

Problem 1: Complex Numbers

Consider the two complex numbers $v_1 = 1 + i$ and $v_2 = 1 - 2i$ where $i^2 = -1$.

a. Calculate the complex numbers $z_1 = v_1 + v_2$ and $z_2 = v_1 - v_2^*$ where z^* denotes the complex conjugate of the complex number z .

Solution: $z_1 = v_1 + v_2 = (1 + i) + (1 - 2i) = 2 - i$. In order to calculate z_2 , we first have to conjugate the number v_2 . Recall that for a complex number $w = a + bi$, its complex conjugate is $w^* = a - bi$. It is then easy to see that $v_2^* = 1 + 2i$ and thus $z_2 = v_1 - v_2^* = (1 + i) - (1 + 2i) = -i$.

b. Let $w = 1 - i$. Calculate wz_1 and $(z_2w)^*$.

Solution: For the first multiplication we have:

$$wz_1 = (1 - i)(2 - i) = 2 - i - 2i - 1 = 1 - 3i$$

since $i^2 = -1$. For the second expression, we should first do the multiplication and then calculate the conjugate of the product. So

$$z_2w = -i(1 - i) = -i + i^2 = -1 - i$$

and then if we conjugate:

$$(z_2w)^* = -1 + i$$

c. Calculate the norm (modulus) $|v_1|$ and $|v_2|$ of v_1 and v_2 .

Solution: The *norm* of a complex number $w = a + bi$ is defined as

$$|w| = \sqrt{a^2 + b^2}$$

In our case, for v_1 :

$$|v_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and for v_2 :

$$|v_2| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Problem 2: Inner products and orthonormal bases

Consider the quantum states

$$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

1. Write $\langle R|$ and $\langle L|$ in vector notation.
2. Prove that both $|R\rangle$ and $|L\rangle$ are normalized, i.e. $\sqrt{\langle R|R\rangle} = \sqrt{\langle L|L\rangle} = 1$.
3. Are $|R\rangle$ and $|L\rangle$ orthogonal?
4. Show that $|R\rangle$ and $|L\rangle$ satisfy all the conditions of an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$.

Solution:

Let $|\psi\rangle$ be a vector in Dirac “ket” notation. If $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ then the conjugate transpose vector, denoted $\langle\psi|$ and called a “bra”, is defined as $\langle\psi| = (|\psi\rangle^T)^* = |\psi\rangle^\dagger = (a^* \ b^*)$. Thus,

$$\langle R| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix}$$

and

$$\langle L| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix}$$

We will prove that both $|R\rangle, |L\rangle$ are normalised.

$$\langle R|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(1+1) = 1$$

and so:

$$\sqrt{\langle R|R\rangle} = 1$$

Same for $|L\rangle$:

$$\langle L|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(1+1) = 1$$

and so:

$$\sqrt{\langle L|L\rangle} = 1$$

Two vectors $|R\rangle, |L\rangle$ are *orthogonal* if their inner product is 0, i.e. $\langle R|L\rangle = 0$. We have,

$$\langle R|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(1-1) = 0$$

So $|R\rangle$ and $|L\rangle$ are *orthogonal*.

Finally, for the last question, in order for $|R\rangle$ and $|L\rangle$ to satisfy all the conditions of an orthonormal basis, they must:

- Be orthogonal, which is true as we proved before.
- Be normalized to one, which we also proved.
- Span the space: the number of basis elements must be equal to the dimension of the vector space, which is true as well. Indeed, orthogonal non-zero vectors are automatically linearly independent, so two of them span $\mathbb{C}^2$.

Problem 3: Matrices and operators

a.

1. One of the most important linear operators in quantum computing is the *Hadamard operator* defined as:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Find the action of the operator on the state $|R\rangle$ from Problem 2,

$$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Solution:

We want to calculate $H |R\rangle$. We have:

$$H |R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$$

2. Consider two of the Pauli matrices:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Calculate XZ and ZX . Compare the two calculations.

Solution: We start by computing XZ . We have:

$$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We continue by computing ZX . We have:

$$ZX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If we observe the two multiplications we see that $ZX = -XZ$. This is a well-known property of the Pauli matrices: any two distinct Pauli matrices *anticommute*. For our case this translates to $\{X, Z\} = XZ + ZX = 0$.

b.

1. Show that for finite-size matrices $(A^\dagger)^\dagger = A$ always holds.

Solution: The elements of the adjoint are given by $(A^\dagger)_{ij} = A_{ji}^*$. Applying this definition twice:

$$((A^\dagger)^\dagger)_{ij} = ((A^\dagger)_{ji})^* = (A_{ij}^*)^* = A_{ij},$$

and thus:

$$(A^\dagger)^\dagger = A \text{ for every matrix } A.$$

2. Prove that for any two matrices A and B of compatible dimensions (so that the product AB is defined) we have $(AB)^\dagger = B^\dagger A^\dagger$.

Solution: Let $(\cdot, \cdot)$ denote the inner product, $(|v\rangle, |w\rangle) = \langle v|w\rangle$. The adjoint $M^\dagger$ of an operator M is defined as the unique operator satisfying

$$(|v\rangle, M|w\rangle) = (M^\dagger|v\rangle, |w\rangle) \quad \text{for all } |v\rangle, |w\rangle.$$

We can now write

$$(|v\rangle, AB|u\rangle) = (|v\rangle, A(B|u\rangle)),$$

where setting $|w\rangle = B|u\rangle$ and $M = A$ in the definition of adjoint operator above, allows us to write

$$(|v\rangle, A(B|u\rangle)) = (A^\dagger|v\rangle, B|u\rangle).$$

Using again the definition of the adjoint operator, now with B , we obtain

$$(A^\dagger|v\rangle, B|u\rangle) = (B^\dagger A^\dagger|v\rangle, |u\rangle).$$

On the other hand, by definition,

$$(|v\rangle, AB|u\rangle) = ((AB)^\dagger|v\rangle, |u\rangle).$$

Since $((AB)^\dagger|v\rangle, |u\rangle) = (B^\dagger A^\dagger|v\rangle, |u\rangle)$ holds for all $|u\rangle, |v\rangle$, and the adjoint is unique, we conclude that

$$(AB)^\dagger = B^\dagger A^\dagger.$$

Alternative (index) proof. Using $(M^\dagger)_{ij} = M_{ji}^*$ and the rule for matrix multiplication,

$$((AB)^\dagger)_{ij} = (AB)_{ji}^* = \sum_k A_{jk}^* B_{ki}^* = \sum_k (B^\dagger)_{ik} (A^\dagger)_{kj} = (B^\dagger A^\dagger)_{ij}.$$

3. Prove that the Hadamard operator defined above is a self-adjoint operator, i.e. $H^\dagger = H$.

Solution: As we already mentioned, the elements of the adjoint Hadamard operator $H^\dagger$ are related to those of the Hadamard operator H as $(H^\dagger)_{ij} = H_{ji}^*$. Since all the entries of H are real, $H^* = H$, and since H is symmetric, $H^T = H$. Therefore

$$H^\dagger = (H^T)^* = H^T = H,$$

so the Hadamard operator is a *self-adjoint operator*.

- c. Compute the eigenvalues and eigenvectors of X and Z .

Solution: We will work with the matrix X . The eigenvectors $|v\rangle$ of the matrix X are such that when X acts on the vectors $|v\rangle$ they are only scaled by a factor λ (which is called an eigenvalue of the matrix), i.e. $X|v\rangle = \lambda|v\rangle$.

The eigenvalues λ of the matrix X must satisfy:

$$\det(X - \lambda I) = 0 \implies \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 - 1 = 0 \implies \lambda = \pm 1$$

Thus, we found that the eigenvalues of X are ± 1 . In order to find the eigenvectors, we replace the eigenvalues in the equation $X|v\rangle = \lambda|v\rangle$. Let's also write the vectors $|v\rangle$ as $|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$. For $\lambda = 1$ we have:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

We can then conclude that the eigenvector corresponding to the eigenvalue $\lambda = 1$ is $|v\rangle = \begin{pmatrix} a \\ a \end{pmatrix}$. If we impose the condition that the vector is normalized, $\| |v\rangle \| = 1$, and choose a real and positive (an eigenvector is only defined up to a phase), then we get $a = \frac{1}{\sqrt{2}}$. So the eigenvector becomes $|v\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

By working in the same manner for the second eigenvalue ($\lambda = -1$) it is easy to see that the second eigenvector is $|u\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. In the quantum computing literature you will find that these two vectors are usually denoted as $|+\rangle$ and $|-\rangle$.

It is trivial to see that for Z the eigenvectors are the states of the computational basis $|0\rangle$ and $|1\rangle$ with eigenvalues 1 and -1 respectively. The reason is that Z is a *diagonal* matrix: its eigenvalues are its diagonal elements and the corresponding eigenvectors are the computational basis states.

Problem 4: Euler's formula for complex numbers

Euler's formula is very handy when working with complex numbers in quantum computation. Any complex number z can be written in terms of its norm $|z|$ and a term $e^{i\phi}$, i.e. $z = |z|e^{i\phi}$. The term $e^{i\phi}$ is also a complex number, of norm 1, and is usually referred to in quantum computation, quantum mechanics and other fields as a *phase*.

a. Use *Euler's formula*, i.e. $e^{i\phi} = \cos \phi + i \sin \phi$, to calculate $e^{i\pi/2}$ and $e^{i\pi}$.

Solution: For the first case, $\phi = \pi/2$ and thus:

$$e^{i\pi/2} = \cos(\pi/2) + i \sin(\pi/2) = 0 + i = i$$

For the second case, $\phi = \pi$:

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0i = -1$$

b. Let $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. First calculate $|z| = \sqrt{zz^*}$ and then use Euler's formula to obtain ϕ so that $z = |z|e^{i\phi}$.

Solution: The complex conjugate of z is $z^* = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, so

$$zz^* = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \frac{1}{2} + \frac{1}{2} = 1 \quad \implies \quad |z| = \sqrt{zz^*} = 1.$$

Equivalently, using the definition of the norm from Problem 1c, $|w| = \sqrt{a^2 + b^2}$ for $w = a + bi$:

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

So we need to find the angle ϕ so that $z = |z|e^{i\phi} = e^{i\phi}$ (since $|z| = 1$). Using Euler's formula:

$$\begin{aligned} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i &= \cos \phi + i \sin \phi \\ \implies \left(\frac{1}{\sqrt{2}} - \cos \phi\right) - \left(\frac{1}{\sqrt{2}} + \sin \phi\right)i &= 0 \end{aligned}$$

For a complex number $w = a + bi$ to be equal to zero, both its real and imaginary parts must be equal to zero. First, for the real part:

$$\cos \phi = \frac{1}{\sqrt{2}}$$

and for the imaginary part:

$$\sin \phi = -\frac{1}{\sqrt{2}}$$

Thus $\phi = 7\pi/4$ and $z = e^{i7\pi/4}$. Since ϕ is only defined up to multiples of 2π , this is equivalent to $\phi = -\pi/4$, i.e. $z = e^{-i\pi/4}$.

c. In Problem 3a.1 you found that

$$H|R\rangle = \frac{1}{2} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}.$$

Write $1+i$ and $1-i$ in the form $|z|e^{i\phi}$ and use this to show that the result simplifies to $H|R\rangle = e^{i\pi/4}|L\rangle$, where $|L\rangle$ is the state from Problem 2. In other words, up to the phase $e^{i\pi/4}$, the Hadamard operator maps $|R\rangle$ to $|L\rangle$.

Solution: Both numbers have norm $|1 \pm i| = \sqrt{1^2 + 1^2} = \sqrt{2}$. For $1+i$ we need $\cos \phi = \sin \phi = \frac{1}{\sqrt{2}}$, so $\phi = \pi/4$; for $1-i$ we need $\cos \phi = \frac{1}{\sqrt{2}}$, $\sin \phi = -\frac{1}{\sqrt{2}}$, so $\phi = -\pi/4$ (as in part b). Hence

$$1+i = \sqrt{2}e^{i\pi/4}, \quad 1-i = \sqrt{2}e^{-i\pi/4} = \sqrt{2}e^{i\pi/4}e^{-i\pi/2} = \sqrt{2}e^{i\pi/4}(-i),$$

where we used $e^{-i\pi/2} = \cos(\pi/2) - i\sin(\pi/2) = -i$. Substituting and factoring out $e^{i\pi/4}$:

$$H|R\rangle = \frac{1}{2} \begin{pmatrix} \sqrt{2}e^{i\pi/4} \\ \sqrt{2}e^{i\pi/4}(-i) \end{pmatrix} = e^{i\pi/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{i\pi/4}|L\rangle.$$