Problem 1: Tensor Product

a. Consider the quantum state:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

1. Calculate $|\psi\rangle^{\otimes 2}$, where $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$.
2. Calculate $|+\rangle \otimes |-\rangle \otimes |+\rangle$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

Solution: There are two ways to represent the tensor product $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$: one can use either the "bra-ket" notation or the matrix representation. Starting with the "bra-ket" notation, we have:

$$\begin{aligned}|\psi\rangle^{\otimes 2} = |\psi\rangle \otimes |\psi\rangle &= \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \\ &= \frac{3}{4}|0\rangle \otimes |0\rangle + \frac{\sqrt{3}}{4}|0\rangle \otimes |1\rangle + \frac{\sqrt{3}}{4}|1\rangle \otimes |0\rangle + \frac{1}{4}|1\rangle \otimes |1\rangle \\ &= \frac{3}{4}|00\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle,\end{aligned}$$

where we used the simplified notation $|xy\rangle = |x\rangle \otimes |y\rangle$. Alternatively, if we use the matrix representation and recall that $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, we can write $|\psi\rangle$ in matrix form:

$$|\psi\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$$

Thus, we can easily calculate $|\psi\rangle^{\otimes 2}$ as:

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3\\\sqrt{3}\\\sqrt{3}\\1\end{pmatrix}$$

The two representations are equivalent, as can easily be verified by calculating the matrix representations of the basis vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

For the second case, $|+\rangle|-\rangle|+\rangle$, working in exactly the same way we can verify that:

$$|+\rangle|-\rangle|+\rangle = \frac{1}{2^{3/2}}(|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle),$$

where we used the simplified notation $|xyz\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle$.

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b. Consider the four Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Calculate:

1. $Z \otimes X$
2. $X \otimes Y$
3. Using the two tensor products $Z \otimes X$ and $X \otimes Y$ above, compute their matrix product $(Z \otimes X)(X \otimes Y)$.
4. Compute the two 2×2 matrix products ZX and XY, followed by the tensor product of their results, to obtain $ZX \otimes XY$. Show that $(Z \otimes X)(X \otimes Y) = ZX \otimes XY$.

Solution: We will calculate the tensor product $Z \otimes X$ step by step. We have:

$$\begin{aligned} Z \otimes X &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

Similarly, for $X \otimes Y$ we get:

$$X \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

The third question is just a matrix multiplication of two 4×4 matrices. Thus,

$$\begin{aligned} (Z \otimes X)(X \otimes Y) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \end{aligned}$$

Finally, for the last question, the identity $(A \otimes B)(C \otimes D) = AC \otimes BD$ states that we can first multiply the operators that act on the same subsystem and then take the tensor product. Recall the properties of the Pauli matrices:

$$ZX = iY = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad XY = iZ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

So,

$$ZX \otimes XY = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

which coincides with the result of the third question, so indeed $(Z \otimes X)(X \otimes Y) = ZX \otimes XY$.

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c. Consider two linear operators A, B.

1. Prove that if A, B are unitary operators, then $(A \otimes B)$ is also unitary.
2. Prove that if A, B are projector operators, then $(A \otimes B)$ is also a projector.

Solution: Consider any two linear operators A and B. It is easy to prove that:

$$(A \otimes B)^* = A^* \otimes B^*$$

$$(A \otimes B)^T = A^T \otimes B^T$$

If we combine the above equations, we get:

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

An operator U is called unitary if it satisfies $UU^\dagger = U^\dagger U = I$. Consider two unitary operators A, B, which satisfy $AA^\dagger = A^\dagger A = I$ and $BB^\dagger = B^\dagger B = I$. Using the identity of question b.4 we get:

$$\begin{aligned}(A \otimes B)^\dagger (A \otimes B) &= (A^\dagger \otimes B^\dagger)(A \otimes B) \\ &= A^\dagger A \otimes B^\dagger B = I \otimes I = I\end{aligned}$$

In the same way:

$$(A \otimes B)(A \otimes B)^\dagger = AA^\dagger \otimes BB^\dagger = I \otimes I = I,$$

and thus we proved that if A, B are unitary operators, then $(A \otimes B)$ is also unitary.

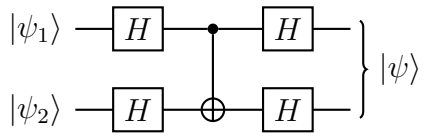
Now for the second question, an operator Π is called a projector if it satisfies $\Pi^2 = \Pi$. Using again the identity of question b.4 we have

$$(A \otimes B)^2 = (A \otimes B)(A \otimes B) = A^2 \otimes B^2 = A \otimes B$$

and so indeed if A, B are projector operators then $(A \otimes B)$ is also a projector. If projectors are additionally required to be Hermitian ($\Pi^\dagger = \Pi$), this also holds, since $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$.

Problem 2: Concatenation and composition of gates

Consider the quantum circuit which consists of two Hadamard gates H, followed by a CNOT and finally two more Hadamard gates:



a. We have seen in the lectures that a CNOT can be written in terms of a linear combination of tensor products of projectors onto the computational basis, $\Pi_0 = |0\rangle\langle 0|$ and $\Pi_1 = |1\rangle\langle 1|$, the identity matrix I and the Pauli matrix X. Using the rules of tensor product and linearity seen in the course, prove that this circuit is equivalent to a reversed CNOT (control on the lower qubit).

Solution: First, recall that the CNOT gate can be written as $\text{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = \Pi_0 \otimes I + \Pi_1 \otimes X$.

The unitary operator corresponding to the action of the circuit can be written as $U = (H \otimes H)\,\text{CNOT}\,(H \otimes H)$. Using linearity and $(A \otimes B)(C \otimes D) = AC \otimes BD$:

$$\begin{aligned} U &= (H \otimes H)(\Pi_0 \otimes I + \Pi_1 \otimes X)(H \otimes H) \\ &= H\Pi_0 H \otimes HIH + H\Pi_1 H \otimes HXH \\ &= \Pi_+ \otimes I + \Pi_- \otimes Z, \end{aligned}$$

where $\Pi_+ = |+\rangle\langle +|$ and $\Pi_- = |-\rangle\langle -|$ are the projectors onto the $\{|+\rangle, |-\rangle\}$ basis. Here we used $H\Pi_0 H = H|0\rangle\langle 0|H = |+\rangle\langle +|$ (since H is Hermitian and $H|0\rangle = |+\rangle$), similarly $H\Pi_1 H = |-\rangle\langle -|$, together with $H^2 = I$ and the identity $HXH = Z$. Now by expanding the unitary in its matrix representation, we get:

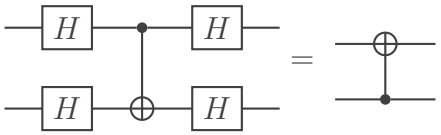
$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\implies U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In order to understand whether this is a well-known 2-qubit gate, we can see how it acts on the computational basis states. We have:

$$\begin{aligned} U|00\rangle &= |00\rangle, \qquad U|01\rangle = |11\rangle, \\ U|10\rangle &= |10\rangle, \qquad U|11\rangle = |01\rangle. \end{aligned}$$

It is clear then that the circuit U is actually a CNOT gate with the control and target qubits interchanged. Thus:



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b. Calculate the output state $|\psi\rangle$ via application of the different quantum gates to the inputs:

1. $|\psi_1\rangle = a\,|0\rangle + b\,|1\rangle$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$, a general quantum state, and $|\psi_2\rangle = |0\rangle$.

   Solution: Since the circuit is equivalent to a CNOT controlled by the lower qubit, and the control qubit is $|\psi_2\rangle = |0\rangle$, the target (upper) qubit is left unchanged. The output state is therefore

   $$|\psi\rangle = (a\,|0\rangle + b\,|1\rangle) \otimes |0\rangle = a\,|00\rangle + b\,|10\rangle\,.$$

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2. $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = a\,|0\rangle + b\,|1\rangle$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

   Solution: We start with the input state:

   $$|\psi_{\text{in}}\rangle = |0\rangle \otimes (a\,|0\rangle + b\,|1\rangle) = a\,|00\rangle + b\,|01\rangle$$

   Then the action of the reversed CNOT is just:

   $$|\psi\rangle = U\,|\psi_{\text{in}}\rangle = a\,|00\rangle + b\,|11\rangle$$

   Note that this output state is an *entangled state* if both a and b are non-zero, which means that it cannot be written as a tensor product of states of the two subsystems, i.e., $|\psi\rangle \neq |\phi_1\rangle \otimes |\phi_2\rangle$ for any single-qubit states $|\phi_1\rangle$ and $|\phi_2\rangle$.

Problem 3: Measurement in the Bell basis

a. Consider the four Bell quantum states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

1. Verify that the Bell states form an orthonormal basis of the Hilbert space of the composite (two-qubit) system.
2. Prove the *completeness relation* $\sum_{i=1}^{4} |u_i\rangle \langle u_i| = I_4$, where $\{|u_i\rangle\}$ is the set of four Bell states.

Solution: First of all, recall that for computational basis vectors $|ij\rangle \equiv |i\rangle \otimes |j\rangle$:

$$\langle kl|ij\rangle = \delta_{ki}\delta_{lj}$$

We will prove that $|\Phi^+\rangle$ is orthogonal to $|\Phi^-\rangle$. You are advised to do the same for all other states. We have:

$$\begin{aligned}\langle\Phi^+|\Phi^-\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle - |11\rangle) \\ &= \frac{1}{2}(\langle 00|00\rangle + \langle 11|00\rangle - \langle 00|11\rangle - \langle 11|11\rangle) = 0\end{aligned}$$

so indeed $|\Phi^+\rangle$ and $|\Phi^-\rangle$ are orthogonal. The same applies to every other pair of Bell states. (Note that any $|\Phi^\pm\rangle$ is trivially orthogonal to any $|\Psi^\pm\rangle$, since they are supported on different computational basis states.) We can also easily prove that all Bell states are normalised to 1. We work again with $|\Phi^+\rangle$:

$$\begin{aligned}\sqrt{\langle\Phi^+|\Phi^+\rangle} &= \sqrt{\frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle)} \\ &= \sqrt{\frac{1}{2}(\langle 00|00\rangle + \langle 11|00\rangle + \langle 00|11\rangle + \langle 11|11\rangle)} = 1\end{aligned}$$

Thus, the four Bell quantum states form an orthonormal basis for $\mathcal{H} = \mathbb{C}^4$ (i.e. $\mathcal{H} = \text{span}\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$) as:

- They are orthogonal.
- They are normalised to one.
- The number of basis vectors equals the dimension of the Hilbert space.

For the completeness relation, we must calculate the outer products $|u_i\rangle \langle u_i|$ for every Bell state $|u_i\rangle$. We have:

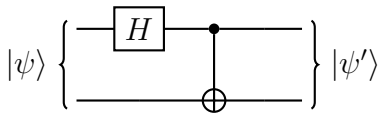
$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}, \quad |\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}\begin{pmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{pmatrix},$$

$$|\Psi^+\rangle\langle\Psi^+| = \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}, \quad |\Psi^-\rangle\langle\Psi^-| = \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{pmatrix}.$$

Adding them, the off-diagonal terms cancel and we obtain $|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-| = I_4$, which proves the *completeness relation*.

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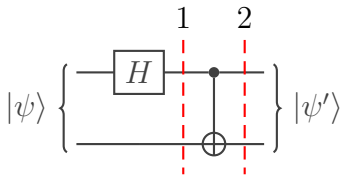
b. Consider the quantum circuit:



Calculate the output state when:

- $|\psi\rangle = |0\rangle|0\rangle$
- $|\psi\rangle = |0\rangle|1\rangle$
- $|\psi\rangle = |1\rangle|0\rangle$
- $|\psi\rangle = |1\rangle|1\rangle$

Solution: We will find the output state for the first case, where $|\psi\rangle = |0\rangle|0\rangle$. We split the circuit into two steps:



Step 1: In the first step, the operator $H \otimes I$ acts on the state $|\psi\rangle$ and gives us:

$$|\psi\rangle_1 = (H \otimes I)(|0\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

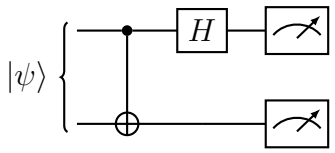
Step 2: In the second step, we act with the CNOT and get:

$$|\psi'\rangle = \text{CNOT}\,|\psi\rangle_1 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

Working in exactly the same way for the rest of the inputs, you can verify that:

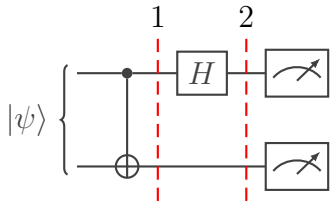
- If $|\psi\rangle = |0\rangle|1\rangle$ then $|\psi'\rangle = |\Psi^+\rangle$.
- If $|\psi\rangle = |1\rangle|0\rangle$ then $|\psi'\rangle = |\Phi^-\rangle$.
- If $|\psi\rangle = |1\rangle|1\rangle$ then $|\psi'\rangle = |\Psi^-\rangle$.

c. Consider the quantum circuit ending in a measurement of both qubits in the computational basis, leading to four possible outcomes $00, 01, 10$, and 11:



1. If we use the Bell state $|\Psi^+\rangle$ as input to the circuit, what is the probability of each of the 4 possible outcomes?
2. And what about when we use $|\Psi^-\rangle$ and $|\Phi^\pm\rangle$ as input?
3. What are the outcome probabilities resulting from the input state $|00\rangle$?
4. What will be the outcome probabilities when we input any state of the computational basis of the two qubits, i.e., $|x_1\rangle \otimes |x_2\rangle$ where $x_i \in \{0, 1\}$?

Solution: First of all, we consider as input the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Similarly to the previous exercises, we break the quantum circuit into three parts, the first two corresponding to the action of the quantum gates and the third one to the measurement:



Step 1: We act with the CNOT gate and thus we get:

$$|\psi\rangle_1 = \text{CNOT}\,|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |1\rangle = |+\rangle \otimes |1\rangle$$

Step 2: We act with the Hadamard operator on the first qubit, $H \otimes I$:

$$|\psi\rangle_2 = (H \otimes I)\,|\psi\rangle_1 = H\,|+\rangle \otimes |1\rangle = |0\rangle \otimes |1\rangle = |01\rangle$$

So we obtain the outcome 01 with probability 1, and the outcomes 00, 10 and 11 with probability 0. For the other three input states $|\Psi^-\rangle$, $|\Phi^+\rangle$ and $|\Phi^-\rangle$ we obtain, with probability 1, the outcomes 11, 00 and 10 respectively (and the other outcomes with probability 0).

Now for the case where $|\psi\rangle = |00\rangle$ we have:

Step 1:

$$|\psi\rangle_1 = |00\rangle$$

Step 2:

$$|\psi\rangle_2 = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

In this case, we obtain the outcomes 00 or 10, each with probability 1/2. For question 4, we can work in the same manner to get:

- If the input state is $|\psi\rangle = |01\rangle$, then the output state is $|\psi'\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$. In this case, we obtain 01 or 11, each with probability 1/2.
- If the input state is $|\psi\rangle = |10\rangle$, then the output state is $|\psi'\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$. In this case, we obtain 01 or 11, each with probability 1/2.
- If the input state is $|\psi\rangle = |11\rangle$, then the output state is $|\psi'\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$. In this case, we obtain 00 or 10, each with probability 1/2.