

Problem 1: Tensor Product

a. Consider the quantum state:

$$|\psi\rangle = \frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle$$

1. Calculate $|\psi\rangle^{\otimes 2}$, where $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$.
2. Calculate $|+\rangle \otimes |-\rangle \otimes |+\rangle$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

b. Consider the four Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Calculate:

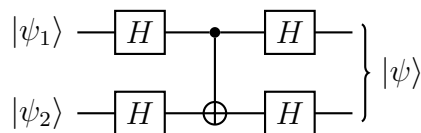
1. $Z \otimes X$
2. $X \otimes Y$
3. Using the two tensor products $Z \otimes X$ and $X \otimes Y$ above, compute their matrix product $(Z \otimes X)(X \otimes Y)$.
4. Compute the two 2×2 matrix products ZX and XY , followed by the tensor product of their results, to obtain $ZX \otimes XY$. Show that $(Z \otimes X)(X \otimes Y) = ZX \otimes XY$.

c. Consider two linear operators A, B .

1. Prove that if A, B are unitary operators, then $(A \otimes B)$ is also unitary.
2. Prove that if A, B are projector operators, then $(A \otimes B)$ is also a projector.

Problem 2: Concatenation and composition of gates

Consider the quantum circuit which consists of two Hadamard gates H , followed by a CNOT and finally two more Hadamard gates:



a. We have seen in the lectures that a CNOT can be written in terms of a linear combination of tensor products of projectors onto the computational basis, $\Pi_0 = |0\rangle\langle 0|$ and $\Pi_1 = |1\rangle\langle 1|$, the identity matrix I and the Pauli matrix X . Using the rules of tensor product and linearity seen in the course, prove that this circuit is equivalent to a reversed CNOT (control on the lower qubit).

b. Calculate the output state $|\psi\rangle$ via application of the different quantum gates to the inputs:

1. $|\psi_1\rangle = a|0\rangle + b|1\rangle$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$, a general quantum state, and $|\psi_2\rangle = |0\rangle$.
2. $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = a|0\rangle + b|1\rangle$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

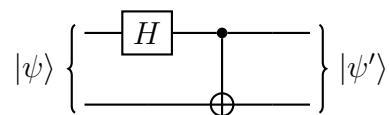
Problem 3: Measurement in the Bell basis

a. Consider the four Bell quantum states:

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), & |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), & |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned}$$

1. Verify that the Bell states form an orthonormal basis of the Hilbert space of the composite (two-qubit) system.
2. Prove the *completeness relation* $\sum_{i=1}^4 |u_i\rangle\langle u_i| = I_4$, where $\{|u_i\rangle\}$ is the set of four Bell states.

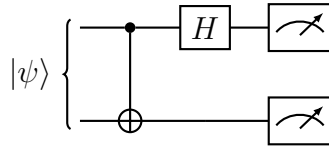
b. Consider the quantum circuit:



Calculate the output state when:

- $|\psi\rangle = |0\rangle|0\rangle$
- $|\psi\rangle = |0\rangle|1\rangle$
- $|\psi\rangle = |1\rangle|0\rangle$
- $|\psi\rangle = |1\rangle|1\rangle$

c. Consider the quantum circuit ending in a measurement of both qubits in the computational basis, leading to four possible outcomes 00, 01, 10, and 11:



1. If we use the Bell state $|\Psi^+\rangle$ as input to the circuit, what is the probability of each of the 4 possible outcomes?
2. And what about when we use $|\Psi^-\rangle$ and $|\Phi^\pm\rangle$ as input?
3. What are the outcome probabilities resulting from the input state $|00\rangle$?
4. What will be the outcome probabilities when we input any state of the computational basis of the two qubits, i.e., $|x_1\rangle \otimes |x_2\rangle$ where $x_i \in \{0, 1\}$?