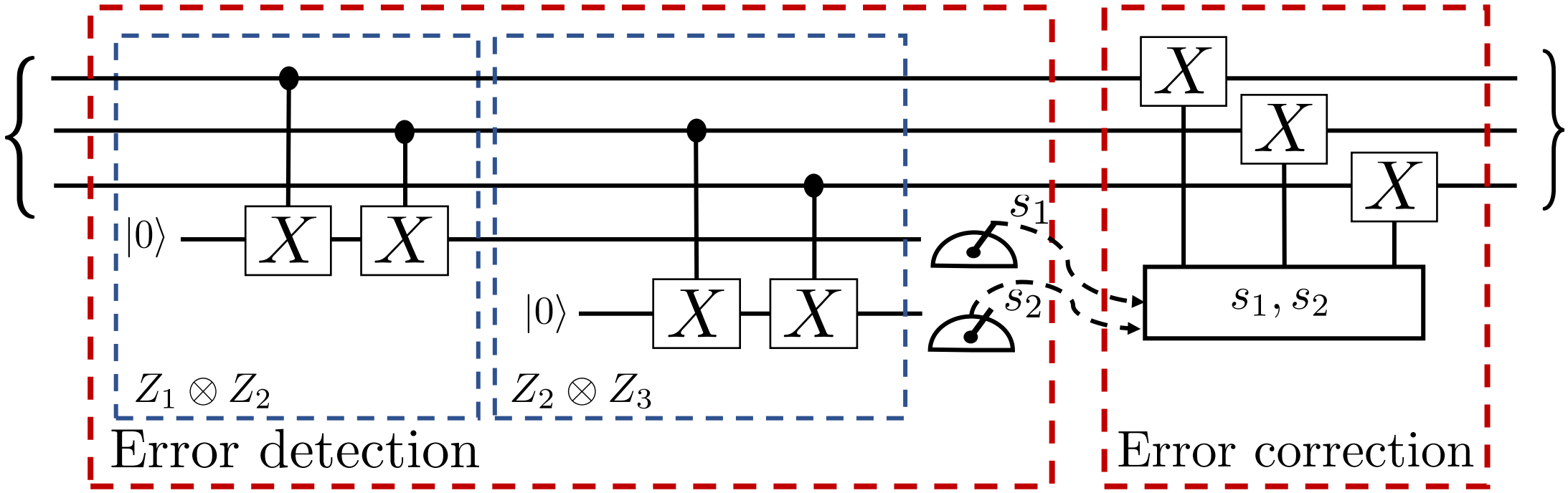


$$|\tilde{\psi}\rangle = a|000\rangle + b|111\rangle$$

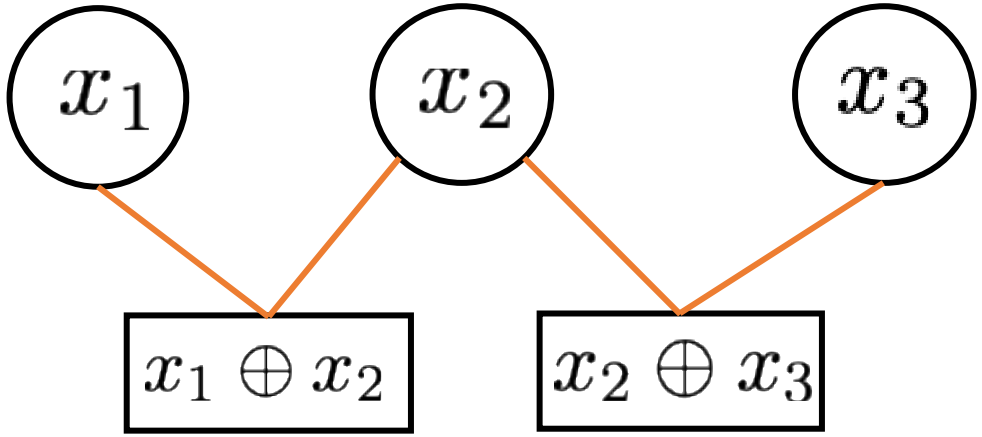


Syndrome measurement

- Error detection: measure $Z_1 \otimes Z_2$ and $Z_2 \otimes Z_3$



Measuring the parities reveals which qubit flipped without revealing or collapsing the state, i.e., it preserves the superposition (assuming only one error). Logical states and its single bit-flips are eigenstates of both parity check operators.





Parity check of phase

The intuition



$$H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) = |\pm\rangle$$

Bit flip error



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

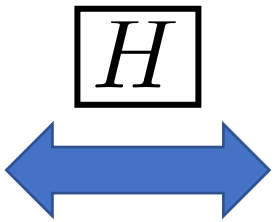
$$X|x\rangle = |x \oplus 1\rangle$$

- Basis $Z|x\rangle = (-1)^x|x\rangle$
- Parity check $Z \otimes Z$

$$\Pi_e = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\Pi_o = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$(Z \otimes Z)\Pi_{e/o} = \pm\Pi_{e/o}$$



Phase flip error



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|\pm\rangle = |\mp\rangle$$

$$X|\pm\rangle = \pm|\pm\rangle$$

- Parity check $X \otimes X$

$$\tilde{\Pi}_e = |++\rangle\langle++| + |--\rangle\langle--|$$

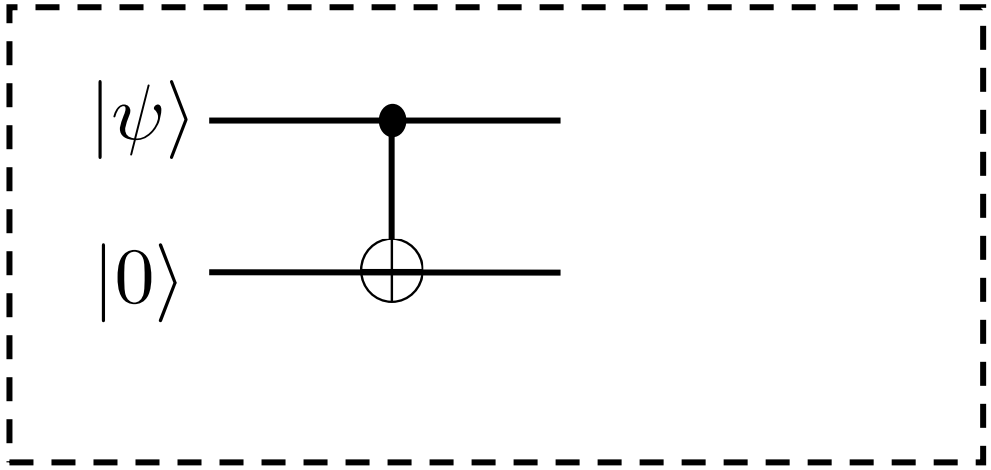
$$\tilde{\Pi}_o = |+-\rangle\langle+-| + |-+\rangle\langle-+|$$

$$(X \otimes X)\tilde{\Pi}_{e/o} = \pm\tilde{\Pi}_{e/o}$$

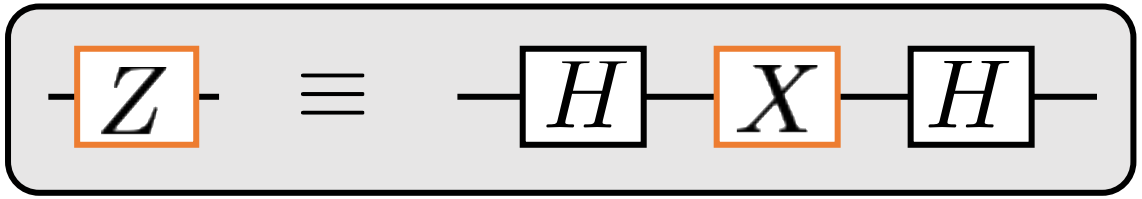


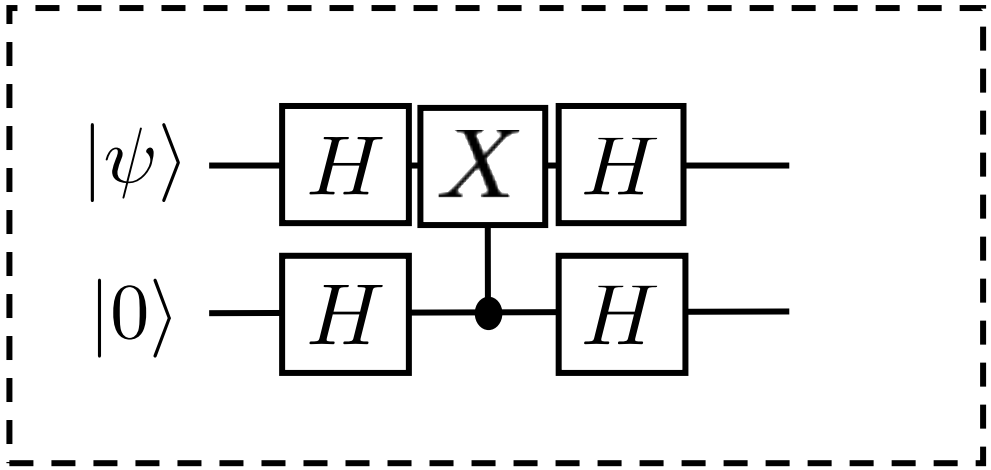
Parity Check as a Hadamard Test

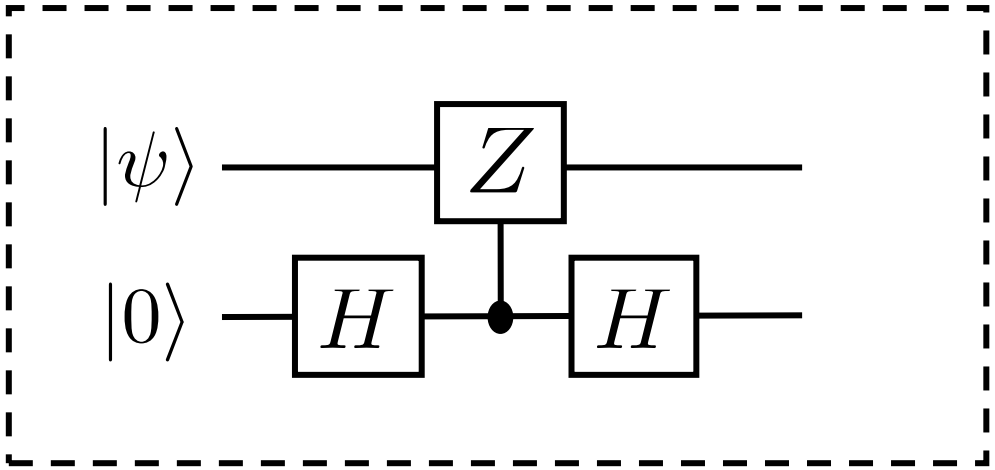
An equivalent circuit



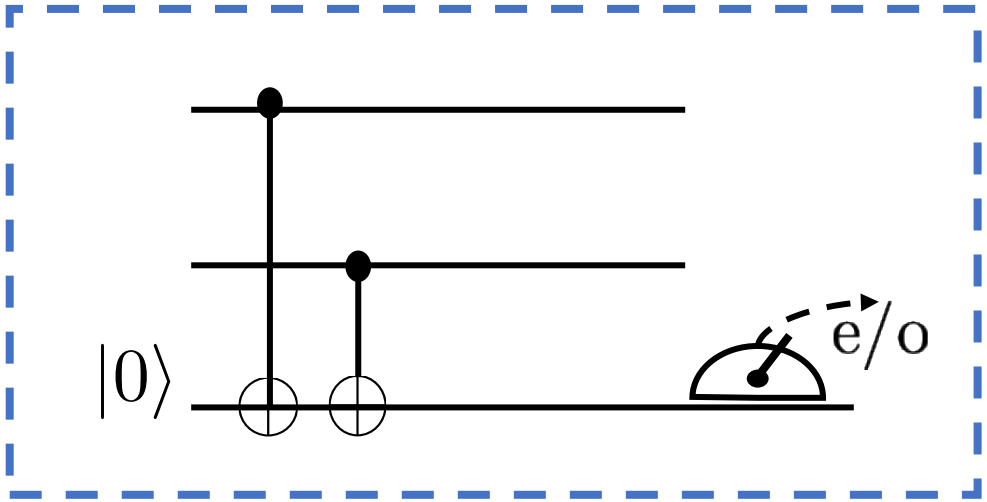
Tutorial 2 problem 2

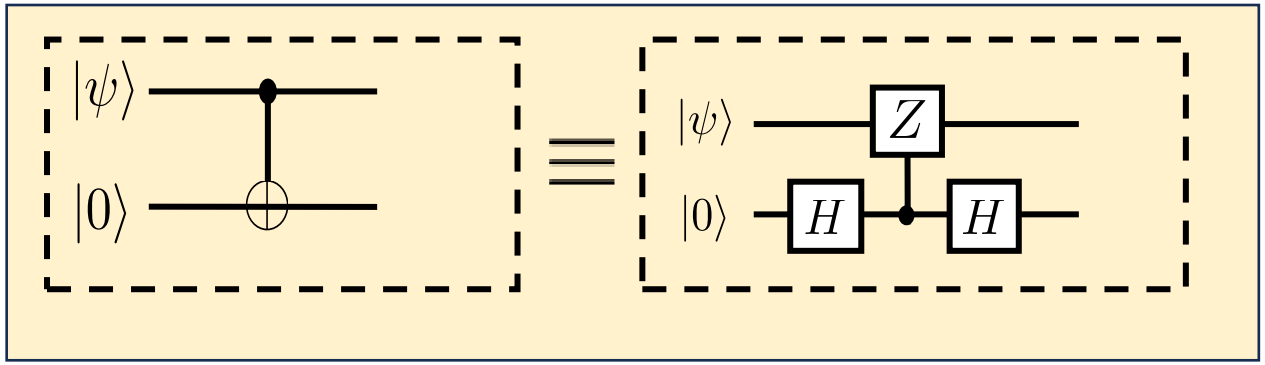


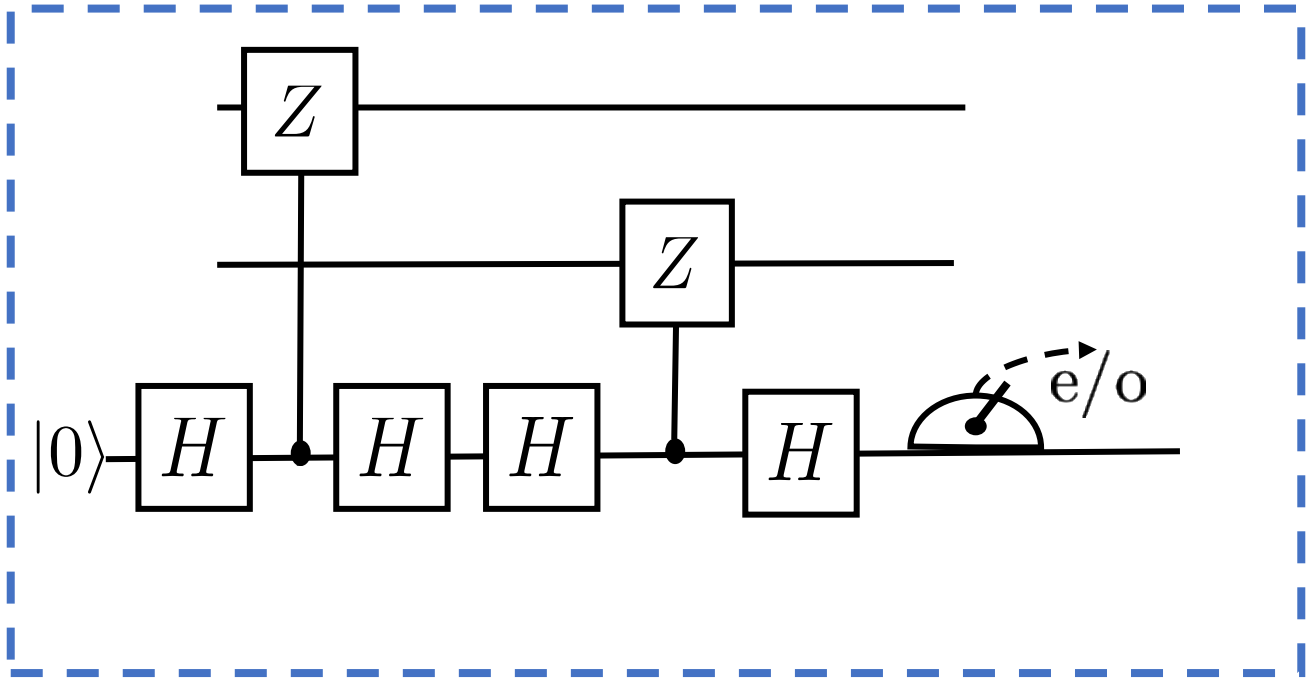


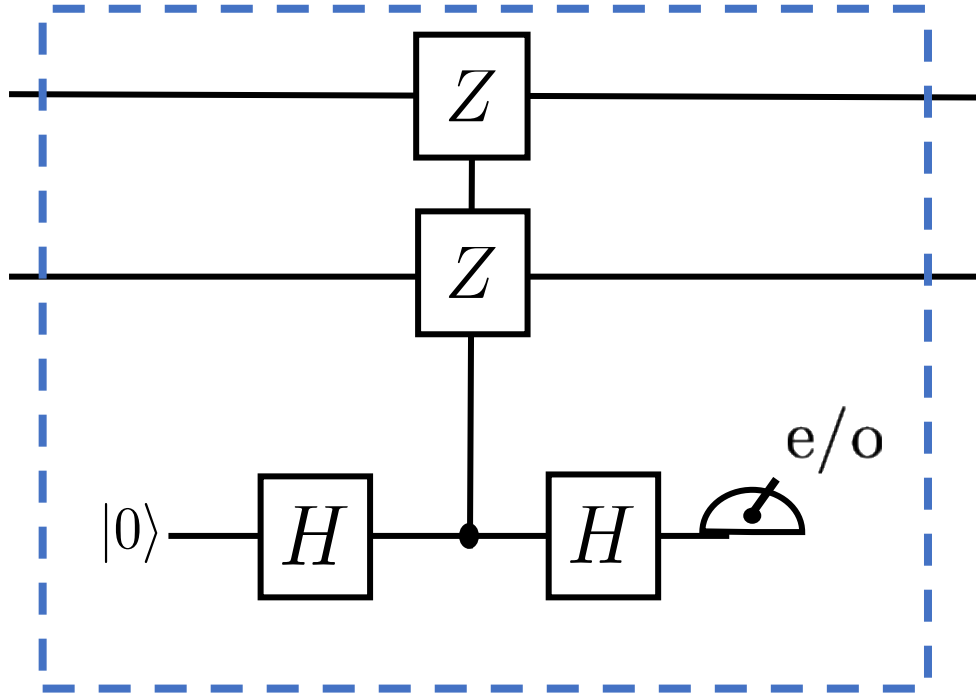


Hadamard test circuit for parity check

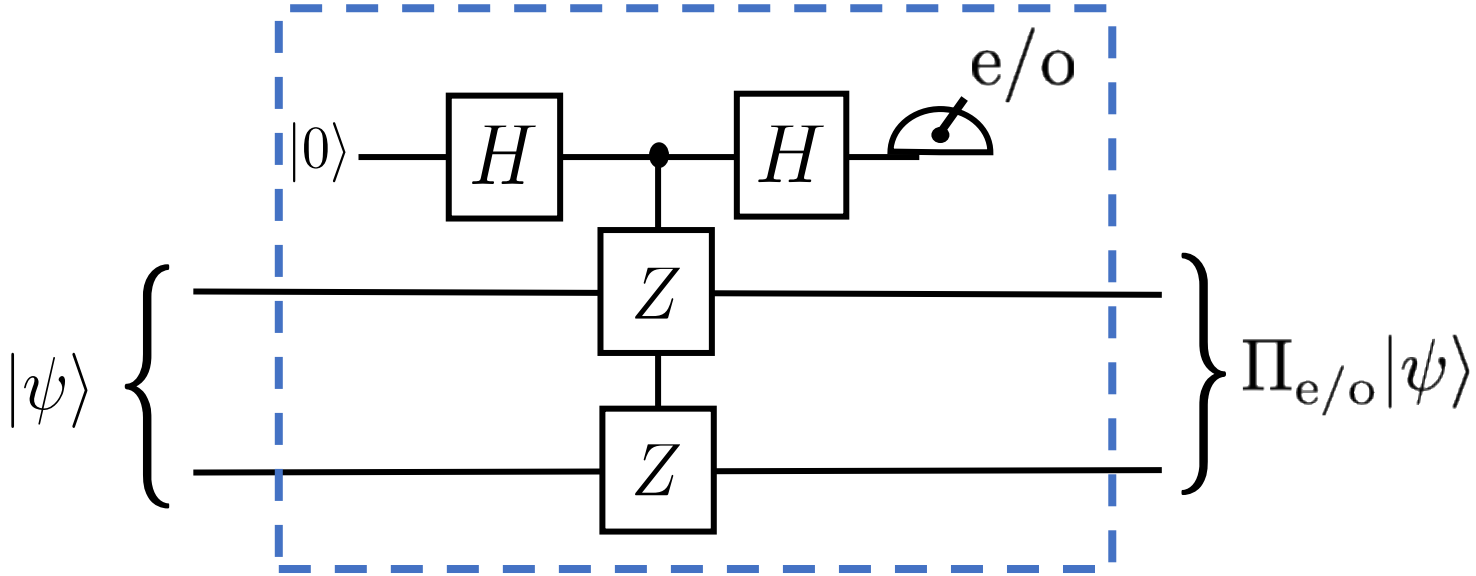








Hadamard test circuit for parity check



$$Z_1 \otimes Z_2 |x_1 x_2\rangle = (-1)^{x_1 \oplus x_2} |x_1 x_2\rangle$$

$$\Pi_e = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\Pi_o = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$(Z \otimes Z)\Pi_{e/o} = \pm \Pi_{e/o}$$

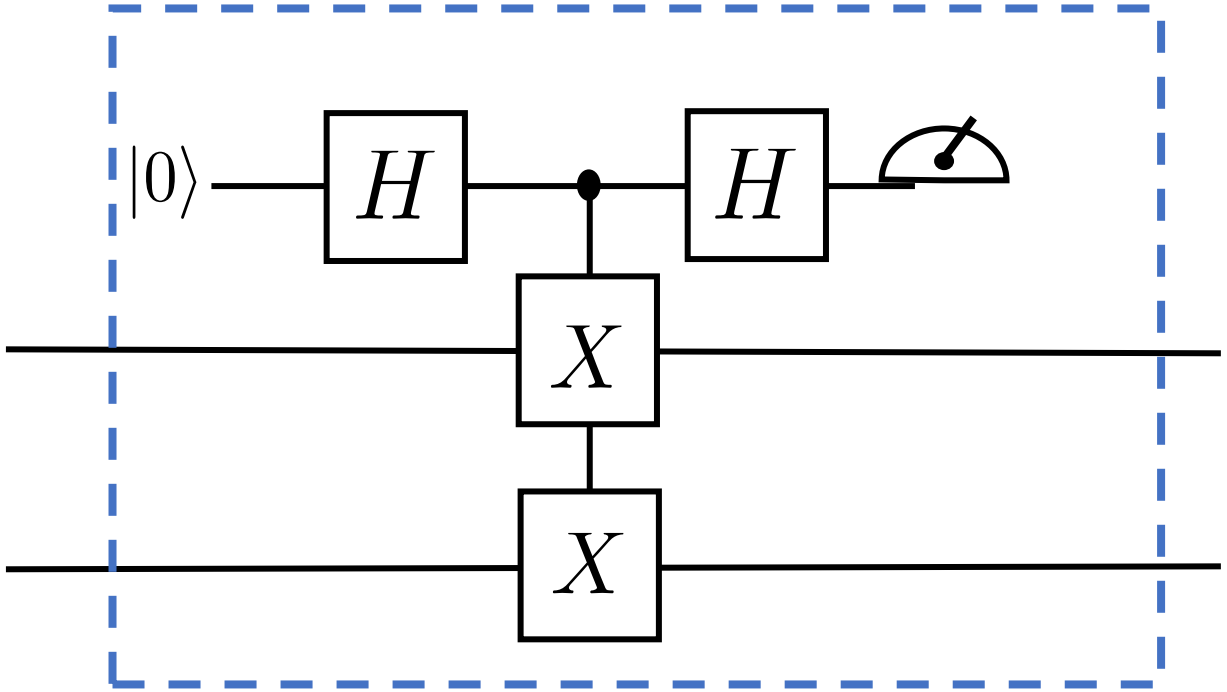
The operator ZZ:

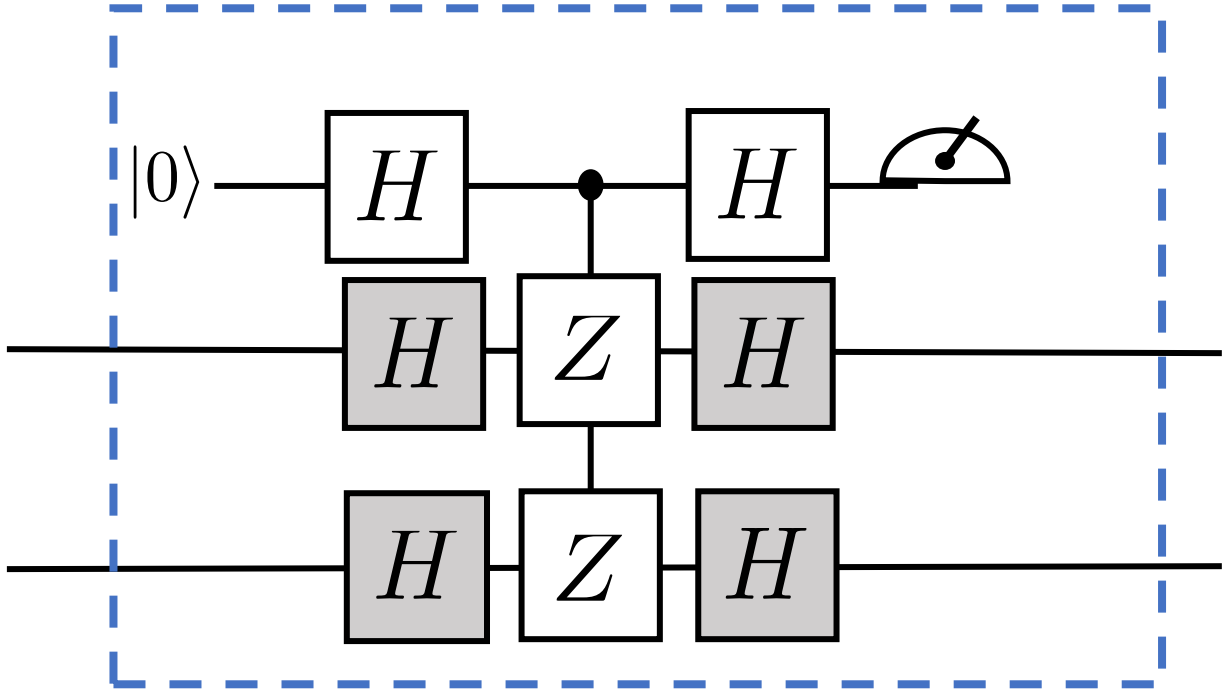
+1 eigenspace of ZZ is the even-parity subspace

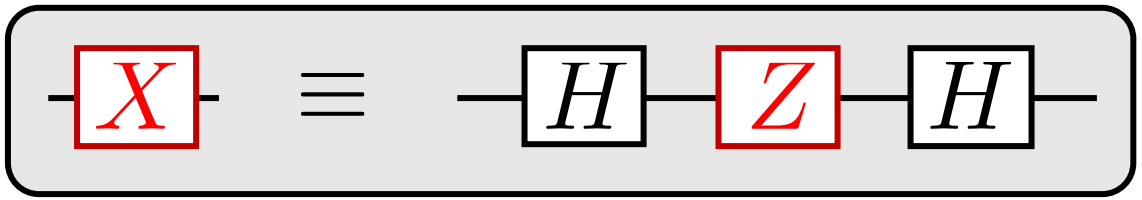
-1 eigenspace of ZZ is the odd-parity subspace

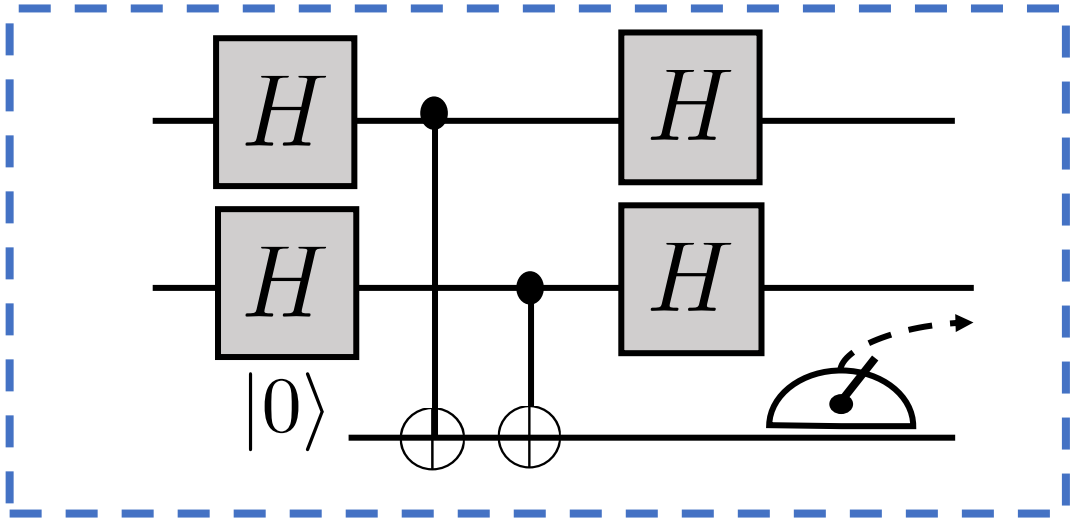
This will generalize to the Hadamard test, see next lecture

Parity in the X basis





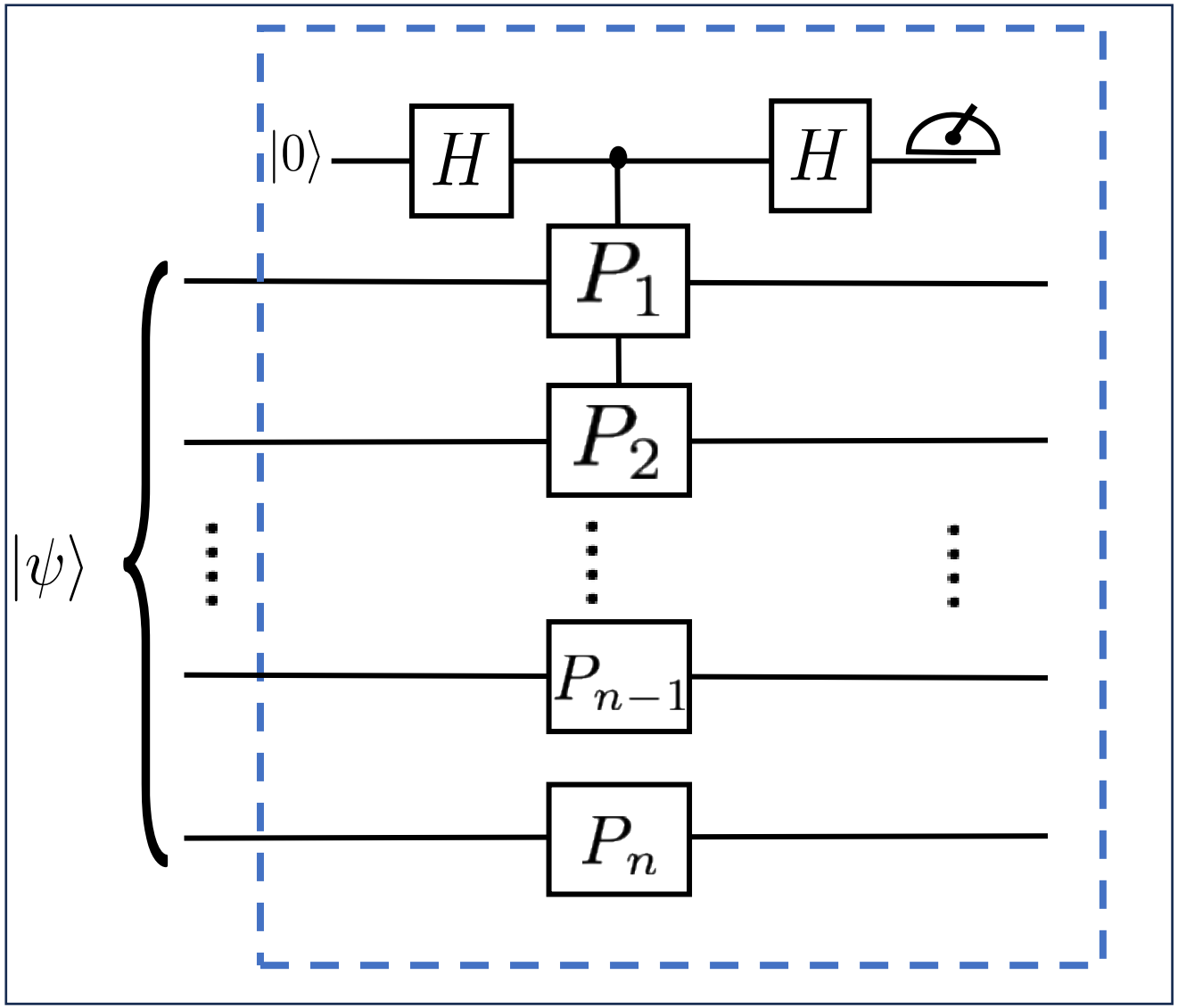






General parity check

Generalized parity check



Parity-check operator

$$P = P_1 \otimes P_2 \otimes \cdots \otimes P_{n-1} \otimes P_n$$

A string of Pauli matrices

$$P_i \in \{I, X, Y, Z\}$$

$$P\Pi_{\pm 1} = \pm\Pi_{\pm 1}$$

$\Pi_{\pm 1}$ eigenspace with eigenvalue ± 1

$$\Pi_{+1} + \Pi_{-1} = I_{2^n}$$

References

Reading references

1. Operators and Observables: NC 2.1.2, 2.1.5, 2.1.6.
2. Projective measurement: NC 2.2.5
3. Parity measurement: NC 10.5.8

NC ≣ Michael Nielsen and Isaac Chuang, Quantum Computation and Quantum Information Cambridge University Press (2010)