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Introduction to Quantum Computing

Lecture4: Postulate IV – Measurement

Raul Garcia-Patron Sanchez

Postulate I: Quantum states

A quantum state with d degrees of freedom is represented by a vector in a complex vector space with inner-product (Hilbert space)

$$|\psi\rangle \in \mathcal{H} \equiv \mathbb{C}^d \qquad \langle\psi|\psi\rangle = 1$$

Hilbert space = Complex Vector Space + Inner-product

State vector $|\psi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix}$

A d-dimensional vector of complex numbers

- Qubit: $d = 2$
- Qudit: $d > 2$
- N qubits: $d = 2^N$

$\psi_i \in \mathbb{C}$: Probability amplitude of degree of freedom i

$$P(i) = |\psi_i|^2$$

Postulate II: Quantum operations

The evolution of a quantum system $|\psi\rangle \in \mathcal{H} \equiv \mathbb{C}^d$ is given by a unitary tranformation $U : \mathcal{H} \to \mathcal{H}$, s.t. $|\psi_{out}\rangle = U|\psi_{in}\rangle$

Unitary matrices

$$\boxed{UU^\dagger = U^\dagger U = I_d}$$

- Linear operator $U : \mathcal{H} \to \mathcal{H}$
- Preserves the inner-product
- Equivalent of orthogonal matrices on real vector spaces

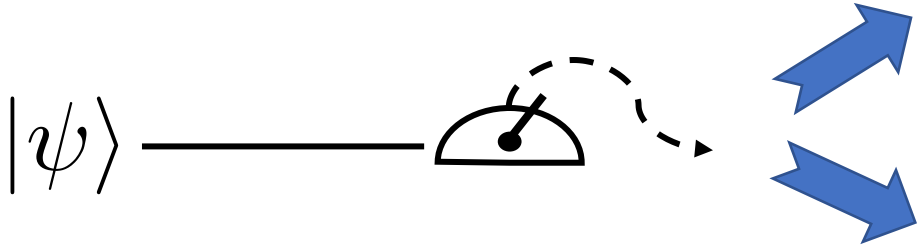
Basis: Computational basis $\mathcal{H} \equiv \mathbb{C}^d$

Computational basis $\mathcal{H}_{\mathcal{Q}} = \mathrm{Span}\{|0\rangle, |1\rangle\}$

- $\forall |\psi\rangle, \exists \psi_0 \text{ and } \psi_1 : |\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$
- $\langle 0|1\rangle = 0$ (Orthogonal basis)
- $|| \, |0\rangle|| = || \, |1\rangle|| = 1$ (Normalized basis)

This ensure logical 0 and 1 is an orthonormal basis of a Hilbert space of dim 2.

Computational basis measurement



$$P(0) = |\langle 0|\psi\rangle|^2 = |\psi_0|^2$$

$$P(1) = |\langle 1|\psi\rangle|^2 = |\psi_1|^2$$

The amplitudes of the quantum state on the logical basis (0 and 1) are associated with the outcome probabilities of the computational basis measurement (logical 0 or 1).

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The effect of measurement

Random coins vs quantum coins

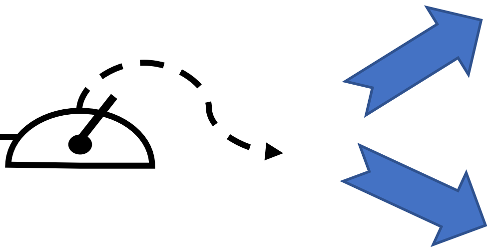
$$R = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

- Classical coin flip

$$\bar{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \boxed{R} \longrightarrow \bar{p}_{out} = R\bar{v}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

- Quantum random number generator circuit

$$|0\rangle \longrightarrow \boxed{H} \longrightarrow$$



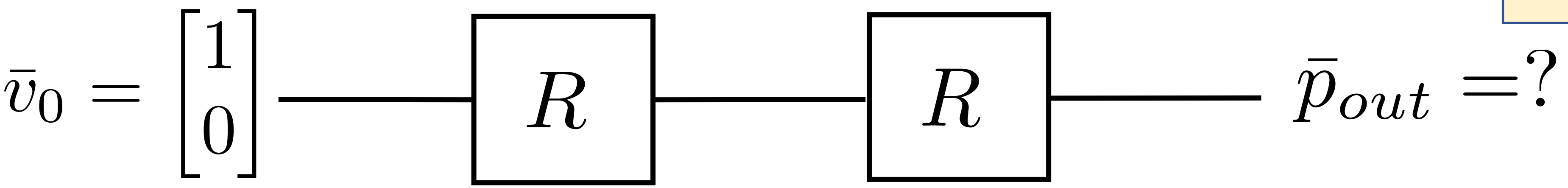
$$P(0) = |\psi_0|^2 = 1/2$$

$$P(1) = |\psi_1|^2 = 1/2$$

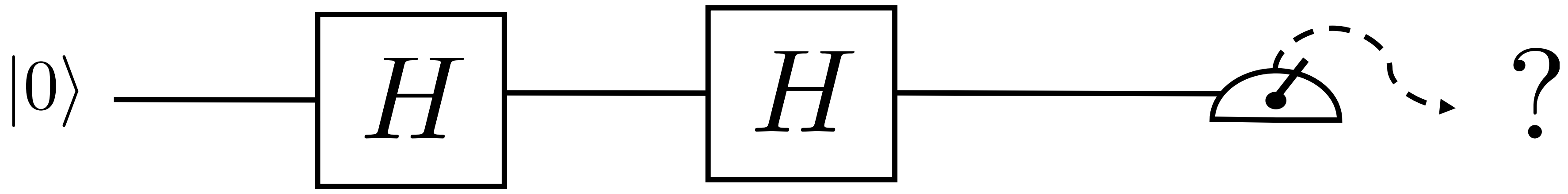
Random coins vs quantum coins: concatenation

$$R = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

- Classical coin flip



- Quantum random number generator circuit



Random coins vs quantum coins: concatenation

$$R = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

- Two classical coin flips

$$\bar{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ — } \boxed{R} \text{ — } \boxed{R} \text{ — } \quad \bar{p}_{out} = R^2 \bar{v}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

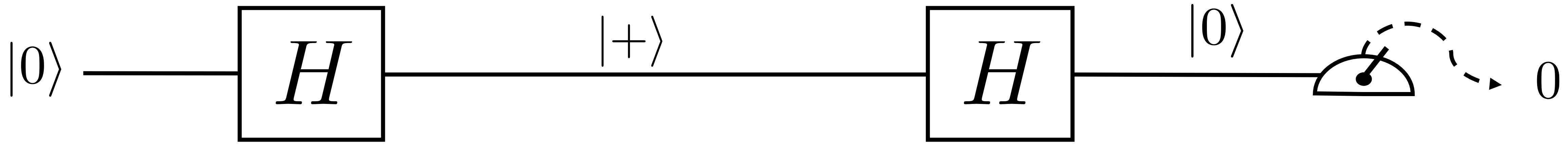
- Two "quantum coin flips"

$$|0\rangle \text{ — } \boxed{H} \text{ — } \boxed{H} \text{ — } |0\rangle \text{ — measurement } \rightarrow 0$$

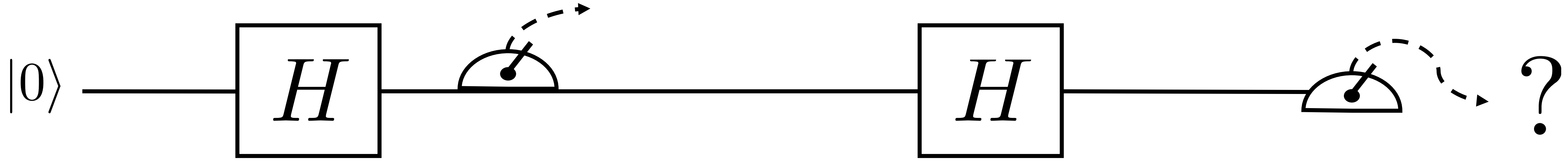
$$H^2 = I$$

To measure or not to measure....

- Two "quantum coin flips"



- Two "quantum coin flips" with intermediate measurement

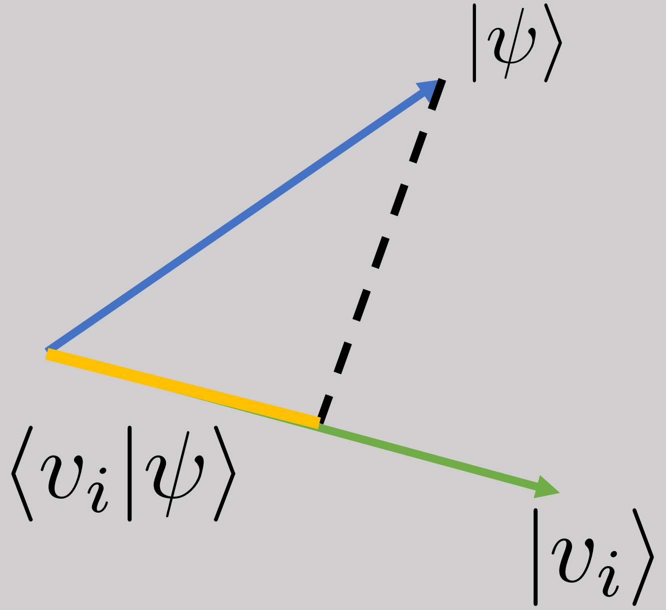


Measurement of orthonormal basis

Any orthonormal basis $\{|v_i\rangle\}$ that span $\mathcal{H}$ has and associated measurement

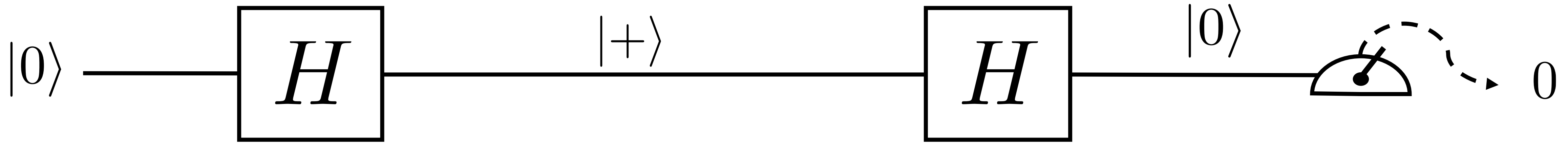
Probability of outcome i reads: $P(i) = |\langle v_i|\psi\rangle|^2$

The quantum state is updated to $|v_i\rangle$

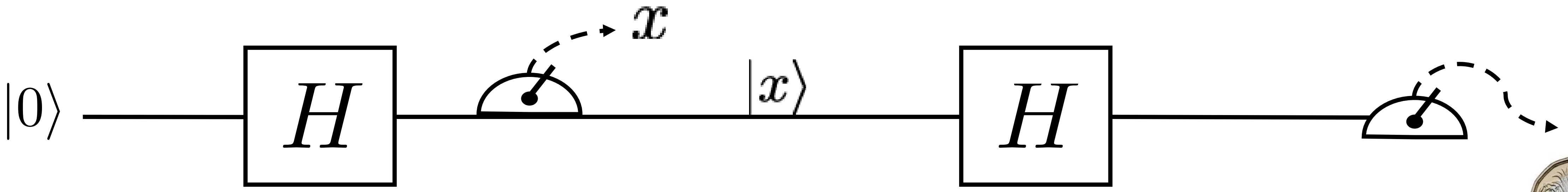


To measure or not to measure....

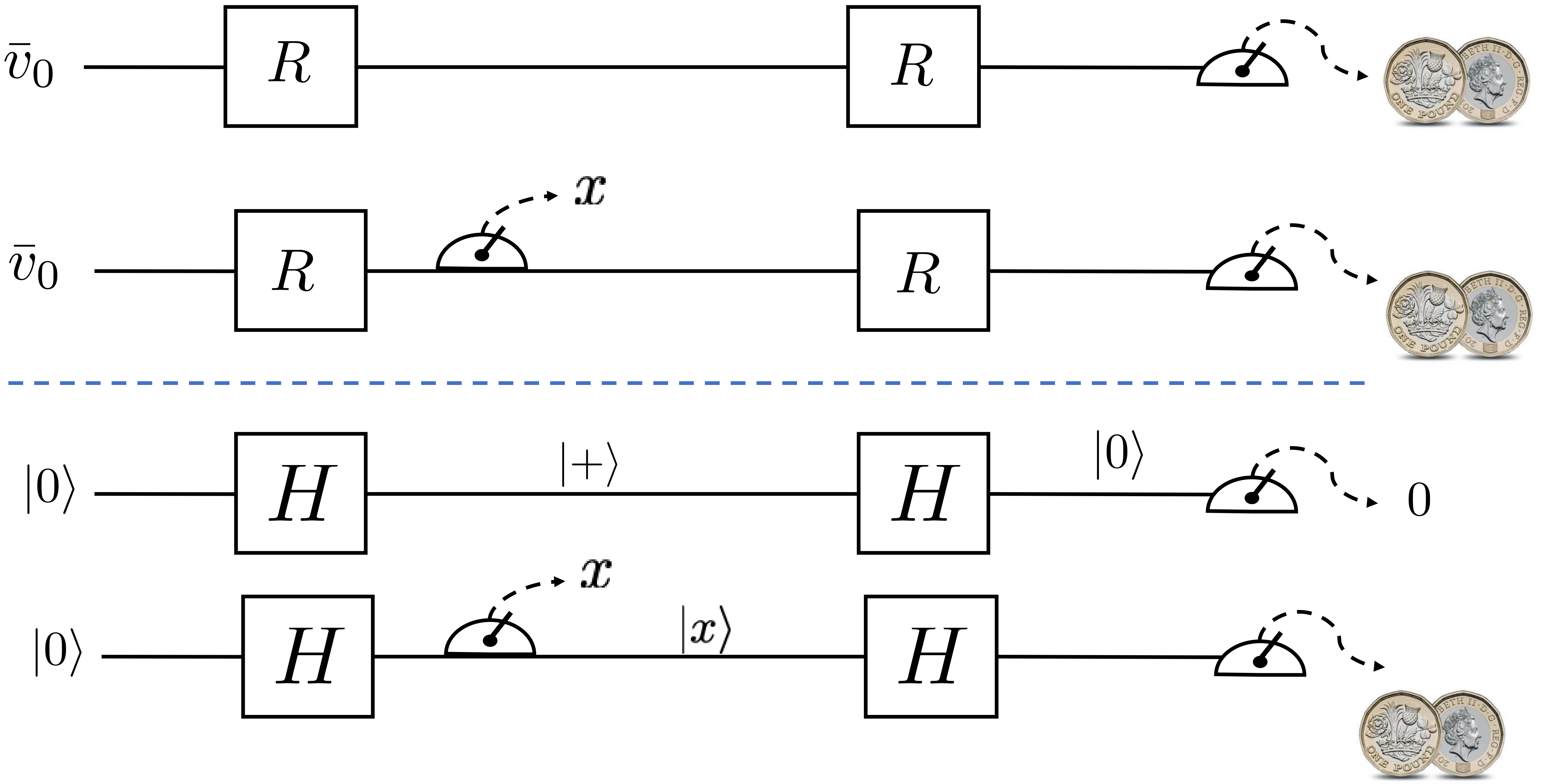
- Two "quantum coin flips"



- Two "quantum coin flips" with intermediate measurement



In quantum world observing the system can change its dynamics!!



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Arbitrary basis qubit measurement

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Basis: General case $\mathcal{H} \equiv \mathbb{C}^d$

- Spanning set of $\mathcal{H}$: set of vectors $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$

  $\forall|\psi\rangle \in \mathcal{H} : |\psi\rangle = \sum_{i=1}^{n} \psi_{v_i}|v_i\rangle$ where amplitudes are given by: $\psi_{v_i} = \langle\psi|v_i\rangle$

- Linearly independent: $\nexists\, a_1, a_2, ..., a_n \neq 0$ complex numbers

  $a_1|v_1\rangle + ... + a_n|v_n\rangle = 0$

- Basis: $\text{Span}\{|v_i\rangle\} = \mathcal{H} \Leftrightarrow n = d$

  $+$ Lin. ind.

- Orthonormal: $\forall i, j \in \{1, ..., d\}\,, \langle v_i|v_j\rangle = \delta_{i,j}$

- Has an associated measurement

Example

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

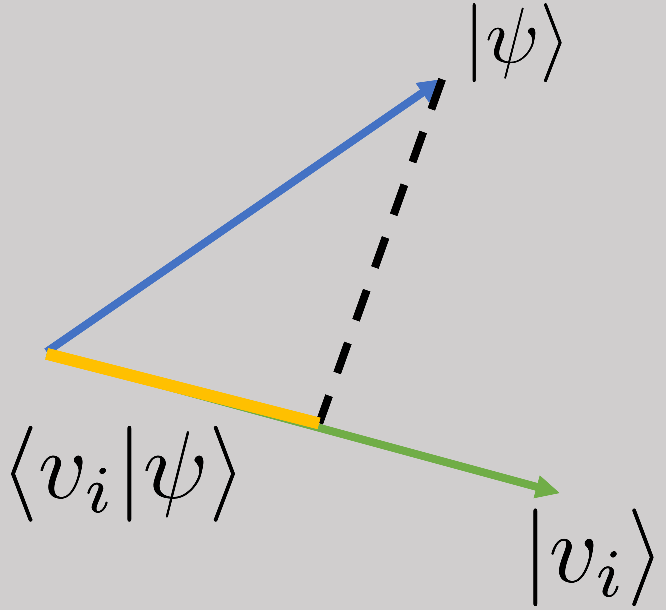
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Measurement of orthonormal basis

Any orthonormal basis $\{|v_i\rangle\}$ that span $\mathcal{H}$ has and associated measurement

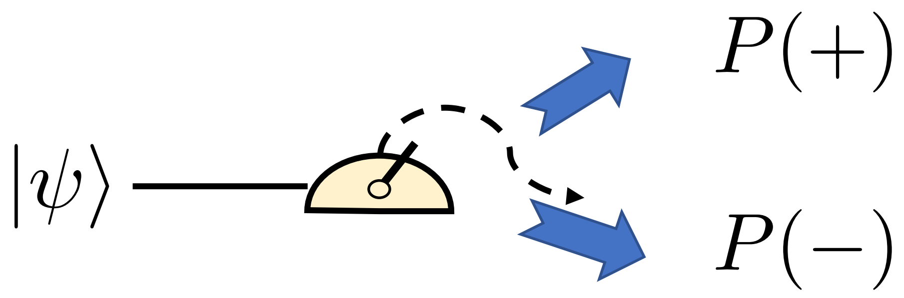
Probability of outcome i reads: $P(i) = |\langle v_i|\psi\rangle|^2$

The quantum state is updated to $|v_i\rangle$



Example:+/- basis

- $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$



$$\mathcal{H}_{\mathcal{Q}} = \mathrm{Span}\{|+\rangle, |-\rangle\}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

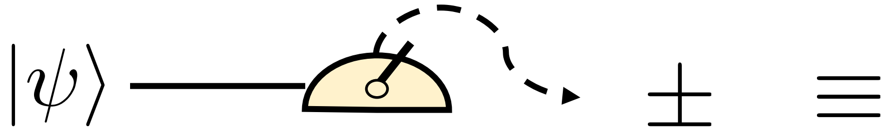
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- $P(+) = |\langle + |\psi\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 1| + \langle 0|)(\psi_0|0\rangle + \psi_1|1\rangle)|^2 = |\psi_0 + \psi_1|^2/2$

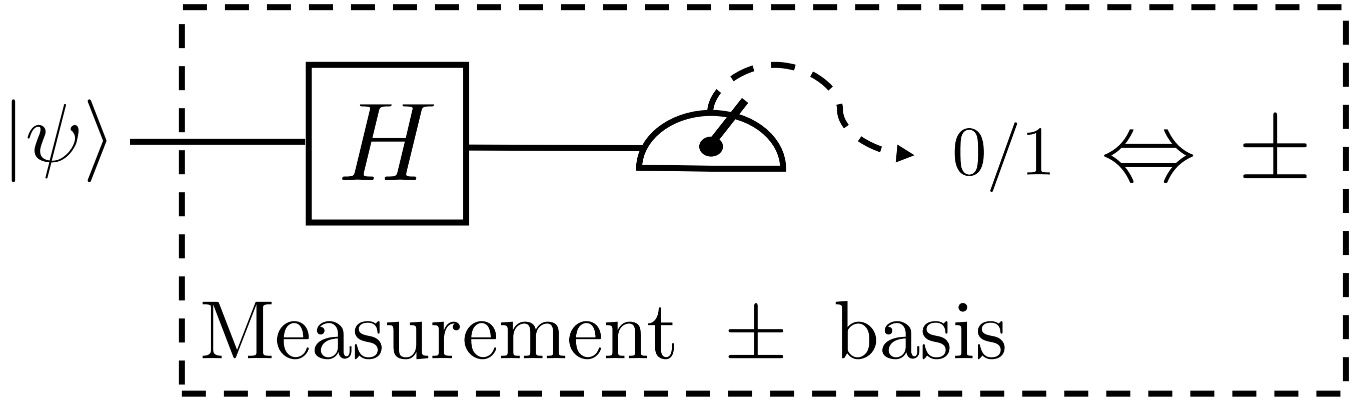
- $P(-) = |\langle - |\psi\rangle|^2 = |\psi_0 - \psi_1|^2/2$

Arbitrary basis measurement

- Measurement basis $\{|\pm\rangle\}$:



$$P(\pm) = |\langle \pm|\psi\rangle|^2$$

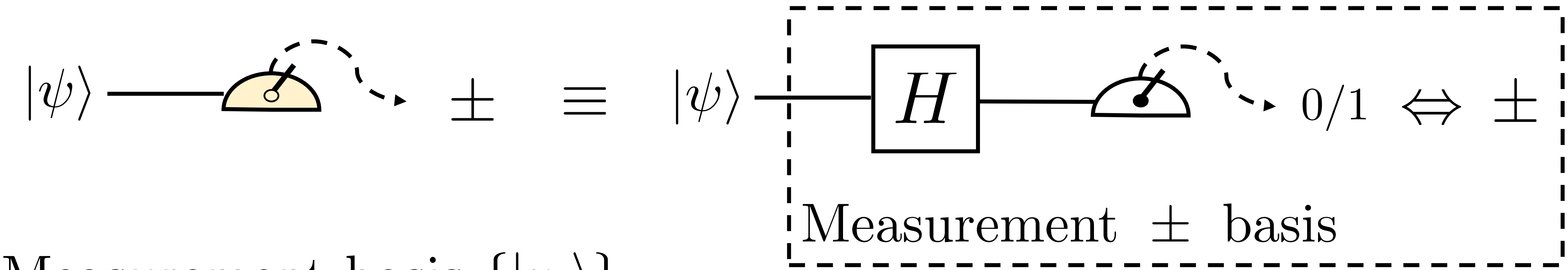


$$|+\rangle = H|0\rangle$$
$$|-\rangle = H|1\rangle$$
$$(A|\psi\rangle)^\dagger = \langle\psi|A^\dagger$$
$$H^\dagger = H$$

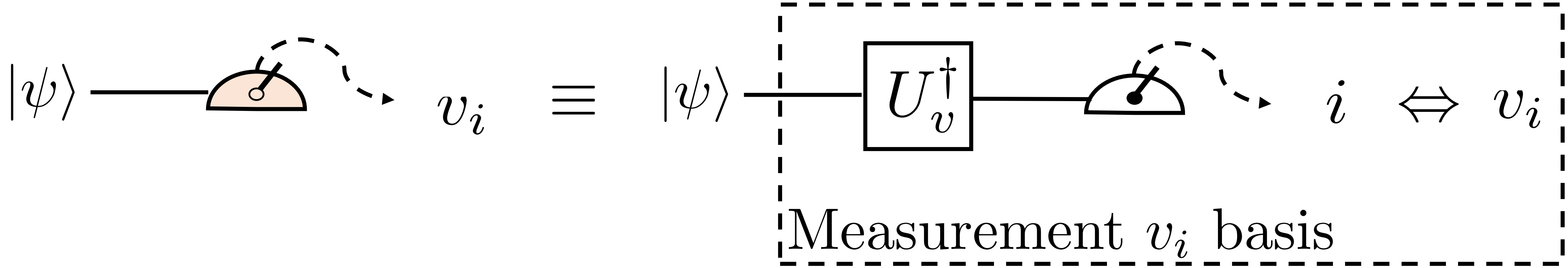
$$\langle +|\psi\rangle = \langle 0|H|\psi\rangle$$
$$\langle -|\psi\rangle = \langle 1|H|\psi\rangle$$

Arbitrary basis measurement

- Measurement basis $\{|\pm\rangle\}$:



- Measurement basis $\{|v_i\rangle\}$:



$\forall$ basis $\{|v_i\rangle\}, \exists U_v$ s.t. $|v_i\rangle = U_v|i\rangle$ $\quad \langle v_i|\psi\rangle = \langle i|U_v^\dagger|\psi\rangle$

References

Reading references

1. NC 2.2.3 and 2.2.5

NC ≡ Michael Nielsen and Isaac Chuang, Quantum Computing and Quantum Information
Cambridge University Press (2010)

Lecture 10 we will revisit the measurement in terms of projectors

There is even a more general concept of measurement in N&C called POVM (2.2.6) that we will not cover in this course.

Even more general is the concept of quantum instrument.

Complex number in a nutshell

- $\mathbb{C} = \{z = a + ib | (a,b) \in \mathbb{R}^2 \text{ and } i^2 = -1\}$

  Trigonometric form: $z = |z|(\cos\theta + i\sin\theta)$

- Norm: $|z|^2 = zz^* = a^2 + b^2$

- Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

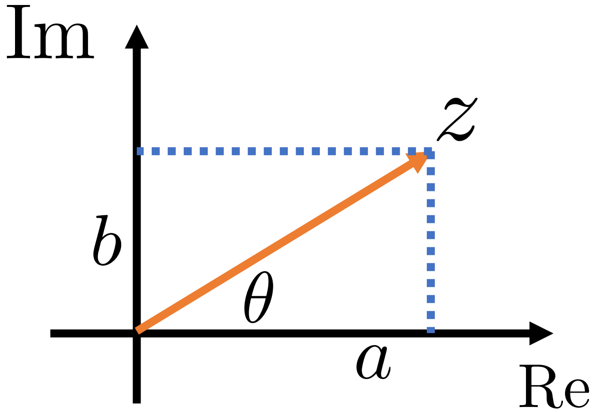
- Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$

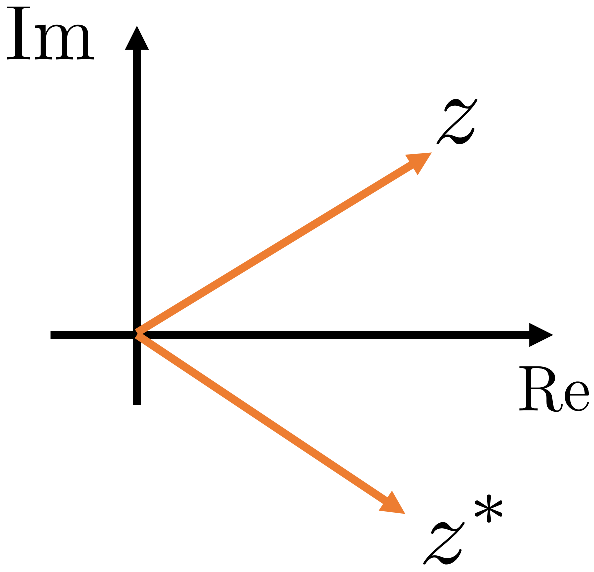
- Conjugation: $z^* = a - ib$

  $(z_1 + z_2)^* = z_1^* + z_2^*$ $\quad$ $(z_1 z_2)^* = z_1^* z_2^*$

- Euler equation: $e^{i\theta} = (\cos\theta + i\sin\theta) \Rightarrow z = |z|e^{i\theta}$

- Multiplication: $z_1 z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}$

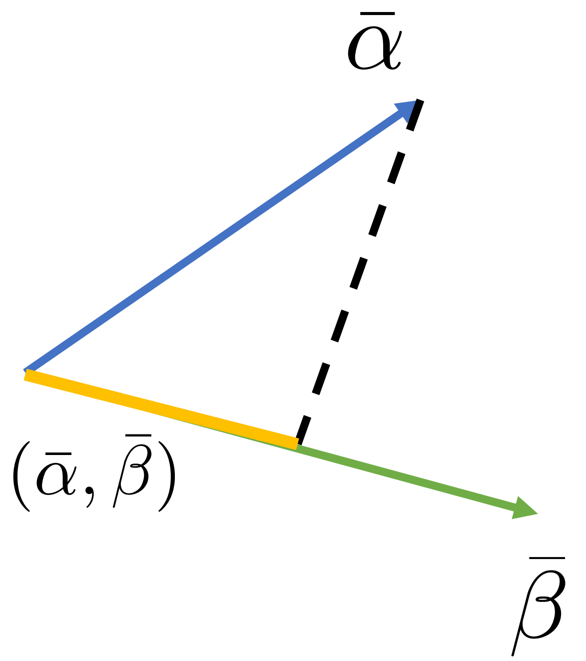




Inner-product (Math notation) $\mathcal{H} \equiv \mathbb{C}^d$

Inner-product $(\cdot,\cdot) : V \times V \to \mathbb{F}$

$$(\bar{\alpha}, \bar{\beta}) = \begin{bmatrix} \beta_0^* & \alpha_1^* & \dots & \alpha_{d-1}^* \end{bmatrix} \times \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{bmatrix} = \sum_{i=0}^{d-1} \alpha_i^* \beta_i$$



Linearity: $(\lambda\bar{\alpha}, \bar{\beta}) = \lambda^*(\bar{\alpha}, \bar{\beta})$

$$(\bar{\alpha}_1 + \bar{\alpha}_2, \bar{\beta}) = (\bar{\alpha}_1, \bar{\beta}) + (\bar{\alpha}_2, \bar{\beta})$$

Conjugate symmetry: $(\bar{\alpha}, \bar{\beta}) = (\bar{\beta}, \bar{\alpha})^*$

Positive semi-definiteness: $(\bar{\alpha}, \bar{\alpha}) \geq 0$

$$(\bar{\alpha}, \bar{\alpha}) = 0 \Leftrightarrow \bar{\alpha} = \bar{0}$$