

Problem 1: Quantum Operations

The *Hadamard* gate plays a very prominent role in quantum computation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

a. Prove that H is unitary, i.e. that it satisfies $HH^\dagger = H^\dagger H = I$.

Solution: A matrix U is unitary if it satisfies $UU^\dagger = U^\dagger U = I$.

First, we have to calculate the adjoint of the Hadamard operator. Recall that the matrix elements of the adjoint operator are related to those of the operator as $(H^\dagger)_{ij} = H_{ji}^*$. Thus:

$$H^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

So we can see that $H^\dagger = H$. In order for H to be unitary the following must hold:

$$H^\dagger H = HH^\dagger = I$$

We have:

$$HH^\dagger = H^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

and, since $H^\dagger = H$, also $H^\dagger H = H^2 = I$. Thus $H^\dagger H = HH^\dagger = I$.

b. Prove that H is its own inverse by showing $H^2 = I$, where I is the identity operator.

Solution: This is a corollary of the previous result. Since $H^\dagger = H$ and $HH^\dagger = I$, then $H^2 = HH^\dagger = I$, i.e. $H^{-1} = H$.

c. Calculate the action of H on the vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: We will show the action of the Hadamard on the computational basis vector $|0\rangle$ and on the vector $|+\rangle$.

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

We can see how H acts on $|+\rangle$ by doing the matrix multiplication, but we can think of a more “clever” way. We proved in part b that $H^2 = I$. So,

$$H|+\rangle = H(H|0\rangle) = H^2|0\rangle = |0\rangle$$

You can work in the same way with the other two vectors and prove that $H|1\rangle = |-\rangle$ and that $H|-\rangle = |1\rangle$.

Problem 2: Pauli matrices

Consider the four Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

a. Prove that for each Pauli matrix σ_i we have $\sigma_i^2 = I$ and $\sigma_i^\dagger = \sigma_i$.

Solution: For I both properties are trivial. We will only write out the proof for the Pauli matrix Y ; the proofs for X and Z are simpler and follow the same approach. We have:

$$Y^2 = YY = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

We will also prove that Y satisfies $Y^\dagger = Y$ (i.e., it is Hermitian):

$$Y^\dagger = \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^T \right]^* = \left[\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right]^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

b. Show that the Pauli matrices are unitary matrices.

Solution: We proved that for all Pauli matrices $\sigma_i^\dagger = \sigma_i$ and $\sigma_i^2 = I$. Then $\sigma_i^\dagger \sigma_i = \sigma_i \sigma_i^\dagger = \sigma_i \sigma_i = \sigma_i^2 = I$, so they are unitary.

c. Show that $Y = iXZ$.

Solution: We have:

$$iXZ = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

d. Show that $HXH = Z$ and $HZH = X$, where H is the Hadamard gate from Problem 1.

Solution: First, we will prove that $HXH = Z$. We have:

$$\begin{aligned} HXH &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = Z \end{aligned}$$

We can work in exactly the same way for $HZH = X$, or we can prove it differently. We proved that $HXH = Z$. Multiplying on the left and on the right by H we have $HHXH = HZH$. But $H^2 = I$, and so $X = HZH$.

Problem 3: Measurement

Consider the two quantum states $|R\rangle$ and $|L\rangle$ (eigenvectors of the Pauli Y operator, with eigenvalues $+1$ and -1 respectively):

$$\begin{aligned} |R\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \\ |L\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \end{aligned}$$

a. Consider the general quantum state:

$$|\psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle, \quad |\psi_0|^2 + |\psi_1|^2 = 1.$$

What are the probabilities of the outcomes $|R\rangle$ and $|L\rangle$ if we measure $|\psi\rangle$ in the $\{|R\rangle, |L\rangle\}$ basis?

Solution: We start with the probability of the outcome $|R\rangle$. If we measure the state $|\psi\rangle$, the probability of obtaining $|R\rangle$ is given by:

$$\begin{aligned} \text{Pr}[R] &= |\langle R|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)(\psi_0 |0\rangle + \psi_1 |1\rangle) \right|^2 \\ &= \frac{1}{2} |\psi_0 - i\psi_1|^2 \end{aligned}$$

where we used $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$, since the basis vectors are orthonormal. Similarly, for the other probability we have:

$$\begin{aligned} \text{Pr}[L] &= |\langle L|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)(\psi_0 |0\rangle + \psi_1 |1\rangle) \right|^2 \\ &= \frac{1}{2} |\psi_0 + i\psi_1|^2 \end{aligned}$$

As a check, expanding the moduli, the cross terms $\pm i(\psi_0^* \psi_1 - \psi_0 \psi_1^*)/2$ cancel in the sum, so

$$\text{Pr}[R] + \text{Pr}[L] = |\psi_0|^2 + |\psi_1|^2 = 1.$$

b. Show that the states $|R\rangle$ and $|L\rangle$ can be generated from $|0\rangle$ and $|1\rangle$, respectively, using the following circuit:

$$|0/1\rangle \text{---} \boxed{H} \text{---} \boxed{R_{\pi/2}} \text{---} |R/L\rangle$$

where

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

(The gate $R_{\pi/2}$ is also known as the phase gate S .)

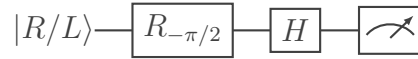
Solution:

$$R_{\pi/2}H|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = |R\rangle$$

$$R_{\pi/2}H|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = |L\rangle$$

c. What circuit will allow implementing a measurement in the $\{|R\rangle, |L\rangle\}$ basis if our hardware only allows for measurement in the computational basis? Use H and R_θ gates.

Solution: We simply run the circuit of part b backwards. In 1b we saw that H is its own inverse, and it can be verified that $R_{-\pi/2}$ is the inverse of the $R_{\pi/2}$ gate, i.e. $R_{-\pi/2}R_{\pi/2} = R_{\pi/2}R_{-\pi/2} = I$. Hence, the circuit



maps $|R\rangle \mapsto |0\rangle$ and $|L\rangle \mapsto |1\rangle$. Indeed, $R_{-\pi/2}|R\rangle = |+\rangle$ and $H|+\rangle = |0\rangle$, while $R_{-\pi/2}|L\rangle = |-\rangle$ and $H|-\rangle = |1\rangle$. A measurement in the computational basis then distinguishes them: outcome 0 corresponds to $|R\rangle$ and outcome 1 to $|L\rangle$.
