Problem 1: Complex Numbers

Consider the two complex numbers $v_1 = 1 + i$ and $v_2 = 1 - 2i$ where $i^2 = -1$.

a. Calculate the complex numbers $z_1 = v_1 + v_2$ and $z_2 = v_1 - v_2^*$ where z^* denotes the complex conjugate of the complex number z.

b. Let $w = 1 - i$. Calculate wz_1 and $(z_2 w)^*$.

c. Calculate the norm (modulus) $|v_1|$ and $|v_2|$ of v_1 and v_2.

Problem 2: Inner products and orthonormal bases

Consider the quantum states

$$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad |L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

1. Write $\langle R|$ and $\langle L|$ in vector notation.
2. Prove that both $|R\rangle$ and $|L\rangle$ are normalized, i.e. $\sqrt{\langle R|R\rangle} = \sqrt{\langle L|L\rangle} = 1$.
3. Are $|R\rangle$ and $|L\rangle$ orthogonal?
4. Show that $|R\rangle$ and $|L\rangle$ satisfy all the conditions of an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$.

Problem 3: Matrices and operators

a.

1. One of the most important linear operators in quantum computing is the *Hadamard operator* defined as:

   $$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

   Find the action of the operator on the state $|R\rangle$ from Problem 2,

   $$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

2. Consider two of the Pauli matrices:

   $$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   Calculate XZ and ZX. Compare the two calculations.

b.

1. Show that for finite-size matrices $(A^\dagger)^\dagger = A$ always holds.
2. Prove that for any two matrices A and B of compatible dimensions (so that the product AB is defined) we have $(AB)^\dagger = B^\dagger A^\dagger$.
3. Prove that the Hadamard operator defined above is a self-adjoint operator, i.e. $H^\dagger = H$.

c. Compute the eigenvalues and eigenvectors of X and Z.

Problem 4: Euler's formula for complex numbers

Euler's formula is very handy when working with complex numbers in quantum computation. Any complex number z can be written in terms of its norm $|z|$ and a term $e^{i\phi}$, i.e. $z = |z|e^{i\phi}$. The term $e^{i\phi}$ is also a complex number, of norm 1, and is usually referred to in quantum computation, quantum mechanics and other fields as a *phase*.

a. Use *Euler's formula*, i.e. $e^{i\phi} = \cos\phi + i\sin\phi$, to calculate $e^{i\pi/2}$ and $e^{i\pi}$.

b. Let $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. First calculate $|z| = \sqrt{zz^*}$ and then use Euler's formula to obtain ϕ so that $z = |z|e^{i\phi}$.

c. In Problem 3a.1 you found that

$$H\,|R\rangle = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}.$$

Write $1+i$ and $1-i$ in the form $|z|e^{i\phi}$ and use this to show that the result simplifies to $H\,|R\rangle = e^{i\pi/4}\,|L\rangle$, where $|L\rangle$ is the state from Problem 2. In other words, up to the phase $e^{i\pi/4}$, the Hadamard operator maps $|R\rangle$ to $|L\rangle$.