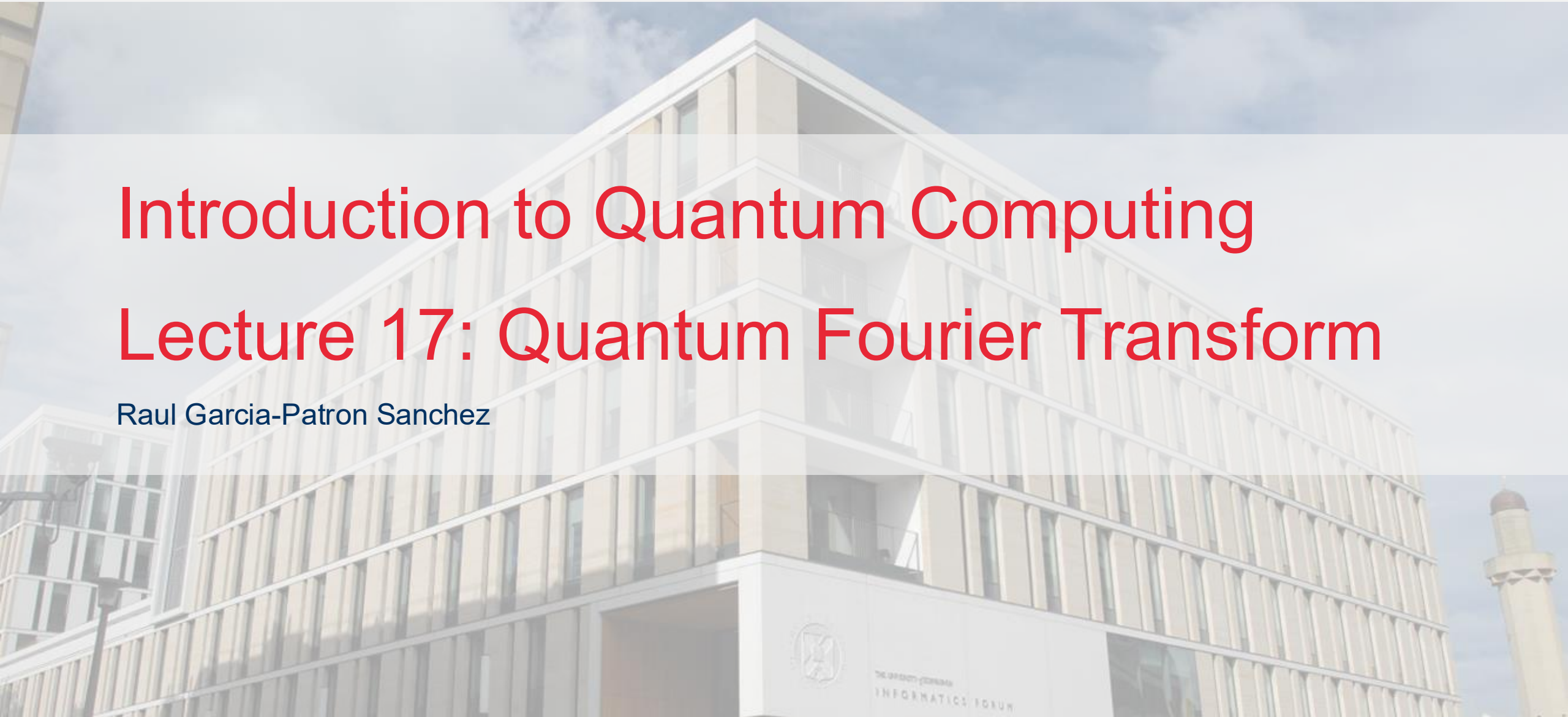


THE UNIVERSITY of EDINBURGH
informatics

# Introduction to Quantum Computing

# Lecture 17: Quantum Fourier Transform

Raul Garcia-Patron Sanchez



# Pauli rotations – last lecture

# Pauli-string rotation

Pauli rotations $e^{-i\theta P}$ are useful subroutines in quantum simulation and beyond.

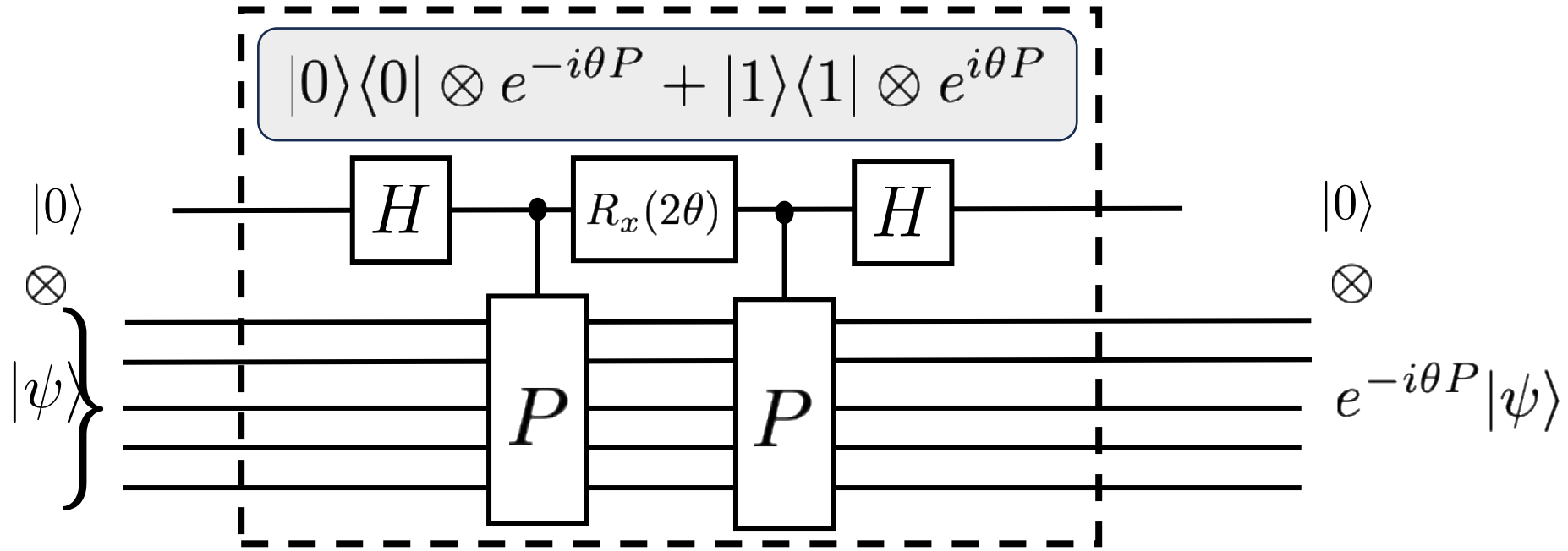
Single-qubit rotations

$$R_x(2\theta) = e^{-i\theta X} \quad R_y(2\theta) = e^{-i\theta Y} \quad R_z(2\theta) = e^{-i\theta Z}$$

$$e^{-i\hat{H}\Delta t} \approx \prod_{w'} e^{-i\theta_{w'} P_{w'}}$$

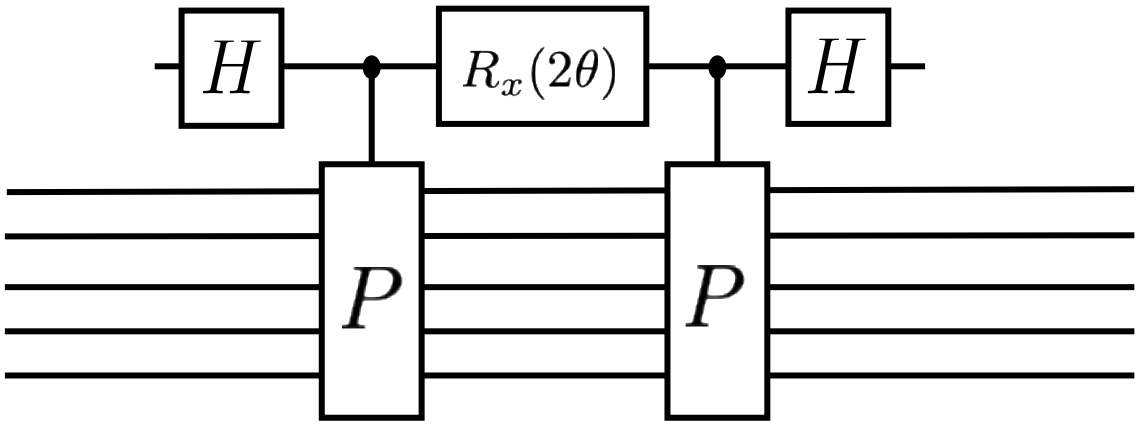
$$e^{-i\theta P} = \cos\theta I_{2^n} - i \sin\theta P$$
$$e^{-i\theta P} = e^{-i\theta}\Pi_0 + e^{i\theta}\Pi_1$$

$$\Pi_0 + \Pi_1 = I_{2^n}$$
$$P = \Pi_0 - \Pi_1$$

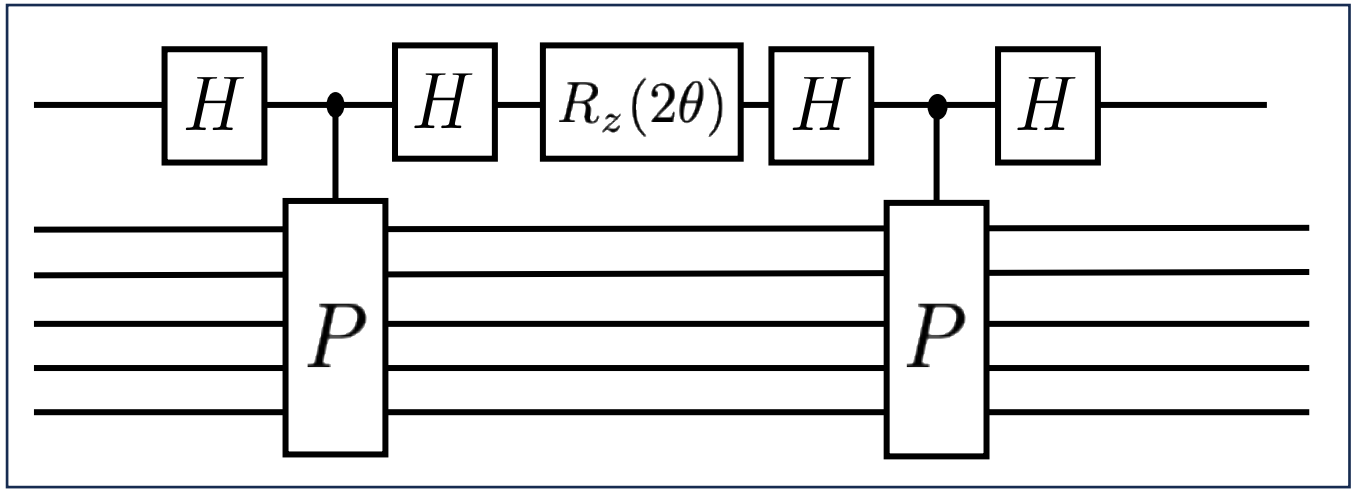


# Pauli-string rotation – Proof

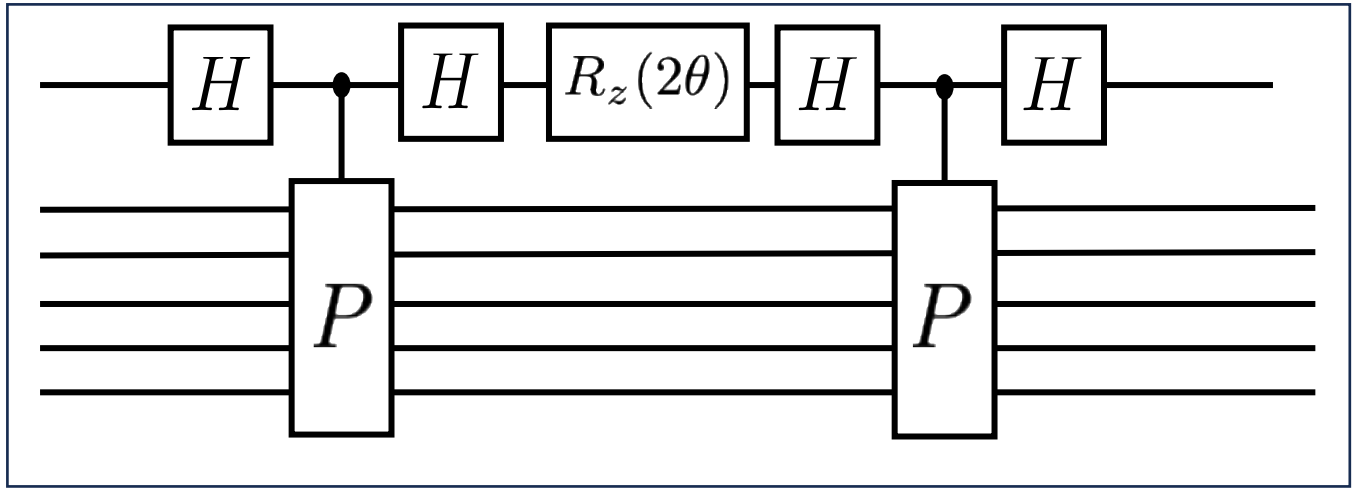
$$|0\rangle\langle 0| \otimes e^{-i\theta P} + |1\rangle\langle 1| \otimes e^{i\theta P}$$




$$\begin{aligned} R_x(2\theta) = e^{-i\theta X} &= \cos\theta I - i\sin\theta X \\ &= \cos\theta H^2 - i\sin\theta HZH \\ &= H(\cos\theta I - i\sin\theta Z)H \\ &= He^{-i\theta Z}H = HR_z(2\theta)H \end{aligned}$$



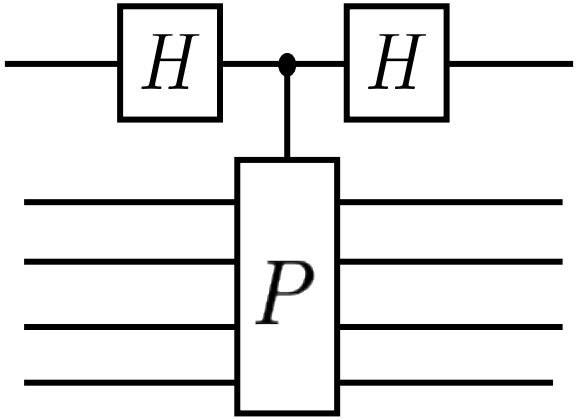
# Pauli-string rotation – Proof




$$|0\rangle\langle 0| \otimes e^{-i\theta P} + |1\rangle\langle 1| \otimes e^{i\theta P}$$

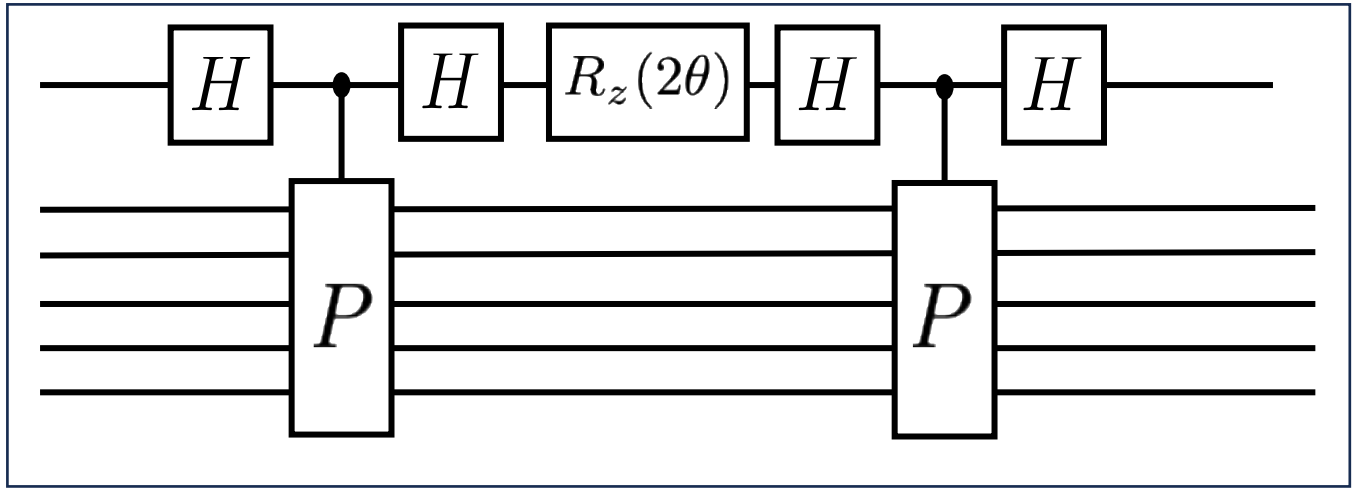
$$|\pm\rangle\langle\pm| = \frac{I \pm X}{2}$$

$$I \otimes \Pi_0 + X \otimes \Pi_1$$




$$\begin{aligned}
&(H \otimes I_{2^n})(|0\rangle\langle 0| \otimes I_{2^n} + |1\rangle\langle 1| \otimes P)(H \otimes I_{2^n}) \\
&\quad = |+\rangle\langle +| \otimes I_{2^n} + |-\rangle\langle -| \otimes P \\
&\quad = \frac{I+X}{2} \otimes (\Pi_0 + \Pi_1) + \frac{I-X}{2} \otimes (\Pi_0 - \Pi_1) \\
&\quad = \frac{I+X}{2} \otimes \Pi_0 + \frac{I+X}{2} \otimes \Pi_1 + \frac{I-X}{2} \otimes \Pi_0 - \frac{I-X}{2} \otimes \Pi_1 \\
&\quad = \left(\frac{I+X}{2} + \frac{I-X}{2}\right) \otimes \Pi_0 + \left(\frac{I+X}{2} - \frac{I-X}{2}\right) \otimes \Pi_1 \\
&\quad = I \otimes \Pi_0 + X \otimes \Pi_1
\end{aligned}$$

# Pauli-string rotation – Proof

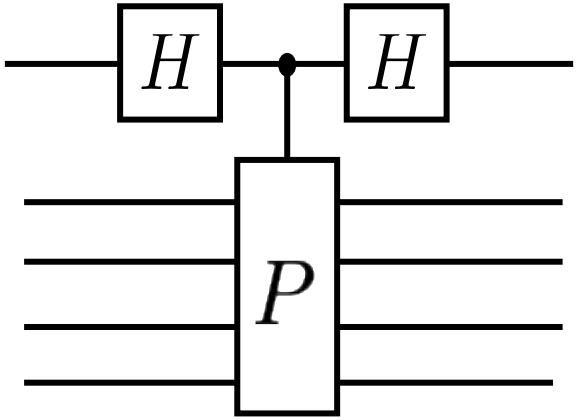



$$|0\rangle\langle 0| \otimes e^{-i\theta P} + |1\rangle\langle 1| \otimes e^{i\theta P}$$

$$e^{-i\theta P} = \cos\theta I_{2^n} - i\sin\theta P$$
$$e^{-i\theta P} = e^{-i\theta}\Pi_0 + e^{i\theta}\Pi_1$$

$$e^{-i\theta Z} = e^{-i\theta}|0\rangle\langle 0| + e^{i\theta}|1\rangle\langle 1|$$

$$I \otimes \Pi_0 + X \otimes \Pi_1$$




$$|0\rangle\langle 0| \otimes e^{-i\theta P} + |1\rangle\langle 1| \otimes e^{i\theta P}$$

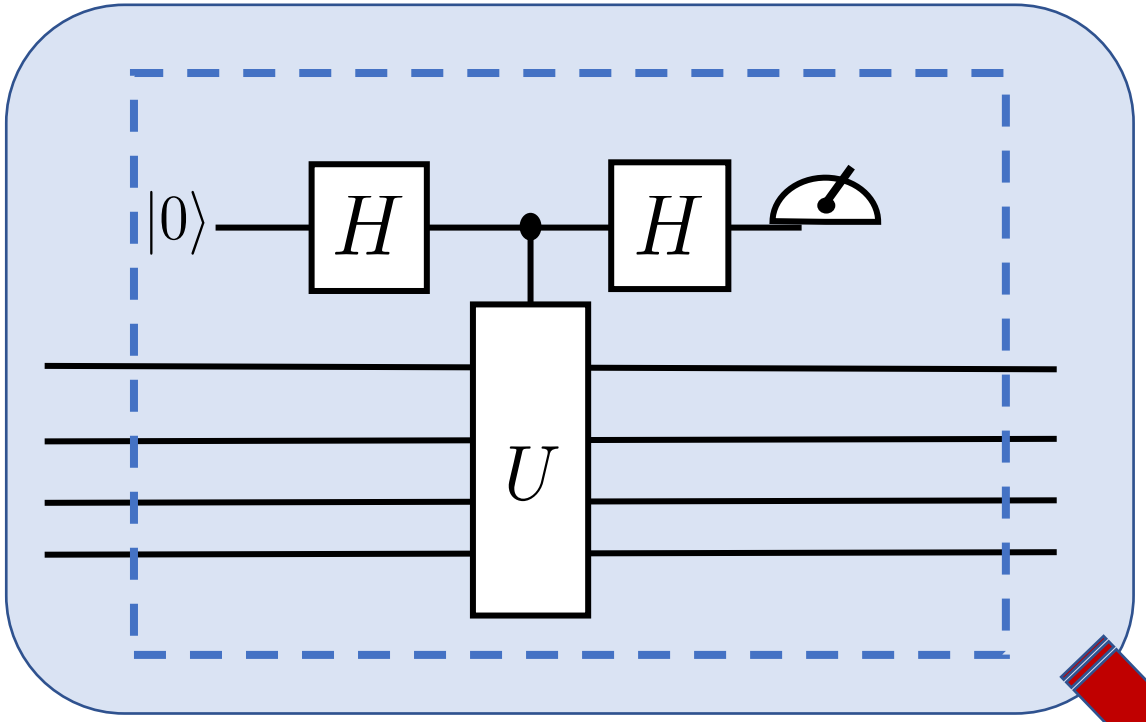
$$\Pi_0\Pi_1 = 0$$

$$(I \otimes \Pi_0 + X \otimes \Pi_1)(e^{-i\theta}|0\rangle\langle 0| \otimes I_{2^n} + e^{i\theta}|1\rangle\langle 1| \otimes I_{2^n})(I \otimes \Pi_0 + X \otimes \Pi_1)$$
$$= e^{-i\theta}|0\rangle\langle 0| \otimes \Pi_0 + e^{i\theta}|1\rangle\langle 1| \otimes \Pi_0 + e^{-i\theta}|1\rangle\langle 1| \otimes \Pi_1 + e^{+i\theta}|0\rangle\langle 0| \otimes \Pi_1$$
$$= |0\rangle\langle 0| \otimes (e^{-i\theta}\Pi_0 + e^{i\theta}\Pi_1) + |1\rangle\langle 1| \otimes (e^{i\theta}\Pi_0 + e^{-i\theta}\Pi_1)$$
$$= |0\rangle\langle 0| \otimes e^{-i\theta P} + |1\rangle\langle 1| \otimes e^{i\theta P}$$



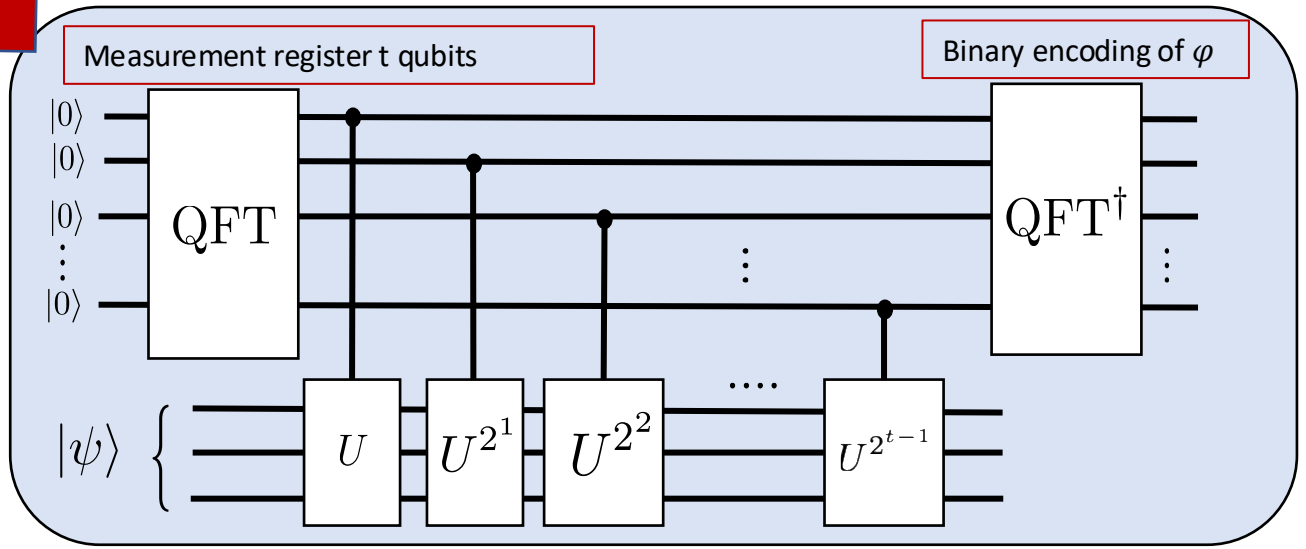
# Quantum Fourier Transform

# Generalization to multiple-outcome measurement




Hadamard test (last lecture)

Phase-estimation circuit (next lecture)

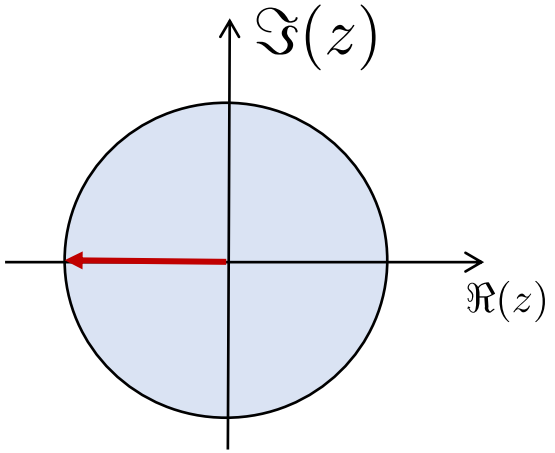


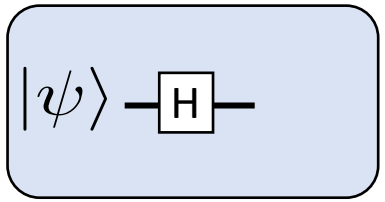
# From Hadamard to QFT

Hadamard gate

$$-1 = e^{i\pi}$$

$$(-1)^2 = 1$$



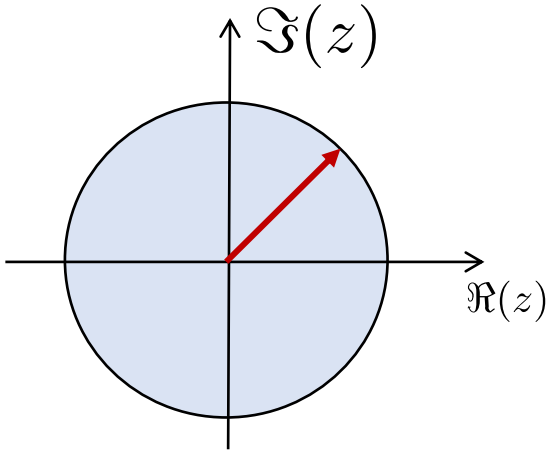

$$|x\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} (-1)^{xy}|y\rangle$$

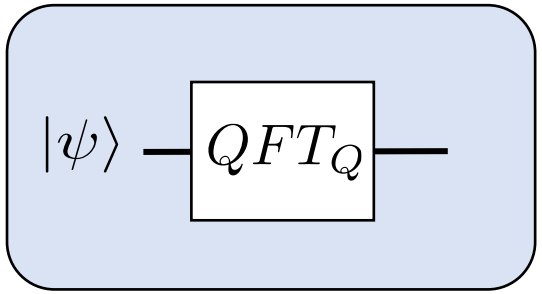
$$(-1)^x(-1)^y = (-1)^{x\oplus y}$$

Fourier transform over $Z_Q$

$$\omega = e^{i2\pi/Q}$$

$$\omega^Q = 1$$




$$|x\rangle \xrightarrow{QFT_Q} \frac{1}{\sqrt{Q}} \sum_{y\in\mathbb{Z}_Q} \omega^{xy}|y\rangle$$

Basis of size $Q$ : $|0\rangle, |1\rangle, ..., |Q-1\rangle$

$$\omega^x\omega^y = e^{ix2\pi/Q}e^{iy2\pi/Q} = \omega^{(x+y \mod Q)}$$

# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

Binary representation of integers:

$$z \equiv z_1 z_2 \dots z_n$$

$$z = z_1 2^{n-1} + z_2 2^{n-2} + \dots + z_{n-1} 2 + z_n$$

Binary fraction:

$$0.w_l w_{l+1} \dots w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \dots + \frac{w_m}{2^{m-l+1}}$$

# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x\rangle \xrightarrow{QFT_{2^n}} \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

$$H|x_1\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^{x_1}|1\rangle]$$

$$H|x_1\rangle = \frac{1}{2^{1/2}}[|0\rangle + e^{2\pi i 0.x_1}|1\rangle]$$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + ... + z_{n-1} 2 + z_n$

Binary fraction: $0.w_l w_{l+1} \ldots w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \cdots + \frac{w_m}{2^{m-l+1}}$

We will now prove

$$|x_1, x_2, ..., x_{n-1}, x_n\rangle \to \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle) \ldots (|0\rangle + e^{2\pi i 0.x_1 x_2 \ldots x_n}|1\rangle)$$

# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x\rangle \to \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + \ldots + z_{n-1} 2 + z_n$

Binary fraction: $0.w_l w_{l+1} \ldots w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \cdots + \frac{w_m}{2^{m-l+1}}$

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} \cdots \sum_{y_n=0}^{1} e^{2\pi i x (\sum_{l=1}^{n} y_l 2^{-l})} |y_1, \ldots, y_n\rangle$$

re-writing $y$ in binary notation

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} \cdots \sum_{y_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i x y_l 2^{-l}} |y_l\rangle$$

exp. of sum = product of exp.

Property: $c(|u\rangle \otimes |v\rangle) = (c|u\rangle) \otimes |v\rangle = |u\rangle \otimes (c|v\rangle)$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ \sum_{y_l=0}^{1} e^{2\pi i x y_l 2^{-l}} |y_l\rangle \right]$$

Property: $|u\rangle \otimes (|v_1\rangle + |v_2\rangle) = |u\rangle \otimes |v_1\rangle + |u\rangle \otimes |v_2\rangle$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ |0\rangle + e^{2\pi i x 2^{-l}} |1\rangle \right]$$

Expanding each sum

# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x\rangle \to \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + ... + z_{n-1} 2 + z_n$

Binary fraction: $0.w_l w_{l+1} \dots w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \cdots + \frac{w_m}{2^{m-l+1}}$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ \sum_{y_l=0}^{1} e^{2\pi i x y_l 2^{-l}} |y_l\rangle \right]$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ |0\rangle + e^{2\pi i x 2^{-l}} |1\rangle \right] = \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ |0\rangle + e^{2\pi i 0.x_{n-l+1} \dots x_n} |1\rangle \right]$$

$$z = z_1 2^{n-1} + z_2 2^{n-2} + \cdots + z_{n-1} 2 + z_n \Rightarrow \quad e^{2\pi i z} = e^{2\pi i (z_1 2^{n-1} + z_2 2^{n-2} + \cdots + z_{n-1} 2 + z_n)} = 1 (z \in \mathbb{Z})$$

$$\frac{z}{2} = z_1 2^{n-2} + z_2 2^{n-3} + \cdots + z_{n-2} 2 + z_{n-1} + \frac{z_n}{2} \Rightarrow \quad e^{2\pi i \frac{z}{2}} = e^{2\pi i (z_1 2^{n-2} + z_2 2^{n-3} + \cdots + z_{n-2} 2 + z_{n-1})} e^{2\pi i \frac{z_n}{2}} = e^{2\pi i 0.z_n}$$

$$\frac{z}{4} = z_1 2^{n-3} + z_2 2^{n-4} + \cdots + z_{n-2} + \frac{z_{n-1}}{2} + \frac{z_n}{4} \Rightarrow \quad e^{2\pi i \frac{z}{4}} = e^{2\pi i (z_1 2^{n-3} + z_2 2^{n-4} + \cdots + z_{n-2})} e^{2\pi i (\frac{z_{n-1}}{2} + \frac{z_n}{4})} = e^{2\pi i 0.z_{n-1} z_n}$$

$$\vdots$$

$$\frac{z}{2^l} = z_1 2^{n-1-l} + \cdots + z_{n-l} + \frac{z_{n-l+1}}{2} + \cdots + \frac{z_n}{2^l}$$

$$e^{2\pi i z 2^{-l}} = e^{2\pi i (z_1 2^{n-l-1} + z_2 2^{n-l-2} + \cdots + z_{n-l})} e^{2\pi i 0.z_{n-l+1} z_{n-l+2} \dots z_n} = e^{2\pi i 0.z_{n-l+1} z_{n-l+2} \dots z_n}$$

# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

Binary representation: $z = z_1 2^{n-1} + z_2 2^{n-2} + ... + z_{n-1} 2 + z_n$

Binary fraction: $0.w_l w_{l+1} \dots w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \dots + \frac{w_m}{2^{m-l+1}}$

$$|x\rangle \to \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y / 2^n} |y\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} \dots \sum_{y_n=0}^{1} e^{2\pi i x (\sum_{l=1}^{n} y_l 2^{-l})} |y_1, \dots, y_n\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{y_1=0}^{1} \dots \sum_{y_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i x y_l 2^{-l}} |y_l\rangle$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ \sum_{y_l=0}^{1} e^{2\pi i x y_l 2^{-l}} |y_l\rangle \right]$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ |0\rangle + e^{2\pi i x 2^{-l}} |1\rangle \right] = \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ |0\rangle + e^{2\pi i 0.x_{n-l+1} \dots x_n} |1\rangle \right]$$

$$x_1, x_2, ..., x_{n-1}, x_n\rangle \to \frac{1}{2^{n/2}} (|0\rangle + e^{2\pi i 0.x_n} |1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1} x_n} |1\rangle) ... (|0\rangle + e^{2\pi i 0.x_1 x_2 \dots x_n} |1\rangle)$$

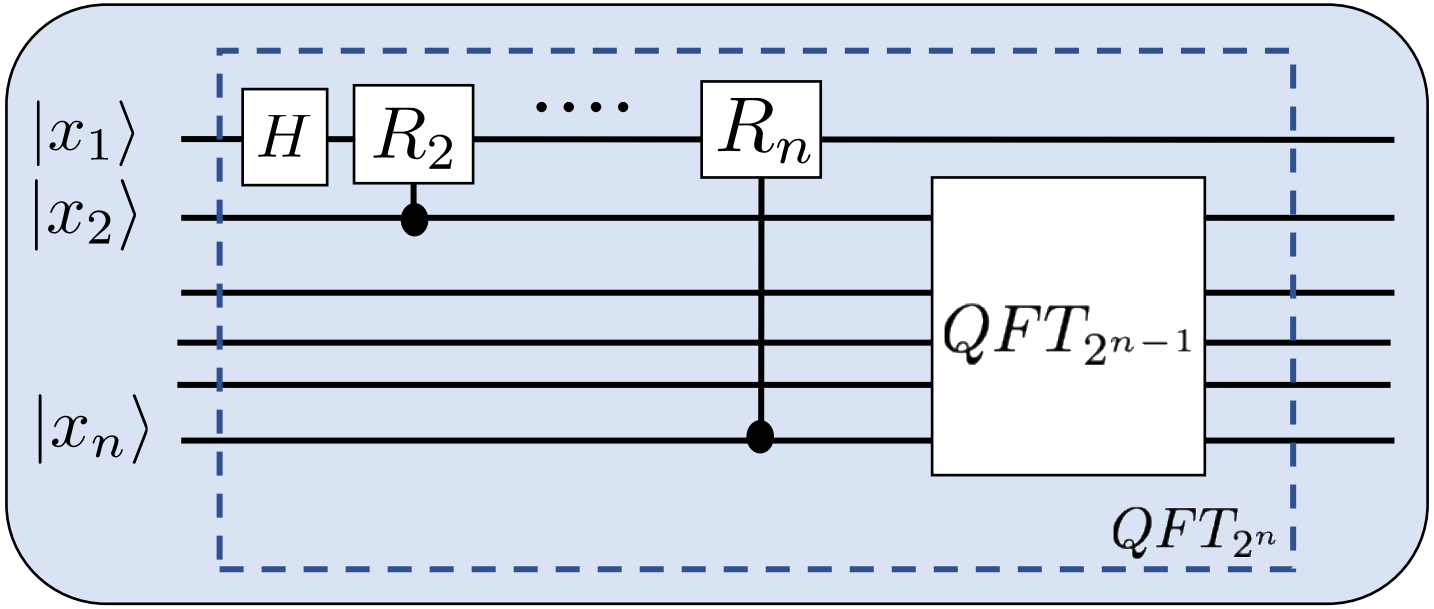
# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x_1, x_2, \ldots, x_{n-1}, x_n\rangle \rightarrow \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle)\cdots(|0\rangle + e^{2\pi i 0.x_1 x_2 \ldots x_n}|1\rangle)$$

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{bmatrix}$$

Binary fraction:

$$0.w_l w_{l+1} \ldots w_m = \frac{w_l}{2} + \frac{w_{l+1}}{2^2} + \cdots + \frac{w_m}{2^{m-l+1}}$$



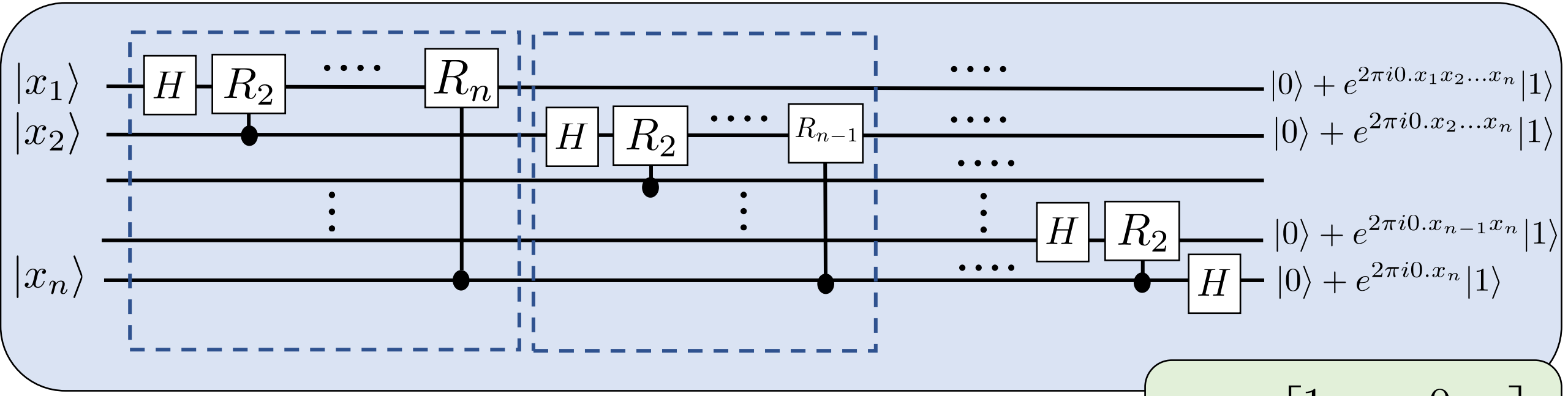

$$|x_1\rangle \otimes |x_2, \ldots, x_n\rangle \xrightarrow{H} \frac{1}{2^{1/2}}\left[|0\rangle + e^{2\pi i 0.x_1}|1\rangle\right] \otimes |x_2, \ldots, x_n\rangle$$

$$\xrightarrow{cR_2} \frac{1}{2^{1/2}}\left[|0\rangle + e^{2\pi i 0.x_1 x_2}|1\rangle\right] \otimes |x_2, \ldots, x_n\rangle$$

$$\xrightarrow{cR_n} \frac{1}{2^{1/2}}\left[|0\rangle + e^{2\pi i 0.x_1 x_2 \ldots x_n}|1\rangle\right] \otimes |x_2, \ldots, x_n\rangle$$

# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

$$|x_1, x_2, \ldots, x_{n-1}, x_n\rangle \rightarrow \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle)\cdots(|0\rangle + e^{2\pi i 0.x_1 x_2 \ldots x_n}|1\rangle)$$



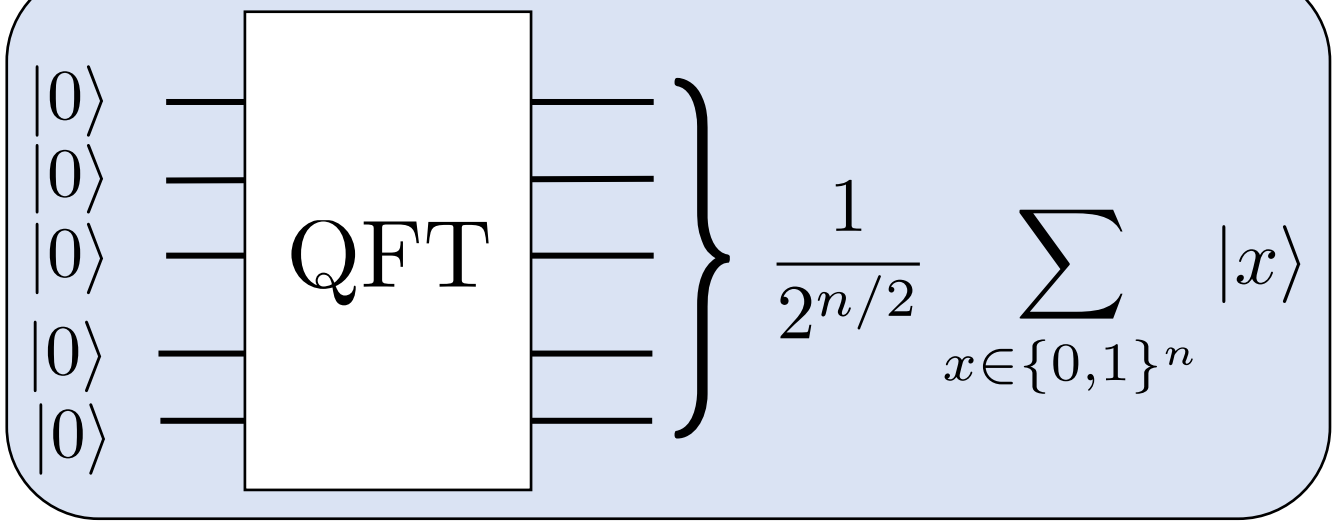

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{bmatrix}$$

- QFT up to a reverse of the order of qubits.
- Unitary as composed of unitary gates.
- $n + (n-1) + \ldots + 1 = n(n+1)/2$ gates are required
  $+\lfloor n/2 \rfloor$ SWAP gates (3 CNOT each)

$\Theta(n^2)$ gates

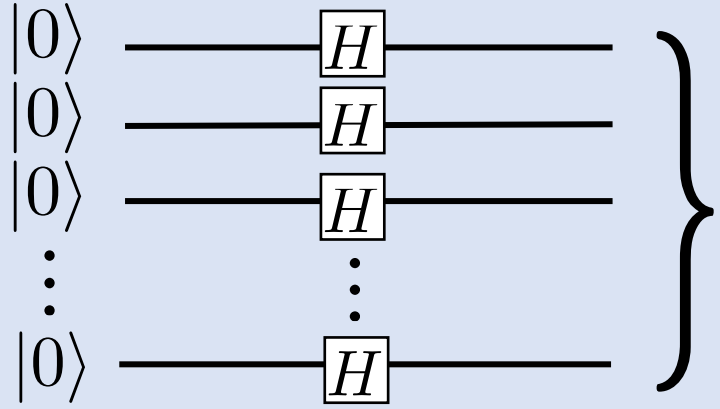
# Parallelization

$$|x_1, x_2, \ldots, x_{n-1}, x_n\rangle \rightarrow \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle)\cdots(|0\rangle + e^{2\pi i 0.x_1 x_2 \ldots x_n}|1\rangle)$$




$$\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$|x\rangle \xrightarrow{QFT_{2^n}} \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y / 2^n} |y\rangle$$




$$\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$$

# Quantum Fourier Transform over $\mathbb{Z}_{2^n}$

- Quantum circuit: $\Theta(n^2)$ gates.
- Classical FFT: $\Theta(n2^n)$ operations on a $2^n$ vector.

the first step in phoneme recognition is to Fourier transform the digitized sound. Can we use the quantum Fourier transform to speed up the computation of these Fourier transforms? Unfortunately, the answer is that there is no known way to do this. The problem is that the amplitudes in a quantum computer cannot be directly accessed by measurement. Thus, there is no way of determining the Fourier transformed amplitudes of the original state. Worse still, there is in general no way to efficiently prepare the original state to be Fourier transformed. Thus, finding uses for the quantum Fourier transform is more subtle than we might have hoped. In this and the next chapter we develop several algorithms based upon a more subtle application of the quantum Fourier transform.

Extract from NC page 220

- A quantum computer does not output the wave-function amplitudes!
  - A quantum computer generates outputs (samples) from a probability distribution $|\psi(x)|^2$
- Quantum mechanics allows you to FFT an exponentially big vector but...

  ... encoding an exponentially big classical dataset in a quantum state is highly non-trivial
- Quantum Machine Learning to be practical needs to deal with a similar issue.

# References

**Reading references**

1. Pauli rotation gadget: NC 4.7.3
2. Quantum simulation: NC 4.7
3. Quantum Fourier Transform: NC 5.1
4. Additional references RdW Ch4, G Ch9

NC ≣ Michael Nielsen and Isaac Chuang, Quantum Computation and Quantum Information Cambridge University Press (2010)

RdW ≣ Quantum Computing Lecture Notes, Ronald de Wolf, https://arxiv.org/abs/1907.09415

G ≣ Introduction to Quantum Computation, Sevag Gharibian, Lecture notes link