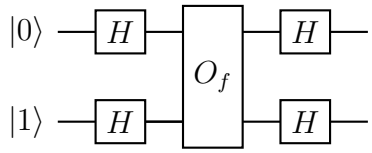
Problem 1: Deutsch's Algorithm

Consider the following circuit:



a. The oracle O_f is a two-qubit gate that maps $|x\rangle\,|y\rangle \to |x\rangle\,|y \oplus x\rangle$. By comparing to what we have seen in the lectures, what is the classical function implemented by the oracle O_f, i.e. $f(x)$? Do you think $f(x)$ will be balanced or constant?

b. What is the circuit of O_f?

c. Compute the two-qubit output state of this circuit and the probability of getting outcome 0 when measuring the upper qubit.

d. Having found the probability of getting outcome 0 on the upper qubit, conclude whether the function is balanced or constant. Justify your answer.

Problem 2: Phase kick-back and the phase oracle

Suppose you have the balanced function $f : \{0,1\}^2 \to \{0,1\}$ such that:

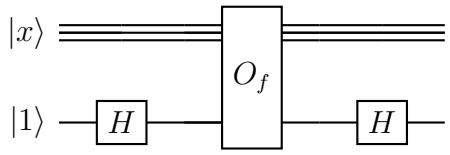
$$f(0,0) = 0, \quad f(0,1) = 1,$$
$$f(1,0) = 1, \quad f(1,1) = 0.$$

a. A classical oracle C_f returns, for a given query input $x \in \{0,1\}^n$, the value $f(x)$ of the Boolean function $f : \{0,1\}^n \to \{0,1\}$. We have seen that for every classical oracle there exists a quantum oracle O_f satisfying

$$O_f\,|x\rangle\,|q\rangle = |x\rangle\,|q \oplus f(x)\rangle\,.$$

Provide a circuit of 3 qubits implementing O_f for the function given above.

b. In the lectures, we have seen that the oracle O_f can be used to implement a phase oracle U_f acting on the address qubits. Show that the circuit below



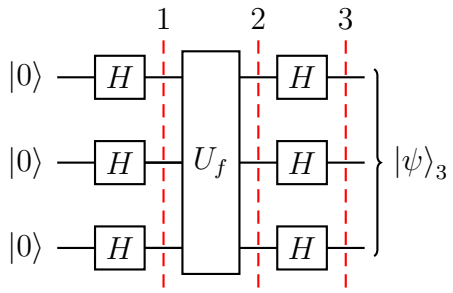
is equivalent to applying the phase oracle U_f to the address register:

$$U_f\,|x\rangle = (-1)^{f(x)}\,|x\rangle\,.$$

c. What is the size of the address register in Deutsch's algorithm in Problem 1?

Problem 3: Bernstein-Vazirani Algorithm

Consider the function $f(x) = a \cdot x \bmod 2$, where $a \cdot x = \sum_i a_i x_i$, with the string $a = 111$. The goal is to find a with a single call to the phase oracle U_f. Consider the quantum circuit implementing the Bernstein-Vazirani algorithm:



a. Write the quantum state at stage 1 of the figure above, i.e. after the first layer of parallel Hadamard gates.

b. What transformation does the phase oracle U_f perform on the state $|x\rangle$, where x is a string of 3 bits encoding the computational basis of 3 qubits?

c. Calculate the state of the composite system at stage 2 of the circuit.

d. Derive the action of a layer of three Hadamard gates (Walsh-Hadamard transform) on a computational basis state $|x_1x_2x_3\rangle$ of three qubits.

e. Provide the quantum state at stage 3 of the computation.

f. Suppose that we perform a measurement in the computational basis. What is the probability of the outcome being the $|000\rangle$ state? What would be the probability of obtaining $|111\rangle$?