



THE UNIVERSITY of EDINBURGH
informatics

Introduction to Quantum Computing

Lecture 2: Postulate I – Quantum States

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The ideal life of a qubit in a nutshell

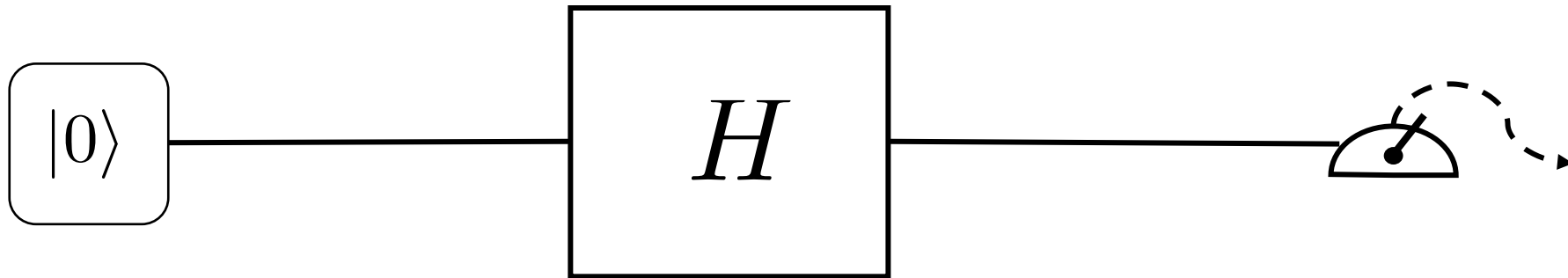
State preparation

Source of quantum states

Operation

Circuit/Gates

Measurement

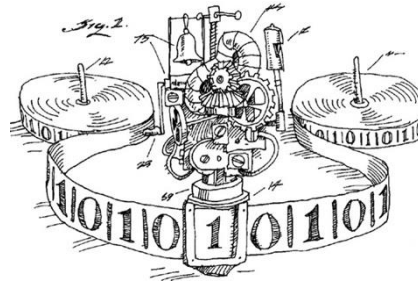


Quantum random number generator circuit

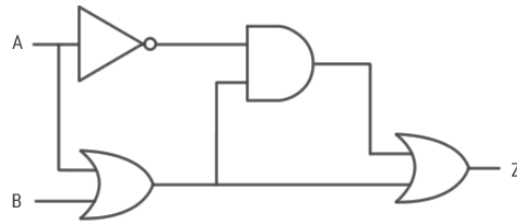
Classical bits

Today computation is based on the computation on binary registers (bits)

- Turing machines

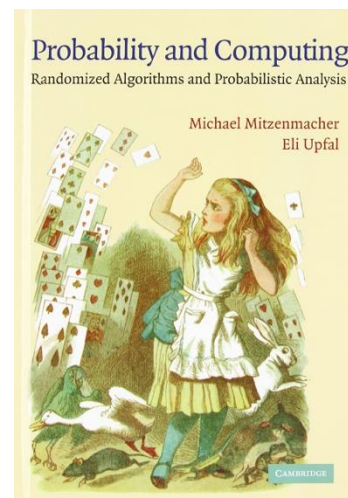


- Circuits



- Probabilistic computation

- Simulated Annealing
- Primality Test



A discrete probability space

- Sample space Ω : the set of all possible outcomes.

$$\Omega = \{0, 1, \dots, d-1\}$$

$$n \text{ bits} : d = 2^n$$

- Axioms (state of the system)

$$\bullet \quad \bar{p} = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{d-1} \end{bmatrix} \in \mathbb{R}^d \quad \bar{p} = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{d-1} \end{bmatrix} = p_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + p_{d-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\bullet \quad p_i \geq 0 \text{ (Non-negativeness)}$$

$$\bullet \quad P(\Omega) = \sum_{i=0}^{d-1} p_i = 1 \text{ (Normalization)}$$

Classical bit

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = p_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Quantum Bit or Qubit

Logical zero $|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



Logical one $|1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



These two states (logical 0 and 1) form an orthonormal basis of a qubit.

General state of a qubit $|\psi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}$

$\psi_i \in \mathbb{C}$: probability amplitudes

where $|\psi_0|^2 + |\psi_1|^2 = 1$ (Normalization)

$$|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle \equiv \psi_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \psi_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

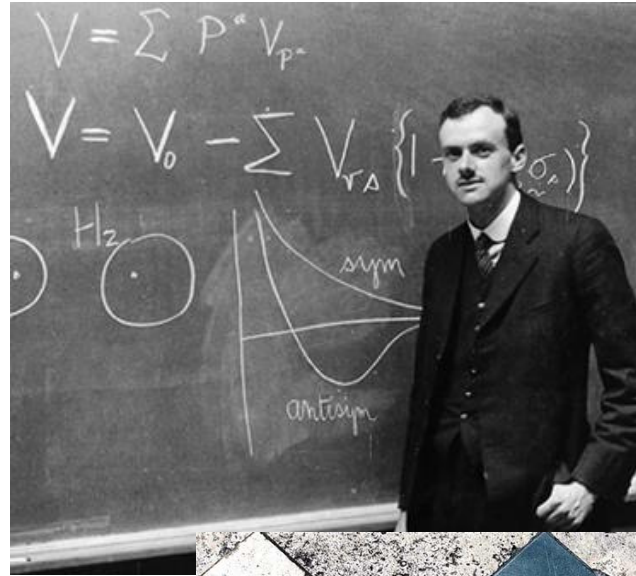
Linearity of QM allows us to write any qubit state as a linear combination of logical 0 and 1, as they are a basis for the qubit state space.

Dirac Notation

Dirac notation

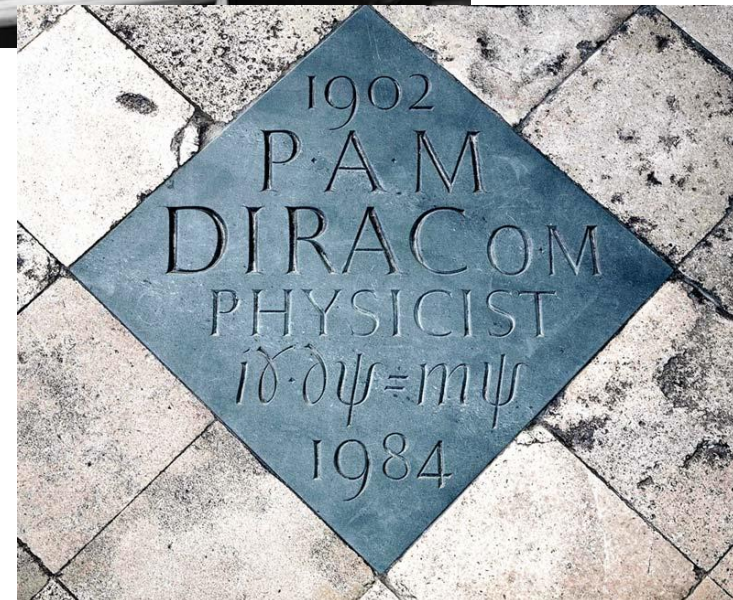
$|\psi\rangle$ ket

$$|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle \equiv \psi_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \psi_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Paul Dirac: Nobel Prize 1933
Fundamental contributions to:

- quantum mechanics
- quantum electrodynamics



Westminster Abbey (London)

Postulate I: Quantum states

A quantum state with d degrees of freedom is a vector belonging to a Hilbert space of dimension d and norm 1.

Hilbert space = Complex vector space + Inner product

State vector $|\psi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix} = \psi_0|0\rangle + \psi_1|1\rangle + \dots + \psi_{d-1}|d-1\rangle$

$\psi_i \in \mathbb{C}$: Probability amplitude of degree of freedom i

Normalization: $\sum_{i=0}^{d-1} \psi_i^* \psi_i = \sum_{i=0}^{d-1} |\psi_i|^2 = 1$ n qubits : $d = 2^n$

Addition: $\mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$

$$|\psi\rangle + |\phi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix} + \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{d-1} \end{bmatrix} = \begin{bmatrix} \psi_0 + \phi_0 \\ \psi_1 + \phi_1 \\ \vdots \\ \psi_{d-1} + \phi_{d-1} \end{bmatrix}$$

Scalar multiplication: $\mathbb{C} \times \mathcal{H} \rightarrow \mathcal{H}$

$$\lambda|\psi\rangle \equiv \lambda \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix} = \begin{bmatrix} \lambda\psi_0 \\ \lambda\psi_1 \\ \vdots \\ \lambda\psi_{d-1} \end{bmatrix}$$

The zero vector: $|\emptyset\rangle \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Inner product in a nutshell

- $(\cdot, \cdot) : V \times V \rightarrow \mathbb{F}$

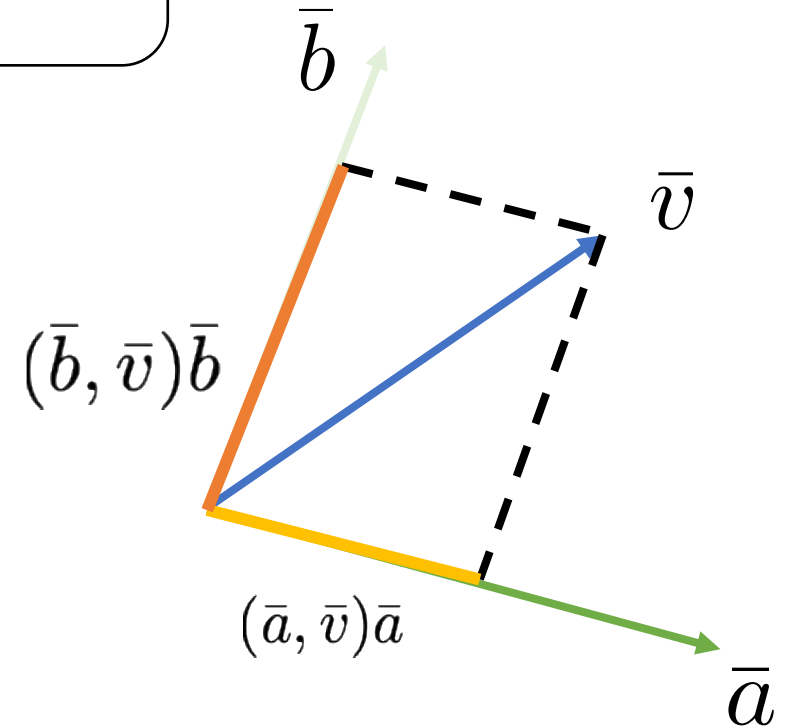
$$(\bar{x}, \bar{y}) = \bar{x} \cdot \bar{y} = \sum_{i=0}^{d-1} x_i y_i$$

$\bar{a}$ and $\bar{b}$ orthonormal basis:

$$(\bar{a}, \bar{a}) = (\bar{b}, \bar{b}) = 1$$

$$(\bar{a}, \bar{b}) = 0$$

$$\bar{v} = (\bar{a}, \bar{v})\bar{a} + (\bar{b}, \bar{v})\bar{b}$$



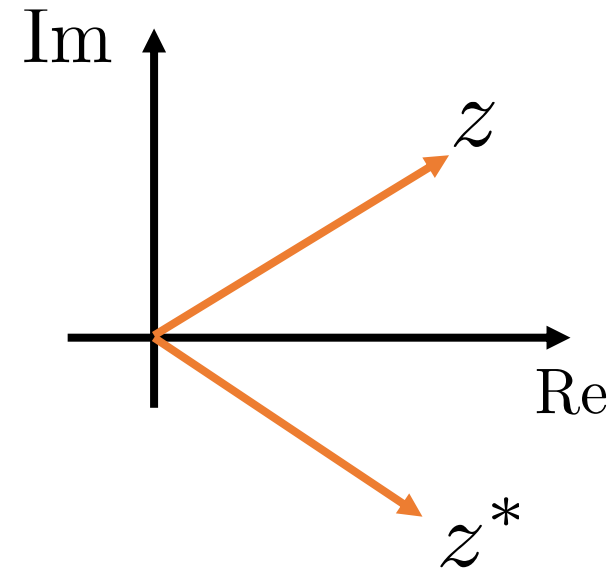
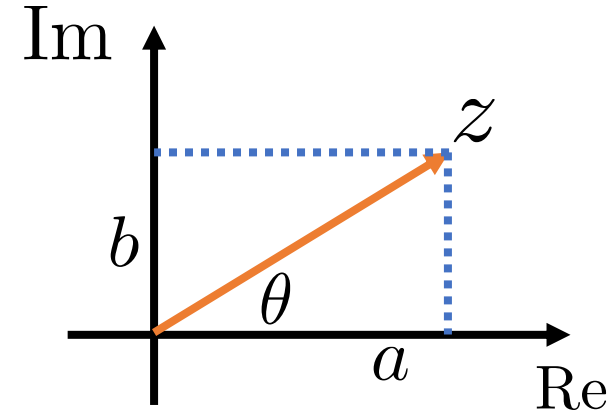
- Norm: $V \rightarrow \mathbb{R}_{\geq 0} \quad \|\bar{v}\| = \sqrt{(\bar{v}, \bar{v})} \geq 0$
- Orthogonal transformation preserves inner product

Complex number in a nutshell

- $\mathbb{C} = \{z = a + ib \mid (a, b) \in \mathbb{R}^2 \text{ and } i^2 = -1\}$

Trigonometric form: $z = |z|(\cos \theta + i \sin \theta)$

- Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
- Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$
- Conjugation: $z^* = a - ib$
- Norm: $|z| = \sqrt{z z^*}$



Inner product: $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$

$$(|\psi\rangle, |\phi\rangle) = \sum_{i=0}^{d-1} \psi_i^* \phi_i = \langle\psi|\phi\rangle$$

Bra

$$\langle\psi| \equiv [\psi_0^* \quad \psi_1^* \quad \dots \quad \psi_{d-1}^*]$$

$$\langle\psi| = |\psi\rangle^\dagger = (|\psi\rangle^T)^*$$

Hilbert Space: Inner product (Dirac notation)

$$\mathcal{H} \equiv \mathbb{C}^d$$

Inner product: $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$

$$(|\psi\rangle, |\phi\rangle) = [\psi_0^* \quad \psi_1^* \quad \dots \quad \psi_{d-1}^*] \times \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{d-1} \end{bmatrix} = \sum_{i=0}^{d-1} \psi_i^* \phi_i$$

Bra

$$\langle\psi| \equiv [\psi_0^* \quad \psi_1^* \quad \dots \quad \psi_{d-1}^*]$$

Ket

$$|\phi\rangle \equiv \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{d-1} \end{bmatrix}$$

Inner product: $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$

$$\langle\psi|\phi\rangle = \sum_{i=0}^{d-1} \psi_i^* \phi_i$$

Inner product: $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$

$$(|\psi\rangle, |\phi\rangle) = \sum_{i=0}^{d-1} \psi_i^* \phi_i = \langle\psi|\phi\rangle$$

Norm: $\mathcal{H} \rightarrow \mathbb{R}_{\geq 0}$

$$|||\psi\rangle|| = \sqrt{\langle\psi|\psi\rangle} \geq 0$$

$$\bullet \quad |||\psi\rangle + |\phi\rangle|| \leq |||\psi\rangle|| + |||\phi\rangle||$$

$$\bullet \quad \text{Quantum states have norm 1: } |||\psi\rangle|| = 1$$

Bra

$$\langle\psi| \equiv [\psi_0^* \quad \psi_1^* \quad \dots \quad \psi_{d-1}^*]$$

$$\langle\psi| = |\psi\rangle^\dagger = (|\psi\rangle^T)^*$$

Triangle inequality

Norm being 1 is associated with the fact that measurement outcome probabilities should sum to one. The QM equivalent of the normalization axiom for classical systems (slide 4).

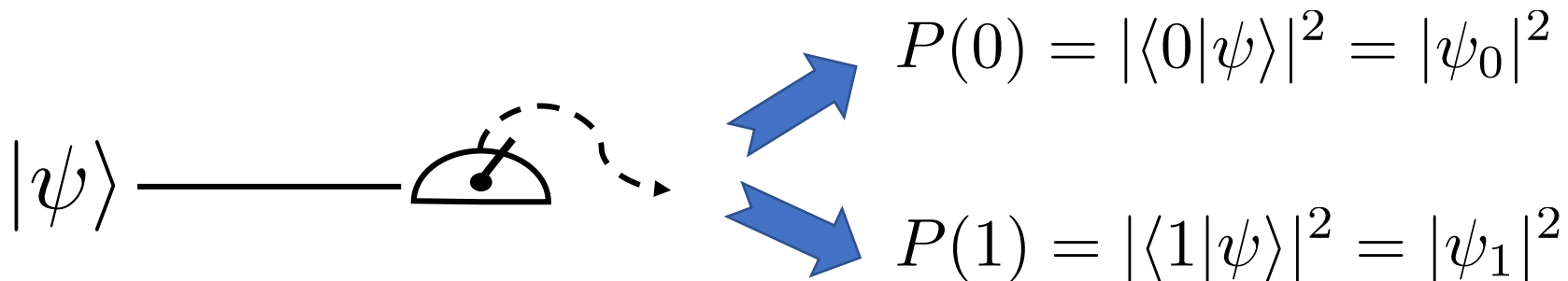
Computational basis

$$\mathcal{H}_Q = \text{Span}\{|0\rangle, |1\rangle\}$$

- $\forall |\psi\rangle, \exists \psi_0 \text{ and } \psi_1 : |\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$
- $\langle 0|1\rangle = 0$ (Orthogonal basis)
- $||0\rangle|| = ||1\rangle|| = 1$ (Normalized basis)

This ensures logical 0 and 1 are an orthonormal basis of a Hilbert space of dim 2.

Computational basis measurement



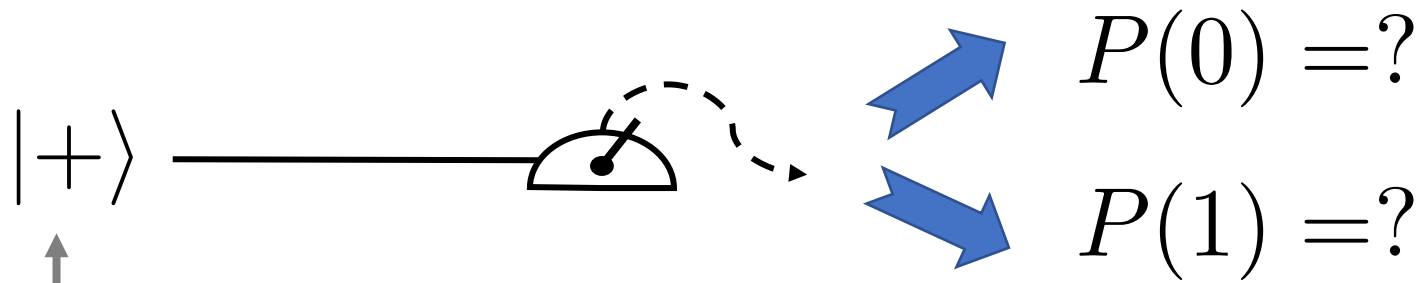
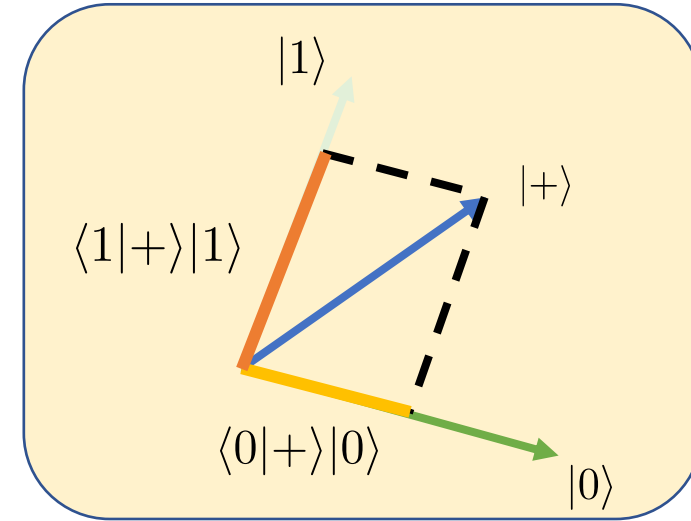
The amplitudes of the quantum state on the logical basis (0 and 1) are associated with the outcome probabilities of the computational basis measurement (logical 0 or 1).

Quantum randomness generation

Computational basis
measurement

$$P(0) = |\langle 0|\psi\rangle|^2 = |\psi_0|^2$$

$$P(1) = |\langle 1|\psi\rangle|^2 = |\psi_1|^2$$



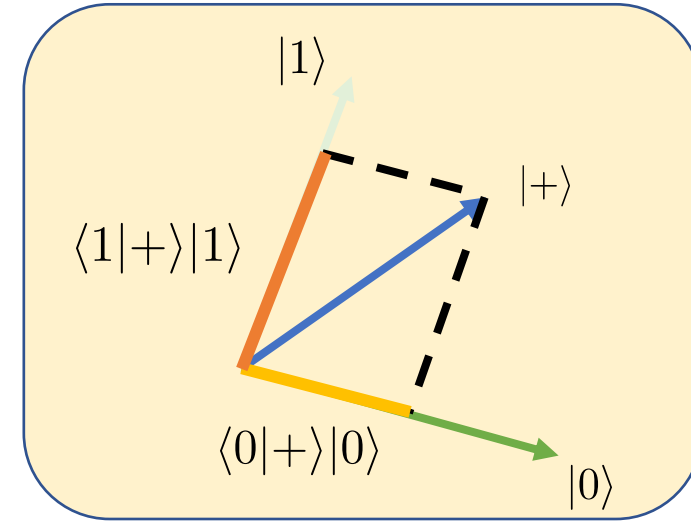
$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

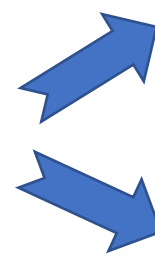
Quantum randomness generation

Computational basis
measurement

$$P(0) = |\langle 0|\psi\rangle|^2 = |\psi_0|^2$$

$$P(1) = |\langle 1|\psi\rangle|^2 = |\psi_1|^2$$



$|+\rangle$ —————  

$$P(0) = |\langle 0|+\rangle|^2 = 1/2$$
$$P(1) = |\langle 1|+\rangle|^2 = 1/2$$

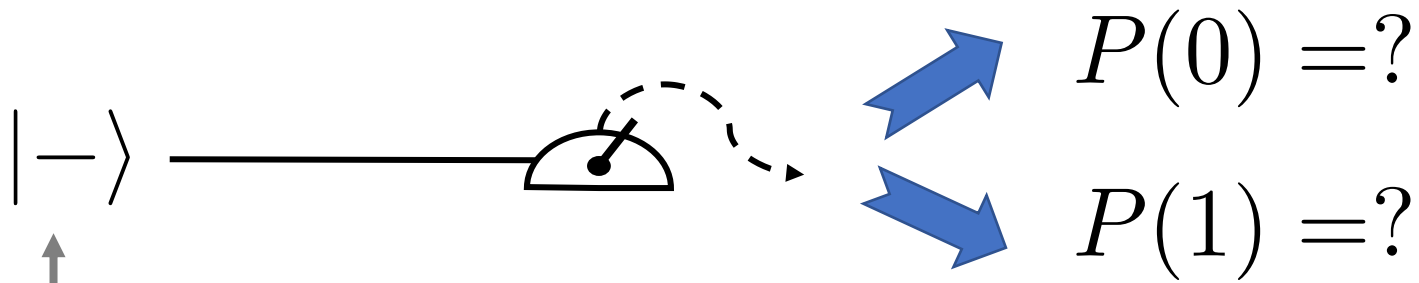
$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Quantum randomness generation

Computational basis
measurement

$$P(0) = |\langle 0|\psi\rangle|^2 = |\psi_0|^2$$

$$P(1) = |\langle 1|\psi\rangle|^2 = |\psi_1|^2$$



$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Quantum randomness generation

Computational basis
measurement

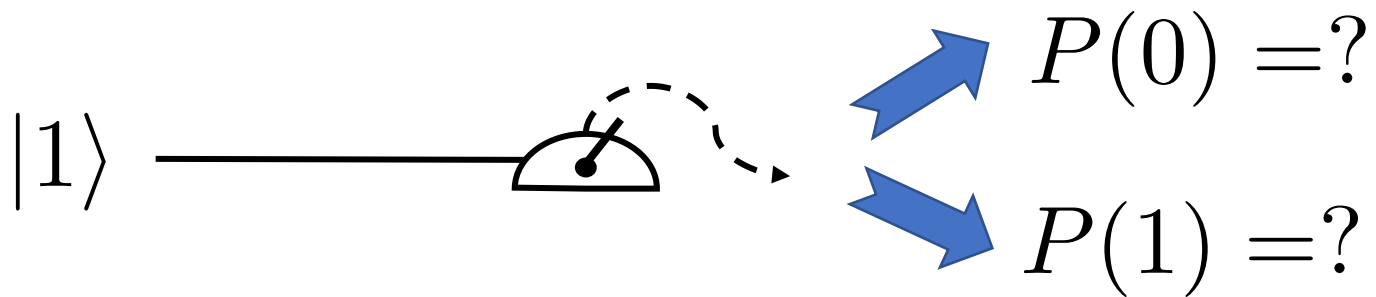
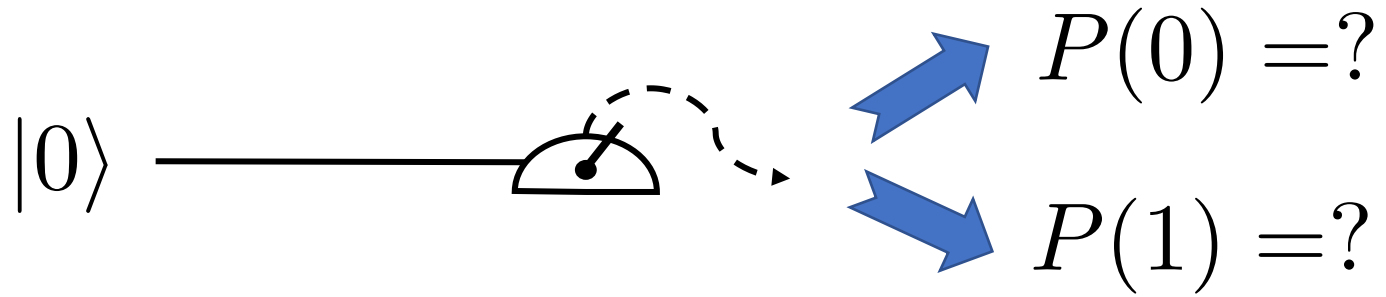
$$P(0) = |\langle 0 | \psi \rangle|^2 = |\psi_0|^2$$

$$P(1) = |\langle 1 | \psi \rangle|^2 = |\psi_1|^2$$

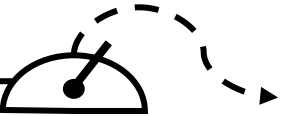
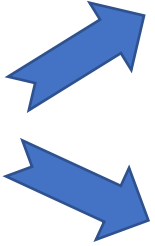
$| - \rangle$ —————  $\begin{cases} P(0) = |\langle 0 | - \rangle|^2 = 1/2 \\ P(1) = |\langle 1 | - \rangle|^2 = 1/2 \end{cases}$

$$| - \rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

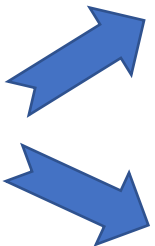
Quantum randomness generation



Quantum randomness generation

$|0\rangle$ —————  

$$P(0) = |\langle 0|0\rangle|^2 = 1$$
$$P(1) = |\langle 1|0\rangle|^2 = 0$$

$|1\rangle$ —————  

$$P(0) = |\langle 0|1\rangle|^2 = 0$$
$$P(1) = |\langle 1|1\rangle|^2 = 1$$

Postulate I: Quantum states

A quantum state with d degrees of freedom is represented by a vector in a complex vector space with inner product (Hilbert space)

$$|\psi\rangle \in \mathcal{H} \equiv \mathbb{C}^d \qquad \langle\psi|\psi\rangle = 1$$

Hilbert space = Complex vector space + Inner product

State vector $|\psi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix}$

A d -dimensional vector of complex numbers

- Qubit: $d = 2$
- Qudit: $d > 2$
- n qubits: $d = 2^n$

$\psi_i = \langle i|\psi\rangle \in \mathbb{C}$: Probability amplitude of computational state i

$$P(i) = |\psi_i|^2$$

References

Course Material

1. *Quantum Computation and Quantum Information*, Michael Nielsen and Isaac Chuang, Cambridge University Press (2010) [Sections 1.2, 2.1.1-2.1.4].

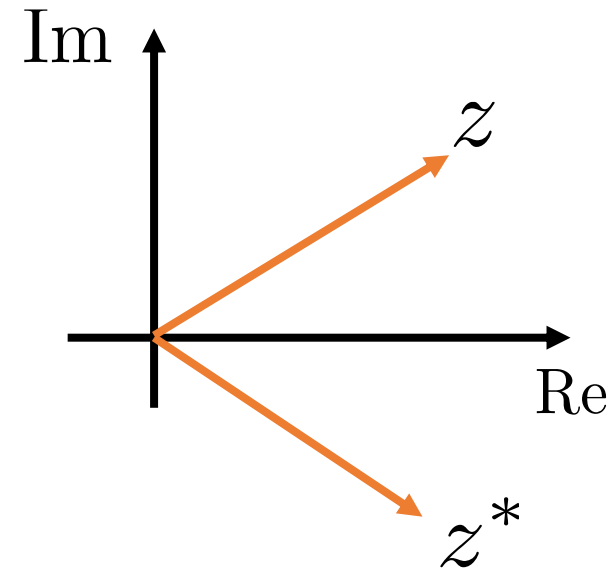
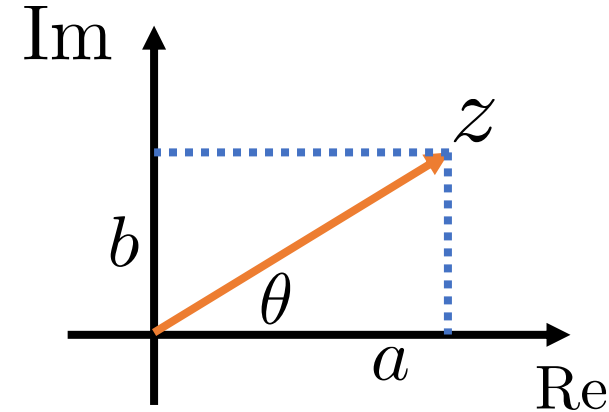
Appendix

Complex number in a nutshell

- $\mathbb{C} = \{z = a + ib | (a, b) \in \mathbb{R}^2 \text{ and } i^2 = -1\}$

Trigonometric form: $z = |z|(\cos \theta + i \sin \theta)$

- Norm: $|z|^2 = zz^* = a^2 + b^2$
- Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
- Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$
- Conjugation: $z^* = a - ib$
 $(z_1 + z_2)^* = z_1^* + z_2^* \quad (z_1 z_2)^* = z_1^* z_2^*$
- Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta \Rightarrow z = |z|e^{i\theta}$
- Multiplication: $z_1 z_2 = |z_1||z_2|e^{i(\theta_1 + \theta_2)}$



Addition: $\mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$

$$|\psi\rangle + |\phi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix} + \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{d-1} \end{bmatrix} = \begin{bmatrix} \psi_0 + \phi_0 \\ \psi_1 + \phi_1 \\ \vdots \\ \psi_{d-1} + \phi_{d-1} \end{bmatrix}$$

Associativity: $|\psi_1\rangle + (|\psi_2\rangle + |\psi_3\rangle) = (|\psi_1\rangle + |\psi_2\rangle) + |\psi_3\rangle$

Commutativity: $|\psi_1\rangle + |\psi_2\rangle = |\psi_2\rangle + |\psi_1\rangle$

Neutral element: $|\psi\rangle + |\emptyset\rangle = |\psi\rangle$

Inverse element: $\forall |\psi\rangle, \exists |\chi\rangle$ s.t. $|\psi\rangle + |\chi\rangle = |\emptyset\rangle$

The zero vector: $|\emptyset\rangle \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Scalar multiplication: $\mathbb{C} \times \mathcal{H} \rightarrow \mathcal{H}$

$$\lambda|\psi\rangle \equiv \lambda \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix} = \begin{bmatrix} \lambda\psi_0 \\ \lambda\psi_1 \\ \vdots \\ \lambda\psi_{d-1} \end{bmatrix}$$

Compatibility: $\lambda(\nu|\psi\rangle) = (\lambda\nu)|\psi\rangle$

(Compatibility of product of scalars with scalar multiplication)

Distributivity (vector addition): $\lambda(|\psi_1\rangle + |\psi_2\rangle) = \lambda|\psi_1\rangle + \lambda|\psi_2\rangle$

Distributivity (field addition): $(\lambda + \nu)|\psi\rangle = \lambda|\psi\rangle + \nu|\psi\rangle$

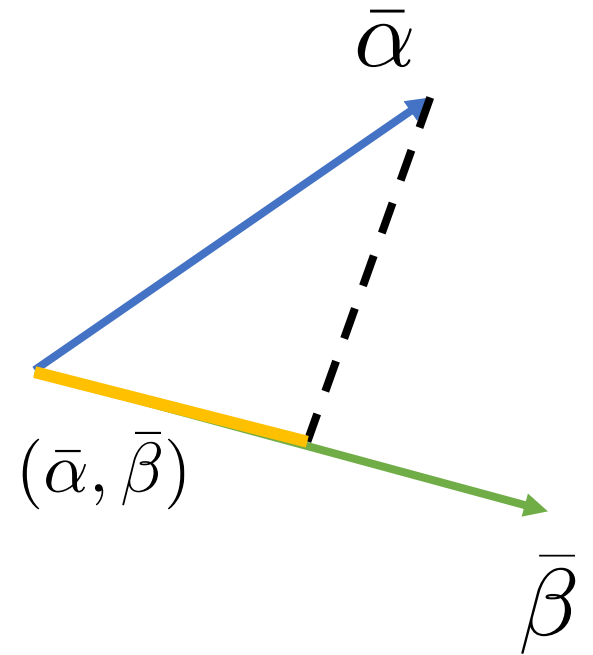
Identity element: $1|\psi\rangle = |\psi\rangle$

Multiplying by zero: $0|\psi\rangle = |\emptyset\rangle$

Inner product (Math notation)

Inner product $(\cdot, \cdot) : V \times V \rightarrow \mathbb{F}$

$$(\bar{\alpha}, \bar{\beta}) = [\alpha_0^* \quad \alpha_1^* \quad \dots \quad \alpha_{d-1}^*] \times \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{bmatrix} = \sum_{i=0}^{d-1} \alpha_i^* \beta_i$$



Linearity: $(\bar{\alpha}, \lambda \bar{\beta}) = \lambda (\bar{\alpha}, \bar{\beta})$

$$(\bar{\alpha}_1 + \bar{\alpha}_2, \bar{\beta}) = (\bar{\alpha}_1, \bar{\beta}) + (\bar{\alpha}_2, \bar{\beta})$$

Conjugate-linearity: $(\lambda \bar{\alpha}, \bar{\beta}) = \lambda^* (\bar{\alpha}, \bar{\beta})$

Conjugate symmetry: $(\bar{\alpha}, \bar{\beta}) = (\bar{\beta}, \bar{\alpha})^*$

Positive semi-definiteness: $(\bar{\alpha}, \bar{\alpha}) \geq 0$ $(\bar{\alpha}, \bar{\alpha}) = 0 \Leftrightarrow \bar{\alpha} = \bar{0}$

Inner product: $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$

$$\langle \psi | \phi \rangle = \sum_{i=0}^{d-1} \psi_i^* \phi_i$$

Ket $|\psi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix}$

Bra $\langle \psi | = |\psi\rangle^\dagger \equiv [\psi_0^* \quad \psi_1^* \quad \dots \quad \psi_{d-1}^*]$

Linearity: $(|\psi\rangle, \lambda|\phi\rangle) = \lambda\langle\psi|\phi\rangle = (|\psi\rangle)^\dagger(\lambda|\phi\rangle)$

$$(|\psi_1\rangle + |\psi_2\rangle, |\phi\rangle) = \langle\psi_1|\phi\rangle + \langle\psi_2|\phi\rangle$$

Conjugate-linearity: $(\lambda|\psi\rangle, |\phi\rangle) = \lambda^*\langle\psi|\phi\rangle = (\lambda|\psi\rangle)^\dagger|\phi\rangle$

Conjugate symmetry: $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

Positive semi-definiteness: $\langle\psi|\psi\rangle \geq 0 \qquad \langle\psi|\psi\rangle = 0 \Leftrightarrow |\psi\rangle = |\emptyset\rangle$

Norm: $\mathcal{H} \rightarrow \mathbb{R}_{\geq 0}$

$$|| |\psi\rangle || = \sqrt{\langle \psi | \psi \rangle} \geq 0$$

- $|| |\psi\rangle || = 0 \Leftrightarrow |\psi\rangle = |\emptyset\rangle$

- $|| \lambda |\psi\rangle || = |\lambda| || |\psi\rangle ||$

- $|| |\psi\rangle + |\phi\rangle || \leq || |\psi\rangle || + || |\phi\rangle ||$

Triangle inequality

- Quantum states have norm 1: $|| |\psi\rangle || = 1$

Norm being 1 is associated with the fact that measurement outcome probabilities should sum to one. The QM equivalent of the normalization axiom for classical systems (slide 4).