

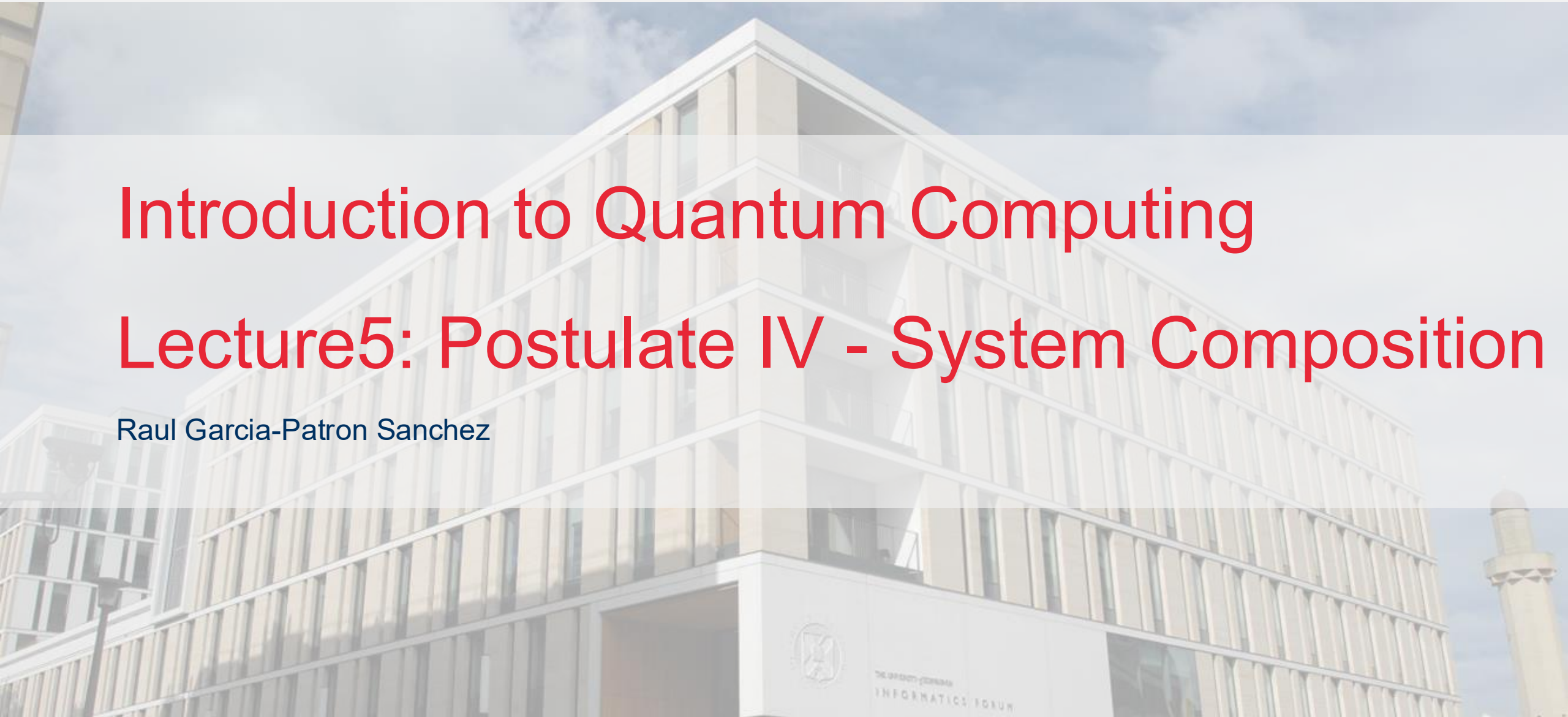


THE UNIVERSITY of EDINBURGH
informatics

Introduction to Quantum Computing

Lecture5: Postulate IV - System Composition

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Composition of quantum systems

Two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ can form a new Hilbert space

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \text{ s.t. } \dim \mathcal{H}_{AB} = \dim \mathcal{H}_A \times \dim \mathcal{H}_B.$$

A basis of $\mathcal{H}_{AB}$ is built via tensor products of bases of $\mathcal{H}_A$ and $\mathcal{H}_B$.

Any operator in $\mathcal{L}(\mathcal{H}_{AB})$ is built via linear combinations of tensor products of operators from $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$.

$$\forall |\psi_{AB}\rangle \in \mathcal{H}_{AB} : |\psi_{AB}\rangle = \sum_{i,j} \psi_{i,j} |i\rangle \otimes |j\rangle$$

$$\forall U \in \mathcal{L}(\mathcal{H}_{AB}), \exists V_i \in \mathcal{L}(\mathcal{H}_A) \text{ and } W_i \in \mathcal{L}(\mathcal{H}_B) : U = \sum_i V_i \otimes W_i$$

Tensor product of vectors/states

- Tensor product of vectors: $\mathcal{H}_A \times \mathcal{H}_B \rightarrow \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
 - $|u\rangle \in \mathcal{H}_A, |v\rangle \in \mathcal{H}_B$ leads to $|u\rangle \otimes |v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
 - $c(|u\rangle \otimes |v\rangle) = (c|u\rangle) \otimes |v\rangle = |u\rangle \otimes (c|v\rangle)$
 - $|u\rangle \otimes (|v_1\rangle + |v_2\rangle) = |u\rangle \otimes |v_1\rangle + |u\rangle \otimes |v_2\rangle$

$$|+\rangle \otimes |+\rangle = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

- Vector notation: $|u\rangle \otimes |v\rangle \equiv \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \otimes \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_0 v_0 \\ u_0 v_1 \\ u_1 v_0 \\ u_1 v_1 \end{bmatrix} = \begin{bmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{bmatrix}$
- $(|v\rangle_A \otimes |w\rangle_B)^\dagger = \langle v|_A \otimes \langle w|_B$

Inner product of tensor products

Inner product of a single system - recap

$$(|\psi\rangle, |\phi\rangle) = (|\psi\rangle)^\dagger |\phi\rangle = \langle\psi|\phi\rangle = \sum_{i=0}^{d-1} \psi_i^* \phi_i$$

Inner product: $(\mathcal{H}_A \otimes \mathcal{H}_B) \times (\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathbb{C}$

$$(|v\rangle_A \otimes |w\rangle_B, |v'\rangle_A \otimes |w'\rangle_B) = \langle v|v'\rangle_A \langle w|w'\rangle_B$$

$$\begin{aligned} (|v\rangle_A \otimes |w\rangle_B, |v'\rangle_A \otimes |w'\rangle_B) &= (\langle v|_A \otimes \langle w|_B)(|v'\rangle_A \otimes |w'\rangle_B) \\ &= \langle v|v'\rangle_A \langle w|w'\rangle_B \end{aligned}$$

Dirac notation

Inner product rules

- Inner product: $(\mathcal{H}_A \otimes \mathcal{H}_B) \times (\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathbb{C}$
 - $(|v\rangle_A \otimes |w\rangle_B)^\dagger = \langle v|_A \otimes \langle w|_B$
 - $(|v\rangle_A \otimes |w\rangle_B, |v'\rangle_A \otimes |w'\rangle_B) = \langle v|v'\rangle_A \langle w|w'\rangle_B$
 - $(|v\rangle_A \otimes |w\rangle_B, \sum_j b_j |v'_j\rangle_A \otimes |w'_j\rangle_B) = \sum_j b_j \langle v|v'_j\rangle_A \langle w|w'_j\rangle_B$

Example of inner product – Bell state and computational basis state

$$(|v\rangle_A \otimes |w\rangle_B, \sum_j b_j |v'_j\rangle_A \otimes |w'_j\rangle_B) = \sum_j b_j \langle v|v'_j\rangle_A \langle w|w'_j\rangle_B$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

State 1

$$|00\rangle = |0\rangle_A \otimes |0\rangle_B$$

State 2

$$\langle 00|\Phi^+\rangle_{AB} = (|00\rangle_{AB}, |\Phi^+\rangle_{AB})$$

$$\langle 00|\Phi^+\rangle_{AB} = (|0\rangle_A \otimes |0\rangle_B, \frac{1}{\sqrt{2}} [|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B])$$

$$= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B, |0\rangle_A \otimes |0\rangle_B) + \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B, |1\rangle_A \otimes |1\rangle_B)$$

$$= \frac{1}{\sqrt{2}}\langle 0|0\rangle_A \langle 0|0\rangle_B + \frac{1}{\sqrt{2}}\langle 0|1\rangle_A \langle 0|1\rangle_B = \frac{1}{\sqrt{2}}$$

Inner product rules

- Inner product: $(\mathcal{H}_A \otimes \mathcal{H}_B) \times (\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathbb{C}$
 - $(|v\rangle_A \otimes |w\rangle_B)^\dagger = \langle v|_A \otimes \langle w|_B$
 - $(|v\rangle_A \otimes |w\rangle_B, |v'\rangle_A \otimes |w'\rangle_B) = \langle v|v'\rangle_A \langle w|w'\rangle_B$
 - $(|v\rangle_A \otimes |w\rangle_B, \sum_j b_j |v'_j\rangle_A \otimes |w'_j\rangle_B) = \sum_j b_j \langle v|v'_j\rangle_A \langle w|w'_j\rangle_B$
 - $(\sum_i a_i |v_i\rangle_A \otimes |w_i\rangle_B, |v'\rangle_A \otimes |w'\rangle_B) = \sum_i a_i^* \langle v_i|v'\rangle_A \langle w_i|w'\rangle_B$
 - $(\sum_i a_i |v_i\rangle_A \otimes |w_i\rangle_B, \sum_j b_j |v'_j\rangle_A \otimes |w'_j\rangle_B) = \sum_{i,j} a_i^* b_j \langle v_i|v'_j\rangle_A \langle w_i|w'_j\rangle_B$

Example of inner product – Orthogonality of two Bell states

$$\left(\sum_i a_i |v_i\rangle_A \otimes |w_i\rangle_B, \sum_j b_j |v'_j\rangle_A \otimes |w'_j\rangle_B \right) = \sum_{i,j} a_i^* b_j \langle v_i | v'_j \rangle_A \langle w_i | w'_j \rangle_B$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

State 1

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$$

State 2

$$\begin{aligned} \langle \Psi^+ | \Phi^+ \rangle_{AB} &= \left(\frac{1}{\sqrt{2}} (\langle 0 |_A \otimes \langle 1 |_B + \langle 1 |_A \otimes \langle 0 |_B) \right) \left(\frac{1}{\sqrt{2}} (|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \right) \\ &= \frac{1}{2} (\langle 0|0\rangle_A \langle 1|0\rangle_B + \langle 1|0\rangle_A \langle 0|0\rangle_B + \langle 0|1\rangle_A \langle 1|1\rangle_B + \langle 1|1\rangle_A \langle 0|1\rangle_B) \\ &= 0 \end{aligned}$$

Summary: inner product and notation

- Inner product: $(\mathcal{H}_A \otimes \mathcal{H}_B) \times (\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathbb{C}$
 - $(|v\rangle_A \otimes |w\rangle_B, |v'\rangle_A \otimes |w'\rangle_B) = \langle v|v'\rangle_A \langle w|w'\rangle_B$
 - $(\sum_i a_i |v_i\rangle_A \otimes |w_i\rangle_B, \sum_j b_j |v'_j\rangle_A \otimes |w'_j\rangle_B) = \sum_{i,j} a_i^* b_j \langle v_i|v'_j\rangle_A \langle w_i|w'_j\rangle_B$
- Additional notation:
 - $|\psi\rangle^{\otimes k} = \underbrace{|\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle}_k$
 - $|0^k\rangle = |0\rangle^{\otimes k} = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$
 - $|\bar{x}\rangle = |x_1, x_2, \dots, x_n\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$

$$\left. \begin{array}{c} |\psi\rangle \\ |\psi\rangle \\ \vdots \\ |\psi\rangle \end{array} \right\} \equiv |\psi\rangle^{\otimes k}$$

Entanglement

- $\exists |\psi_{AB}\rangle \in \mathcal{H}_{AB}$ s.t. $\nexists |\phi_A\rangle \in \mathcal{H}_A$ and $|\varphi_B\rangle \in \mathcal{H}_B : |\psi_{AB}\rangle = |\phi_A\rangle \otimes |\varphi_B\rangle$

Bell states: a basis of $\mathcal{H}_Q \otimes \mathcal{H}_Q$ composed of entangled states.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

- Maximally entangled states of 2 qubits.
- A Bell state is a unit of entanglement, just as an unbiased coin is a unit of randomness.

- $\forall |\psi_{AB}\rangle \in \mathcal{H}_{AB} : |\psi_{AB}\rangle = \sum_{i,j} \psi_{i,j} |i\rangle \otimes |j\rangle$

Tensor product of operators



Tensor product of operators

● Tensor product of operators: $\mathcal{L}(\mathcal{H}_A) \times \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_{AB}) = \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$

● $(A \otimes B)(|v\rangle \otimes |w\rangle) = (A|v\rangle) \otimes (B|w\rangle) = A|v\rangle \otimes B|w\rangle$

● $(A \otimes B)\left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle\right) = \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$

● $\left(\sum_i c_i A_i \otimes B_i\right)|v\rangle \otimes |w\rangle = \sum_i c_i A_i |v\rangle \otimes B_i |w\rangle$

● Matrix notation: $A \otimes B = \begin{bmatrix} A_{00}B & A_{01}B & \dots & A_{0(d-1)}B \\ A_{10}B & A_{11}B & \dots & A_{1(d-1)}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{(d-1)0}B & A_{(d-1)1}B & \dots & A_{(d-1)(d-1)}B \end{bmatrix}$

Example (Dirac notation)

$$\left. \begin{array}{l} |+\rangle \text{ --- } [H] \text{ ---} \\ |-\rangle \text{ --- } [H] \text{ ---} \end{array} \right\}$$

$|\pm\rangle$ basis:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\begin{aligned} (H \otimes H)(|+\rangle \otimes |-\rangle) &= (H|+\rangle) \otimes (H|-\rangle) \\ &= |0\rangle \otimes |1\rangle \end{aligned}$$

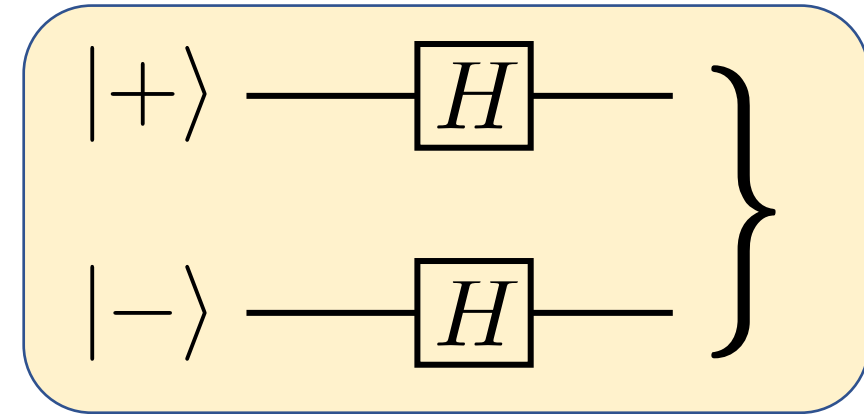
$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

Example (matrix notation)

$$\bullet \quad |+\rangle \otimes |-\rangle \equiv \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

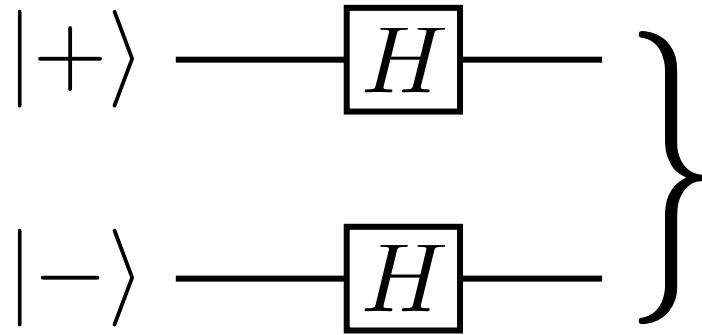
$$\bullet \quad H \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\bullet \quad |\psi\rangle = (H \otimes H)(|+\rangle \otimes |-\rangle) \equiv \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |0\rangle \otimes |1\rangle$$



$$A \otimes B = \begin{bmatrix} A_{00}B & A_{01}B & \dots & A_{0(d-1)}B \\ A_{10}B & A_{11}B & \dots & A_{1(d-1)}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{(d-1)0}B & A_{(d-1)1}B & \dots & A_{(d-1)(d-1)}B \end{bmatrix}$$

Example (Dirac notation)



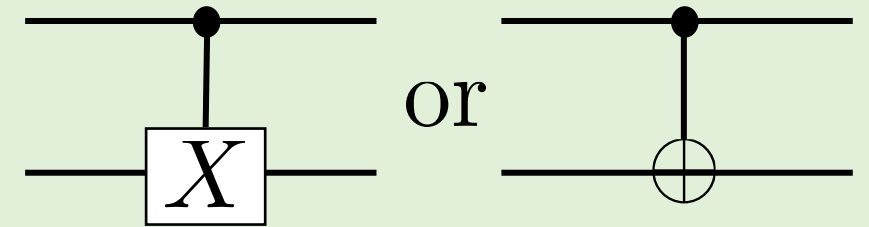
$$(H \otimes H)(|+\rangle \otimes |-\rangle) = (H|+\rangle) \otimes (H|-\rangle) = |0\rangle \otimes |1\rangle$$

Entangling operations

- $\exists U_{AB} \in \mathcal{L}(\mathcal{H}_{AB})$ s.t. $\nexists U_A \in \mathcal{L}(\mathcal{H}_A)$ and $U_B \in \mathcal{L}(\mathcal{H}_B) : U_{AB} = U_A \otimes U_B$

Controlled-NOT gate (CNOT gate):

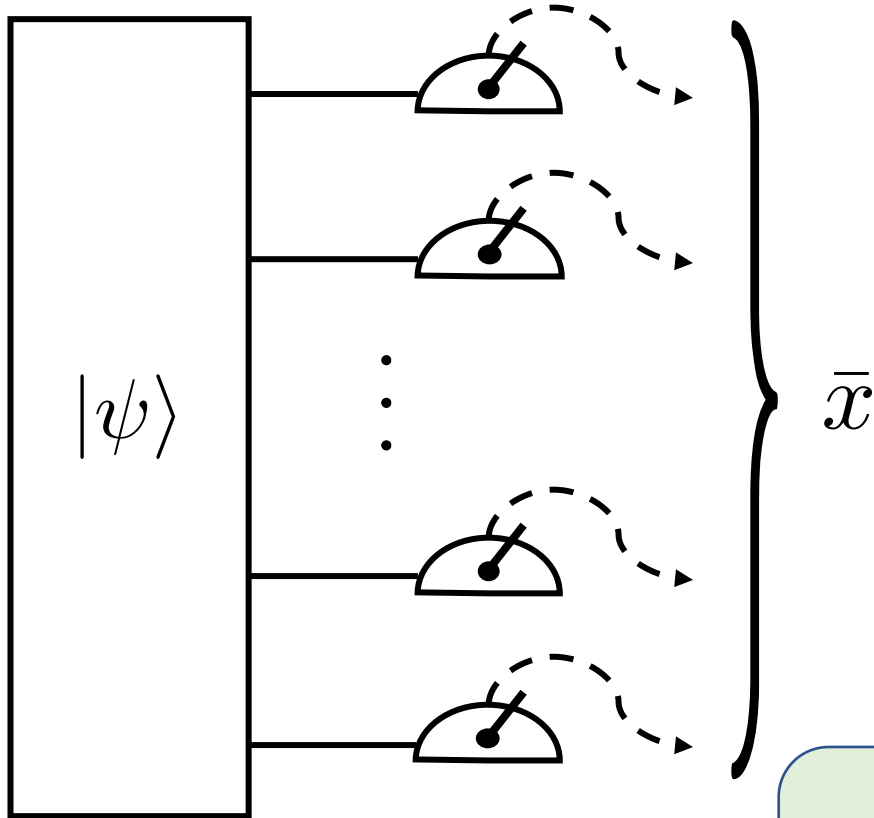
$$U_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



- $\forall U \in \mathcal{L}(\mathcal{H}_{AB}), \exists V_i \in \mathcal{L}(\mathcal{H}_A)$ and $W_i \in \mathcal{L}(\mathcal{H}_B) : U = \sum_i V_i \otimes W_i$

Composition of measurement: computational basis

- Let $\bar{x}$ encode the n -bit outcome

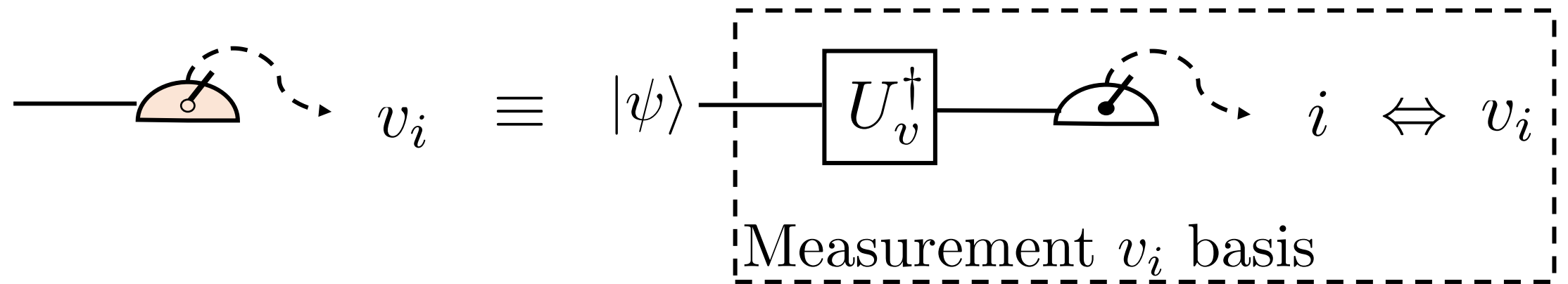


$$|\bar{x}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$

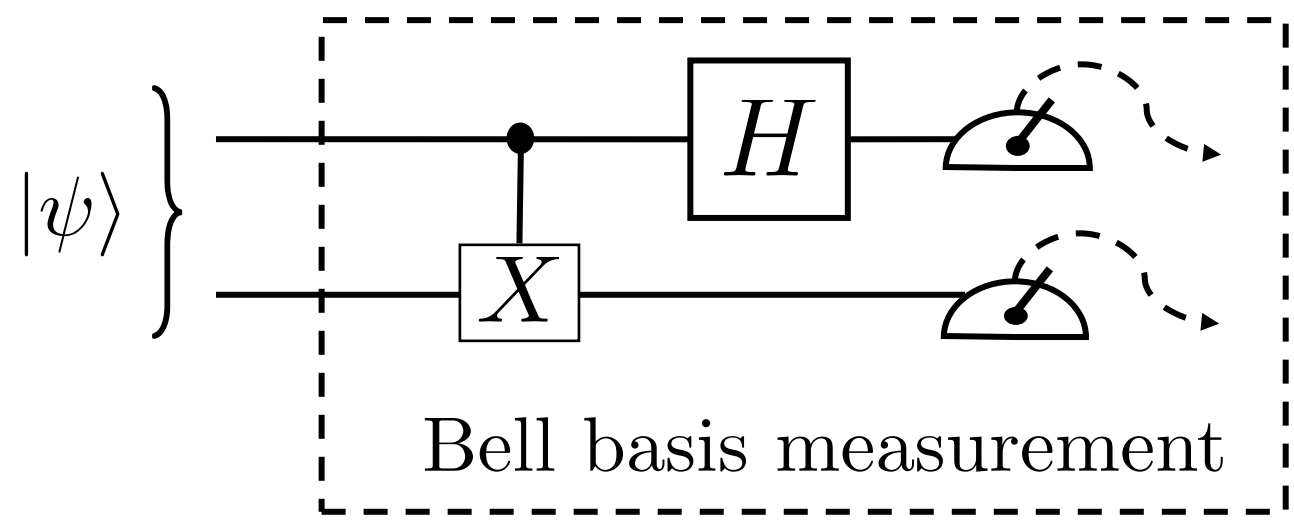
$$|\langle \bar{x} | \psi \rangle|^2 = |\langle \bar{x} | \sum_{\bar{y} \in \{0,1\}^n} \psi_{\bar{y}} |\bar{y}\rangle|^2 = |\psi_{\bar{x}}|^2$$

General multi-qubit basis measurement

- Measurement basis $\{|v_i\rangle\}$:



- Bell basis measurement:



Bell basis

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

References

Reading references

1. Tensor product NC 2.1.7
2. Outer product NC 2.1.4 page 67

NC $\equiv$ Michael Nielsen and Isaac Chuang, Quantum Computation and Quantum Information
Cambridge University Press (2010)