



THE UNIVERSITY of EDINBURGH
informatics

Introduction to Quantum Computing

Lecture 4: Postulate III – Measurement

Raul Garcia-Patron Sanchez

Postulate I: Quantum states

A quantum state with d degrees of freedom is represented by a vector in a complex vector space with inner product (Hilbert space)

$$|\psi\rangle \in \mathcal{H} \equiv \mathbb{C}^d \qquad \langle\psi|\psi\rangle = 1$$

Hilbert space = Complex vector space + Inner product

State vector $|\psi\rangle \equiv \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{d-1} \end{bmatrix}$

A d -dimensional vector of complex numbers

- Qubit: $d = 2$
- Qudit: $d > 2$
- n qubits: $d = 2^n$

$\psi_i = \langle i|\psi\rangle \in \mathbb{C}$: Probability amplitude of computational state i

$$P(i) = |\psi_i|^2$$

Postulate II: Quantum operations

The evolution of a quantum system $|\psi\rangle \in \mathcal{H} \equiv \mathbb{C}^d$ is given by a unitary transformation $U : \mathcal{H} \rightarrow \mathcal{H}$, s.t. $|\psi_{\text{out}}\rangle = U|\psi_{\text{in}}\rangle$

Unitary matrices

$$UU^\dagger = U^\dagger U = I_d$$

- Linear operator $U : \mathcal{H} \rightarrow \mathcal{H}$
- Preserves the inner product
- Equivalent of orthogonal matrices on real vector spaces

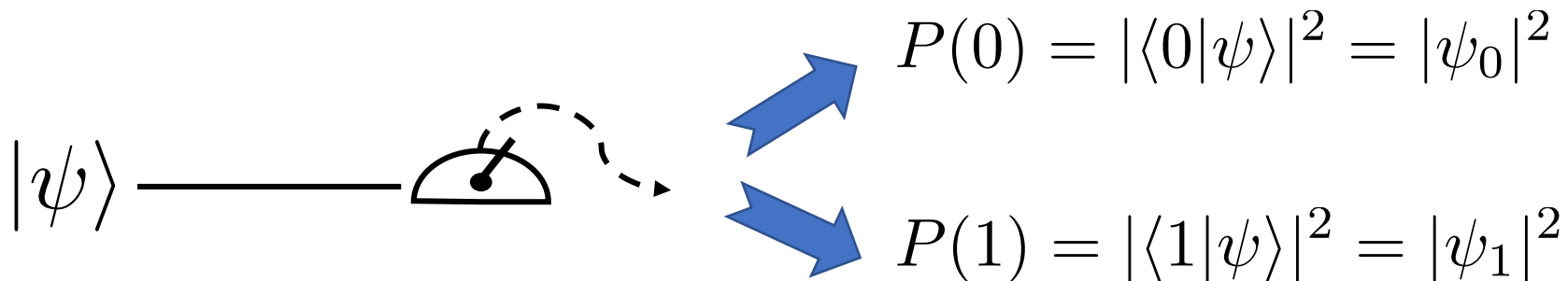
Computational basis

$$\mathcal{H}_Q = \text{Span}\{|0\rangle, |1\rangle\}$$

- $\forall |\psi\rangle, \exists \psi_0 \text{ and } \psi_1 : |\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$
- $\langle 0|1\rangle = 0$ (Orthogonal basis)
- $||0\rangle|| = ||1\rangle|| = 1$ (Normalized basis)

This ensures logical 0 and 1 are an orthonormal basis of a Hilbert space of dim 2.

Computational basis measurement



The amplitudes of the quantum state on the logical basis (0 and 1) are associated with the outcome probabilities of the computational basis measurement (logical 0 or 1).

- Spanning set of $\mathcal{H}$: set of vectors $|v_1\rangle, |v_2\rangle, \dots, |v_n\rangle$

$$\forall |\psi\rangle \in \mathcal{H} : |\psi\rangle = \sum_{i=1}^n \psi_{v_i} |v_i\rangle \quad \text{where amplitudes are given by: } \psi_{v_i} = \langle v_i | \psi \rangle$$

- Linearly independent

$$a_1 |v_1\rangle + \dots + a_n |v_n\rangle = |\emptyset\rangle \Rightarrow a_1 = \dots = a_n = 0$$

- Basis: $\text{Span}\{|v_i\rangle\} = \mathcal{H} \Rightarrow n = d$
+ Lin. ind.

- Orthonormal: $\forall i, j \in \{1, \dots, d\}, \langle v_i | v_j \rangle = \delta_{i,j}$

- Has an associated measurement

Example

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



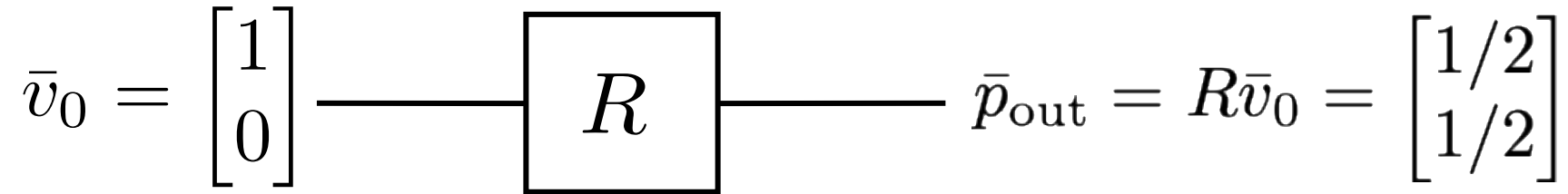
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The effect of measurement



Random coins vs quantum coins

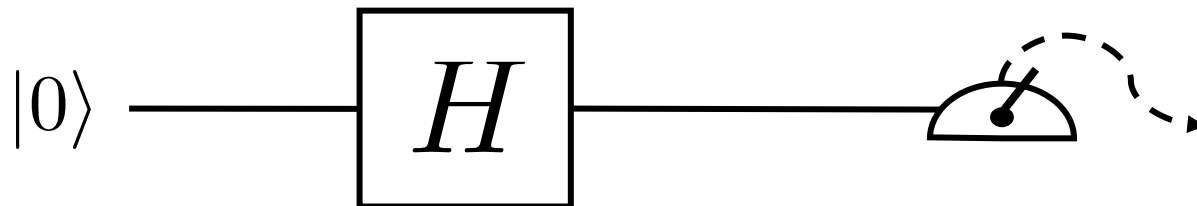
- Classical coin flip



$$R = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$



- Quantum random number generator circuit



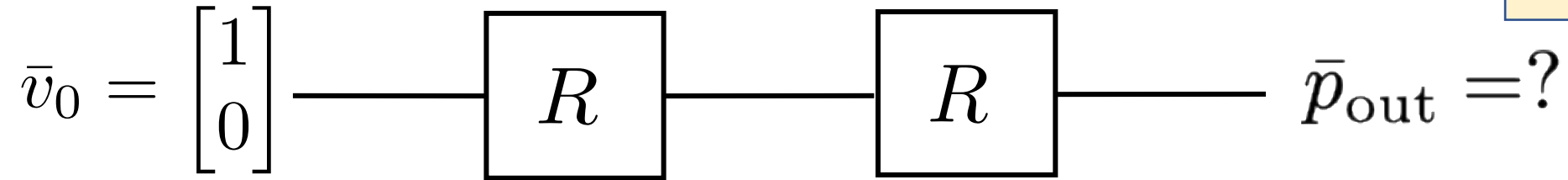
$$P(0) = |\psi_0|^2 = 1/2$$

$$P(1) = |\psi_1|^2 = 1/2$$



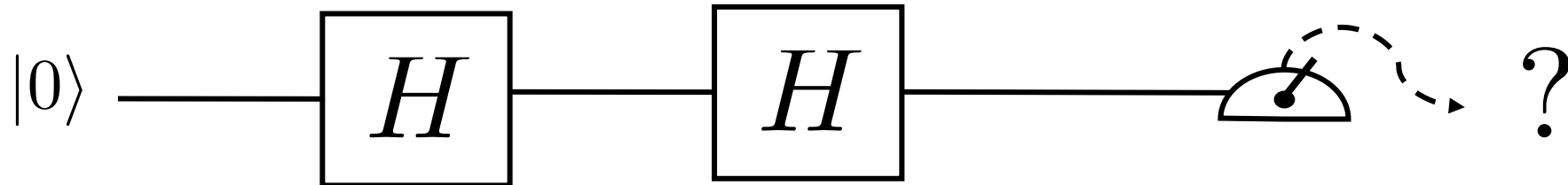
Random coins vs quantum coins: concatenation

- Classical coin flip



$$R = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

- Quantum random number generator circuit



Random coins vs quantum coins: concatenation

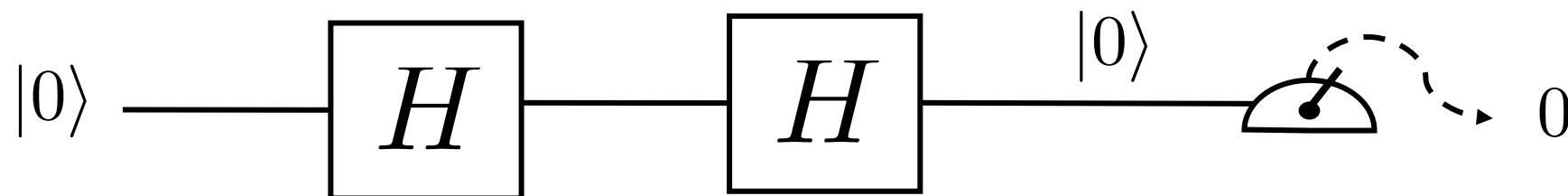
- Two classical coin flips

$$R = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\bar{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \boxed{R} \longrightarrow \boxed{R} \longrightarrow \bar{p}_{\text{out}} = R^2 \bar{v}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$



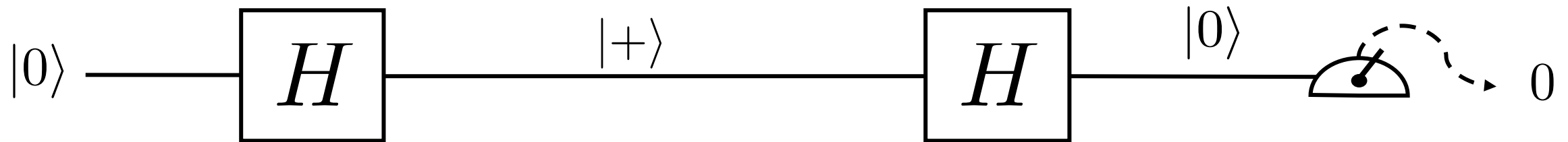
- Two “quantum coin flips”



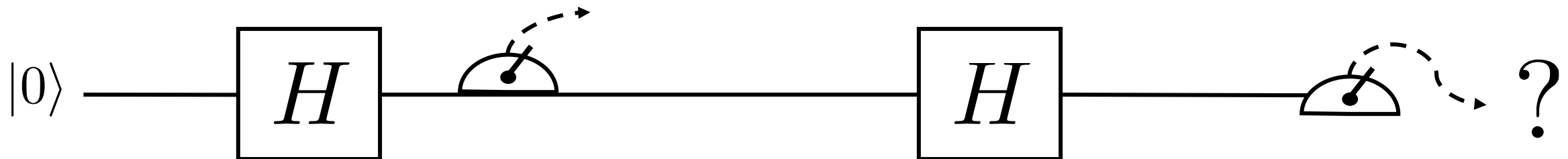
$$H^2 = I$$

To measure or not to measure...

- Two "quantum coin flips"



- Two "quantum coin flips" with intermediate measurement

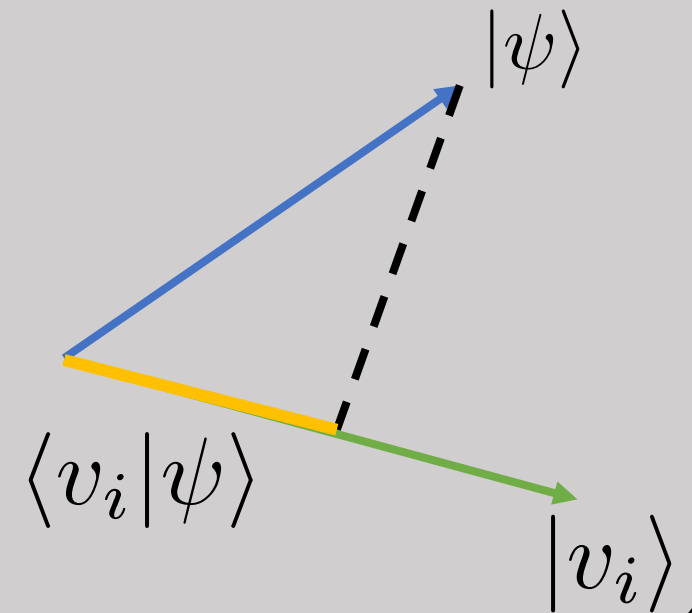


Measurement in an orthonormal basis

Any orthonormal basis $\{|v_i\rangle\}$ that spans $\mathcal{H}$ has an associated measurement

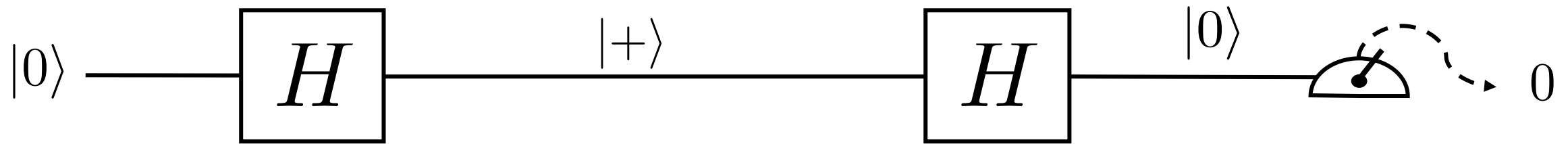
Probability of outcome i reads: $P(i) = |\langle v_i | \psi \rangle|^2$

The quantum state is updated to $|v_i\rangle$

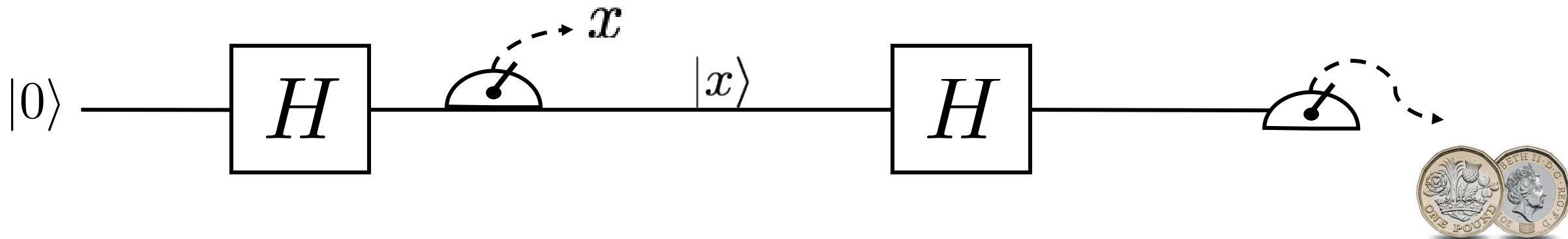


To measure or not to measure...

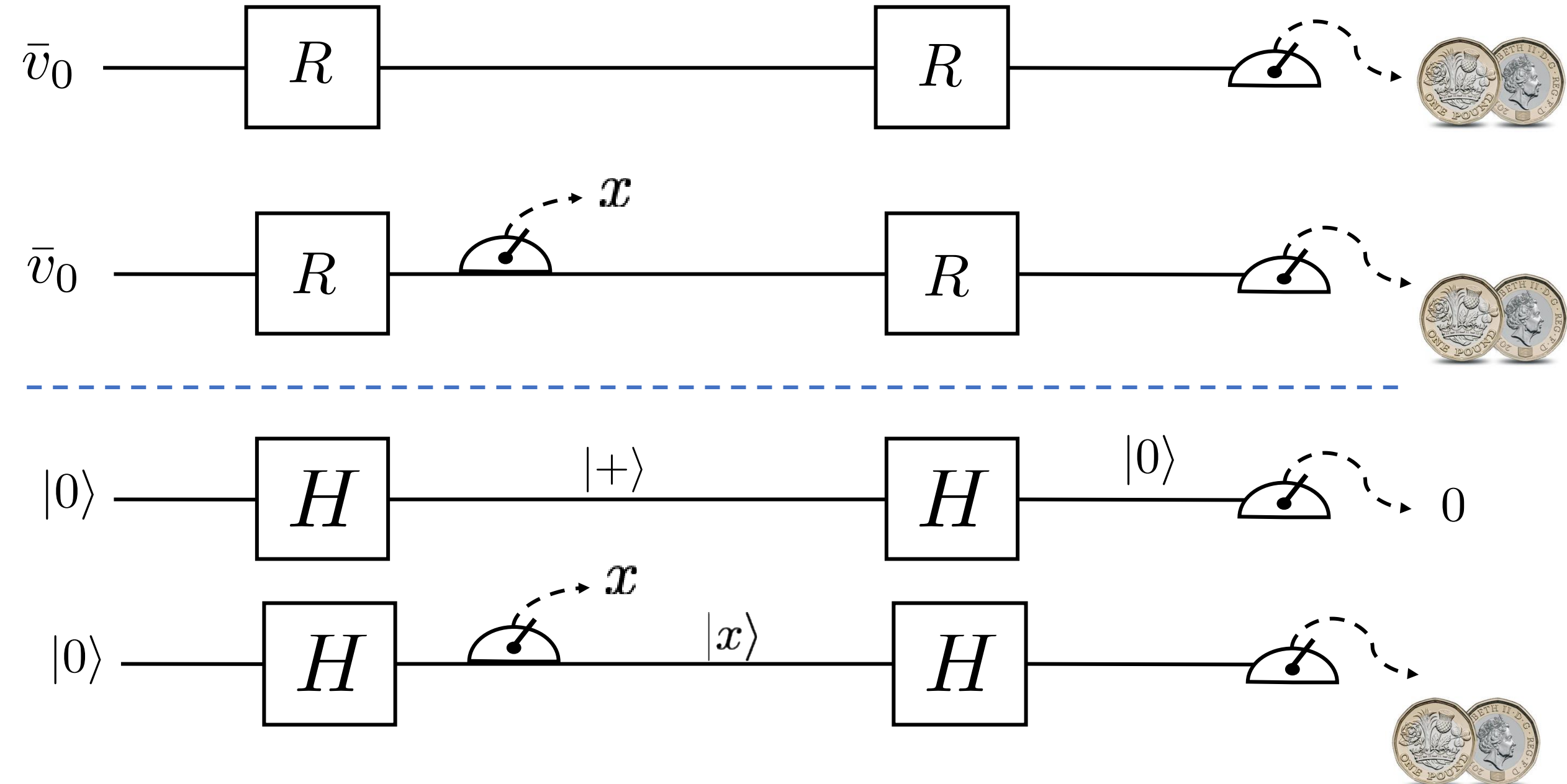
- Two "quantum coin flips"



- Two "quantum coin flips" with intermediate measurement



In the quantum world observing the system can change its dynamics!!





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Arbitrary basis qubit measurement

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- Spanning set of $\mathcal{H}$: set of vectors $|v_1\rangle, |v_2\rangle, \dots, |v_n\rangle$

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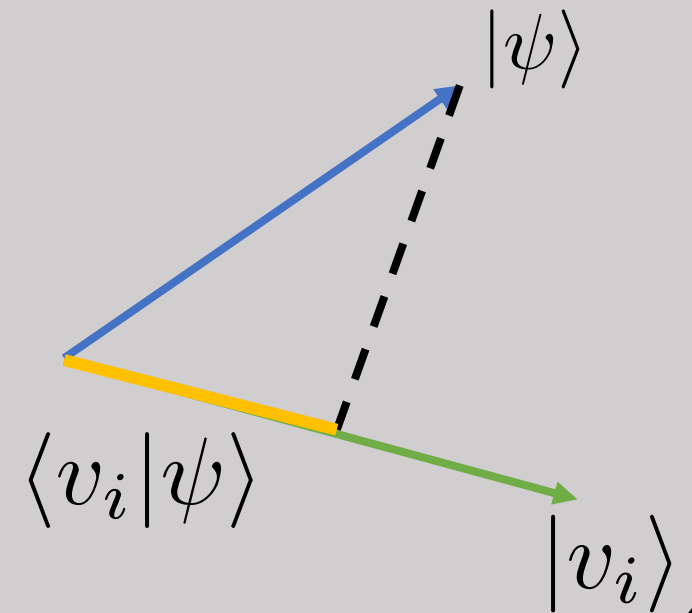
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Measurement in an orthonormal basis

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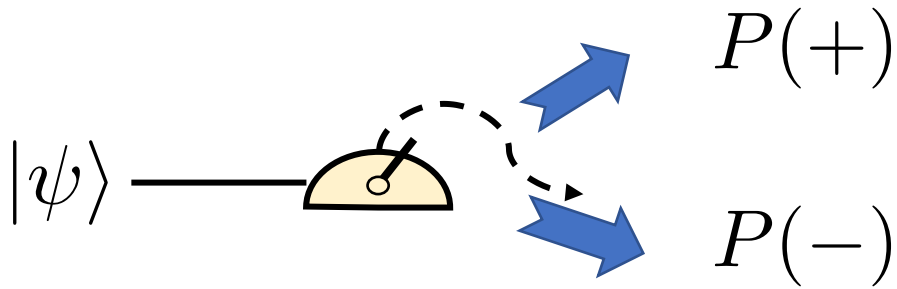
Probability of outcome i reads: $P(i) = |\langle v_i | \psi \rangle|^2$

The quantum state is updated to $|v_i\rangle$



Example: +/- basis

- $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$



$$\mathcal{H}_Q = \text{Span}\{|+\rangle, |-\rangle\}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

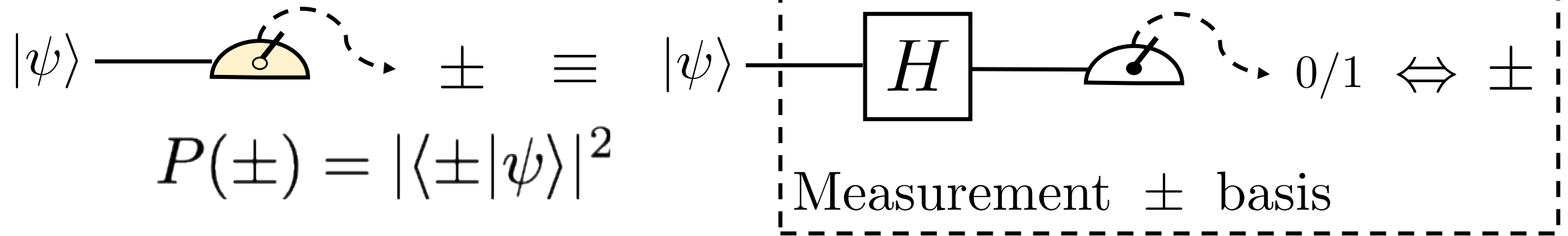
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- $P(+)=|\langle+|\psi\rangle|^2=|\frac{1}{\sqrt{2}}(\langle 1|+\langle 0|)(\psi_0|0\rangle+\psi_1|1\rangle)|^2=|\psi_0+\psi_1|^2/2$

- $P(-)=|\langle-|\psi\rangle|^2=|\psi_0-\psi_1|^2/2$

Arbitrary basis measurement

- Measurement basis $\{|\pm\rangle\}$:



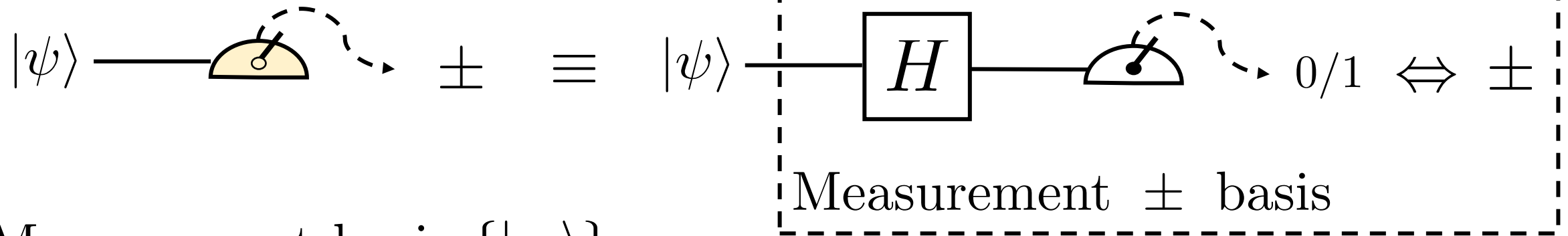
$$\begin{aligned} |+\rangle &= H|0\rangle \\ |-\rangle &= H|1\rangle \\ (A|\psi\rangle)^\dagger &= \langle\psi|A^\dagger \\ H^\dagger &= H \end{aligned}$$



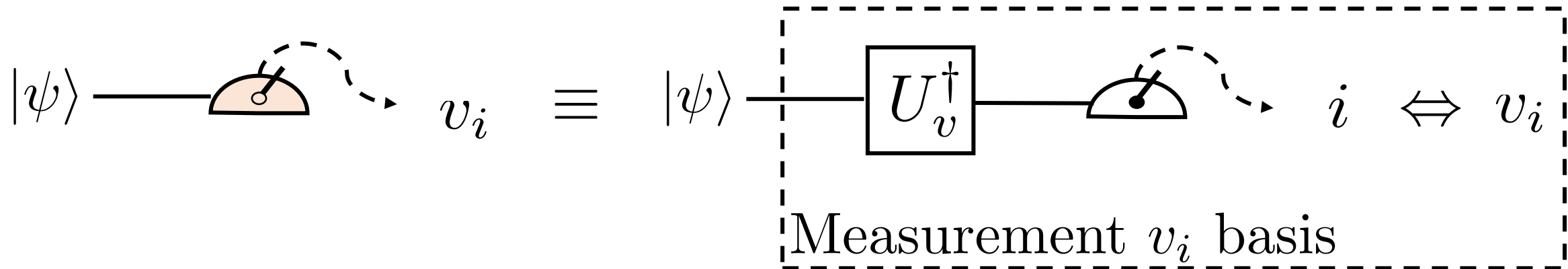
$$\begin{aligned} \langle + | \psi \rangle &= \langle 0 | H | \psi \rangle \\ \langle - | \psi \rangle &= \langle 1 | H | \psi \rangle \end{aligned}$$

Arbitrary basis measurement

- Measurement basis $\{|\pm\rangle\}$:



- Measurement basis $\{|v_i\rangle\}$:



$$\forall \text{ o.n. basis } \{|v_i\rangle\}, \exists U_v \text{ s.t. } |v_i\rangle = U_v |i\rangle \quad \langle v_i | \psi \rangle = \langle i | U_v^\dagger | \psi \rangle$$

References

Reading references

1. NC 2.2.3 and 2.2.5

NC $\equiv$ Michael Nielsen and Isaac Chuang, Quantum Computation and Quantum Information
Cambridge University Press (2010)

In Lecture 10 we will revisit the measurement in terms of projectors

There is an even more general concept of measurement in NC called POVM (2.2.6) that we will not cover in this course.

Even more general is the concept of a [quantum instrument](#).