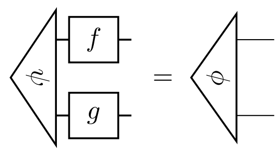
Introduction to Quantum Programming and Semantics 2025
Coursework

This coursework will count for 30% of your overall grade. Upload your answers to Gradescope by 12 noon on Monday 24th February. Make any combination of 5 exercises from the 10 listed below; the first five are the standard curriculum, the last five are for those who want to learn some category theory. Each exercise will be weighed equally.

Exercise 1

We say that two joint states ψ and ϕ of $A \otimes B$ are *locally equivalent*, written $\psi \sim \phi$, if there exist invertible maps $A \xrightarrow{f} A$ and $B \xrightarrow{g} B$ such that



(a) Show that $\sim$ is an equivalence relation.

In **Hilb**, we can write a state $\mathbb{C} \xrightarrow{\phi} \mathbb{C}^2 \otimes \mathbb{C}^2$ as a column vector

$$\phi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

or as a matrix

$$M_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

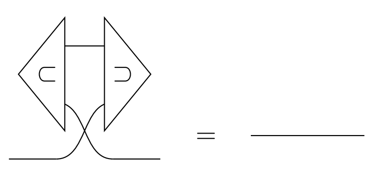
(b) Show that ϕ is an entangled state if and only if M_ϕ is invertible. (Hint: a matrix is invertible if and only if it has nonzero determinant.)

(c) Show that $M_{(\mathrm{id}_{\mathbb{C}^2} \otimes f) \circ \phi} = M_\phi \circ f^{\mathrm{T}}$, where $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}^2$ is any linear map and f^{T} is the transpose of f in the canonical basis of $\mathbb{C}^2$.

(d) Use this to show that there are three families of locally equivalent joint states of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Exercise 2

Prove that



follows from the following 4 equations:

(i)

(ii)

In fact, only 2 equations are needed: prove that either of (i) or (ii) implies the other.

Exercise 3

This excercise is about encoding classical functions as linear maps using orthonormal basis states and effects (see [KW 5.3.4]). For a function $F : \{0,1\}^m \to \{0,1\}^n$, we can define an associated linear map f as

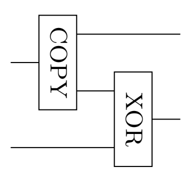
$$f = \sum_{(a_1 \dots a_m \mapsto b_1 \dots b_n) \in F} (|a_1\rangle \langle b_1|) \otimes \cdots \otimes (|a_n\rangle \langle b_n|)$$

where the notation $(a_1 \dots a_m \mapsto b_1 \dots b_n) \in F$ means we are summing over the *graph of* F, i.e. the set of bitstrings $\{(a_1, \dots, a_m, b_1, \dots b_n) \mid F(a_1, \dots, a_m) = (b_1, \dots b_n)\}$. Using this encoding, define:

$$\begin{aligned} \text{XOR} &= |0\rangle \langle 00| + |1\rangle \langle 01| + |1\rangle \langle 10| + |0\rangle \langle 11| \\ \text{CNOT} &= |00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 11| + |11\rangle \langle 10| \\ \text{COPY} &= |00\rangle \langle 0| + |11\rangle \langle 1| \end{aligned}$$

Show that

CNOT $=$



(Hint: try comparing the left-hand side to the right-hand side on all basis states, rather than writing out a big sum.)

Next, find ψ and ϕ such that the following equation holds.

$$|\phi\rangle \langle \psi| \quad = \quad \text{XOR} \circ \text{COPY}$$

Exercise 4

Recall that a set $\{x_0, \dots, x_n\}$ of vectors in a vector space is *linearly independent* if $\sum_{i=0}^n z_i x_i = 0$ for $z_i \in \mathbb{C}$ implies $z_0 = \dots = z_n = 0$. Show that the nonzero copyable states of an associative and unital linear map $\mathbb{C}^n \xrightarrow{d} \mathbb{C}^n \otimes \mathbb{C}^n$ are linearly independent. (Hint: consider a minimal linearly dependent set.)

Exercise 5

Given that a and b are qubits, show that the following three OpenQASM programs are denotationally equivalent.

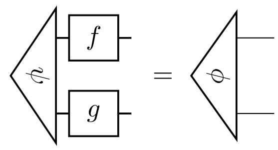
```
ctrl @ X a, b;
```

```
H b;
ctrl @ Z a, b;
H b;
```

```
H b;
ctrl @ Z b, a;
H b;
```

Exercise 6

We say that two joint states ψ and ϕ of $A \otimes B$ are *locally equivalent*, written $\psi \sim \phi$, if there exist invertible maps $A \xrightarrow{f} A$ and $B \xrightarrow{g} B$ such that



(a) Show that $\sim$ is an equivalence relation.

(b) Find all isomorphisms $\{0, 1\} \to \{0, 1\}$ in **Rel**.

(c) Write out all 16 states of the object $\{0, 1\} \times \{0, 1\}$ in **Rel**.

(d) Use your answer to (b) to group the states of (c) into locally equivalent families. How many families are there? Which of these are entangled?

Exercise 7

In a monoidal category, show that:

(a) if an initial object 0 exists and $L \dashv R$, then $L \otimes 0 \simeq 0 \simeq 0 \otimes R$;

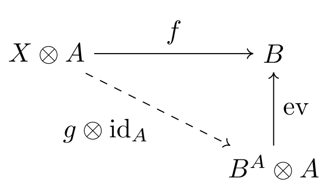
(b) if a terminal object 1 exists and $L \dashv R$, then $R \otimes 1 \simeq 1 \simeq 1 \otimes L$.

Exercise 8

In **Rel**, show that the trace of an endomorphism can be used to identify whether a relation has a fixed point.

Exercise 9

Let A and B be objects in a monoidal category. Their *exponential* is an object B^A together with a map $B^A \otimes A \xrightarrow{\text{ev}} B$ such that every morphism $X \otimes A \xrightarrow{f} B$ allows a unique morphism $X \xrightarrow{g} B^A$ with $f = \text{ev} \circ (g \otimes \text{id}_A)$.



The category is called *left closed* when every pair of objects has an exponential. Show that any monoidal category in which every object has a left dual is left closed.

Exercise 10

This exercise is about *property* versus *structure*.

(a) Suppose that a category **C** has products and terminal objects. You may assume that this implies that any object has a unique comonoid structure with respect to the monoidal structure given by the categorical product. Show that any monoid in **C** has a unique bialgebra structure with respect to the monoidal structure given by the categorical product.

(b) It now follows from [HV, Theorem 6.3.2] that being a Hopf algebra is a property of, rather than a structure on, a bialgebra. Prove directly that a bialgebra can have at most one antipode.