

Problem 1: Quantum Operations

The *Hadamard* gate plays a very prominent role in quantum computation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- a. Prove that H is unitary, i.e. that it satisfies $HH^\dagger = H^\dagger H = I$.
- b. Prove that H is its own inverse by showing $H^2 = I$, where I is the identity operator.
- c. Calculate the action of H on the vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 2: Pauli matrices

Consider the four Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

- a. Prove that for each Pauli matrix σ_i we have $\sigma_i^2 = I$ and $\sigma_i^\dagger = \sigma_i$.
- b. Show that the Pauli matrices are unitary matrices.
- c. Show that $Y = iXZ$.
- d. Show that $HXH = Z$ and $HZH = X$, where H is the Hadamard gate from Problem 1.

Problem 3: Measurement

Consider the two quantum states $|R\rangle$ and $|L\rangle$ (eigenvectors of the Pauli Y operator, with eigenvalues $+1$ and -1 respectively):

$$|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

- a. Consider the general quantum state:

$$|\psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle, \quad |\psi_0|^2 + |\psi_1|^2 = 1.$$

What are the probabilities of the outcomes $|R\rangle$ and $|L\rangle$ if we measure $|\psi\rangle$ in the $\{|R\rangle, |L\rangle\}$ basis?

b. Show that the states $|R\rangle$ and $|L\rangle$ can be generated from $|0\rangle$ and $|1\rangle$, respectively, using the following circuit:

$$|0/1\rangle \longrightarrow \boxed{H} \longrightarrow \boxed{R_{\pi/2}} \longrightarrow |R/L\rangle$$

where

$$R_{\theta} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

(The gate $R_{\pi/2}$ is also known as the phase gate S .)

c. What circuit will allow implementing a measurement in the $\{|R\rangle, |L\rangle\}$ basis if our hardware only allows for measurement in the computational basis? Use H and R_{θ} gates.