Introduction to Quantum Programming and Semantics

Lecture 13: Clifford circuits

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Overview

- Clifford circuits and Clifford diagrams
- Graph states with local Clifford operations
- Local complementation, pivoting
- Strong simulation of Clifford circuits
- Synthesis of Clifford circuits

Clifford circuits and Clifford diagrams

Clifford circuits

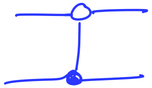
are made from:

$S =$ 

$= \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$

$H =$ 

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

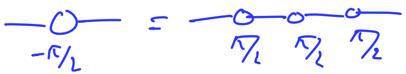
$CNOT =$  $= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$

e.g.:

$Z =$ 

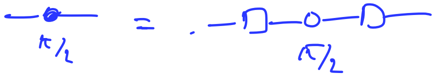
$= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

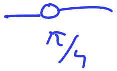
$CZ =$ 

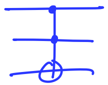
$S^\dagger =$ 

$X =$ 

$= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$\sqrt{X} =$ 

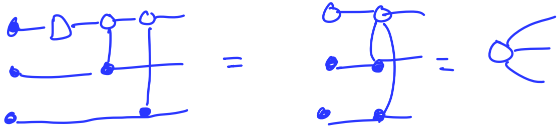
but not: $T =$ 

$TOF = CCX =$ 

Why Clifford circuits interesting?

- contain useful states:

Bell 

GHZ 

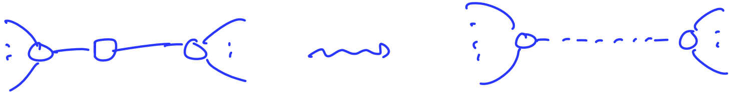
- quantum error correction
- classical simulation efficient
- rich rewrite theory

Clifford diagrams

is a ZX diagram made of "Clifford spiders"

for $k \in \mathbb{Z}$

<u>notation</u>:



"Hadamard edges"

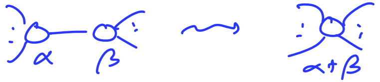
a ZX diagram is <u>graph-like</u> if:

- all spiders are Z spiders
- all edges are Hadamard edges
- no parallel edges or self-loops
- every input/output is connected to spider

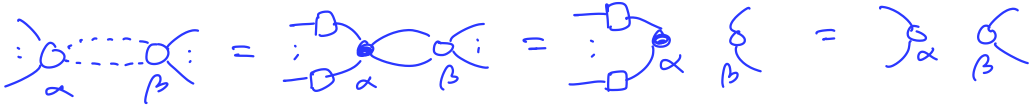
Clifford diagrams

Prop: every ZX diagram is equal to a graph-like one.

Proof: 1. use colour change to eliminate X spiders

2. use spider fusion to eliminate non-H edges 

3. eliminate parallel edges:



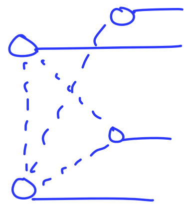
eliminate self-loops:

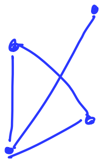
4. use 

Graph states with local Clifford operations

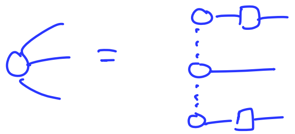
Graph states are graph-like ZX diagrams with no input

e.g.

$|G\rangle =$ 

$G =$ 

but not



def: a graph state w local Cliffords (GSLC) is a state of form $(U_1 \otimes \cdots \otimes U_n)|G\rangle$ for some graph G and 1-qubit Clifford gates U_i

Normal form

Def: a Clifford state is of form $|\psi\rangle = C|0\cdots 0\rangle$ for some Clifford circuit C.

Thm any Clifford state is equal to a GSLC

Local complementation, pivoting

Local complementation of graph G about vertex u

is a new graph $G * u = (V, E')$ where

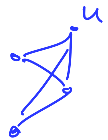
V set of vertices
$E \subseteq V \times V$ set of edges

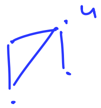
$$\forall v, w \in N_G(u): \quad (v,w) \in E' \iff (v,w) \in E$$

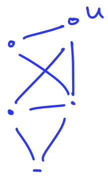
neighborhood

$$\text{and} \quad E' \cap (V \setminus N_G(u)) \times V = E \cap (V \setminus N_G(u)) \times V$$

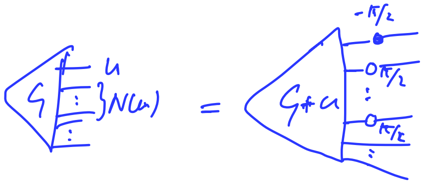
ex:

$G =$ 

$G * u =$ 

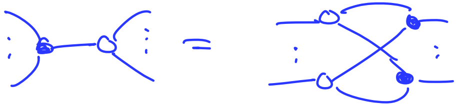
$G =$ 

$G * u =$ 

Prop: 

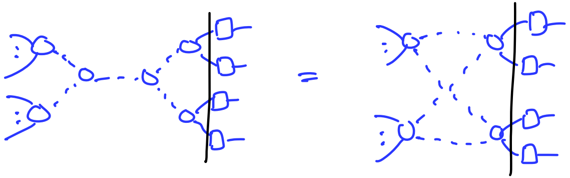
Pivoting

consider strong complementarity:



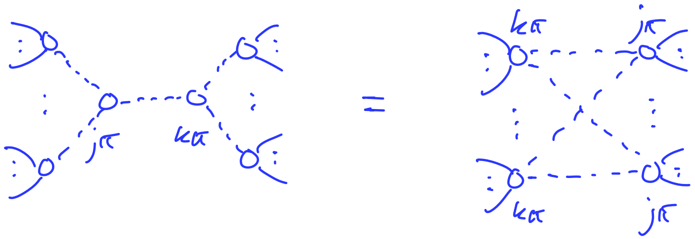
add context:

deletes spiders.



deleted adjacent phase free spiders

<u>pivot rule</u>:

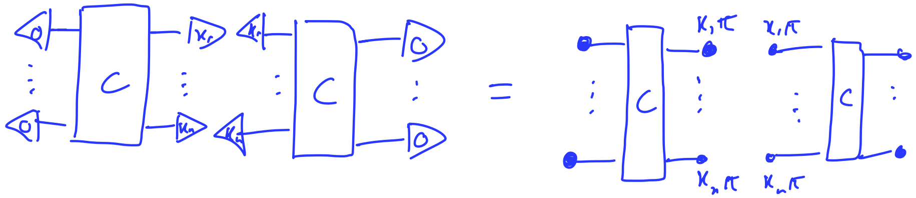


Simulating Clifford circuits

Simulation of Clifford circuits

given Clifford circuit C, compute $\text{Prob}(x_1, \ldots, x_n \mid |\psi\rangle)$, where $|\psi\rangle = C|0 \cdots 0\rangle$

use Born rule $\text{Prob}(x_1 \cdots x_n \mid |\psi\rangle) = |\langle x_1 \cdots x_n | C | 0 \cdots 0 \rangle|^2$

= 

$= D$

algorithm: consider diagram D, run rewrite strat "Clifford-simp":

1. convert to graph-like diagram
2. repeat until impossible: local complementation and pivoting
3. remove all isolated $\{0, \pi\}$-spiders

prop: this terminates

Simulation of Clifford circuits

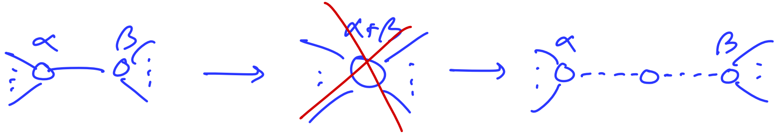
prop: algorithm terminates in polynomial time (in # qubits $=: n$ and # gates of C $=: k$)

$O((n+k)^3)$

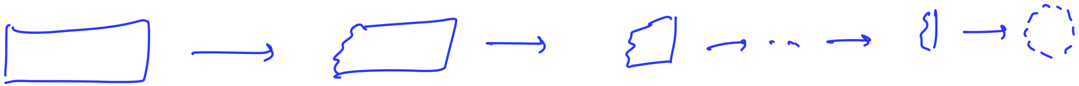
remark this is not optimal: doing step 2 optimally takes $O(n^2 k)$ steps

if $k >> n$, this makes a big difference

idea: 1. avoid big spiders:



2. apply local complementation & pivoting left-to-right:



each step involves at most $O(n)$ spiders (hence $O(n^2)$ wires)

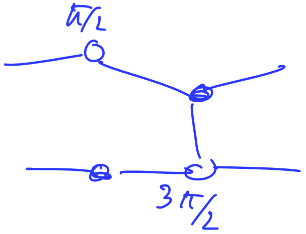
Synthesising Clifford circuits

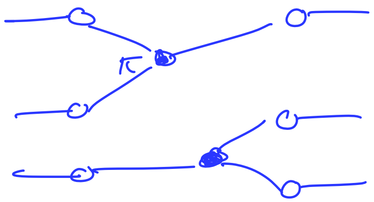
Synthesis of Clifford circuits

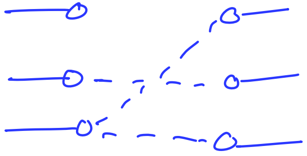
Clifford diagram $\longrightarrow$ AP-form $\longrightarrow$ GSLC

AP-form: all interior spiders
- have phase $0, \pi$
- are only connected to boundary spiders

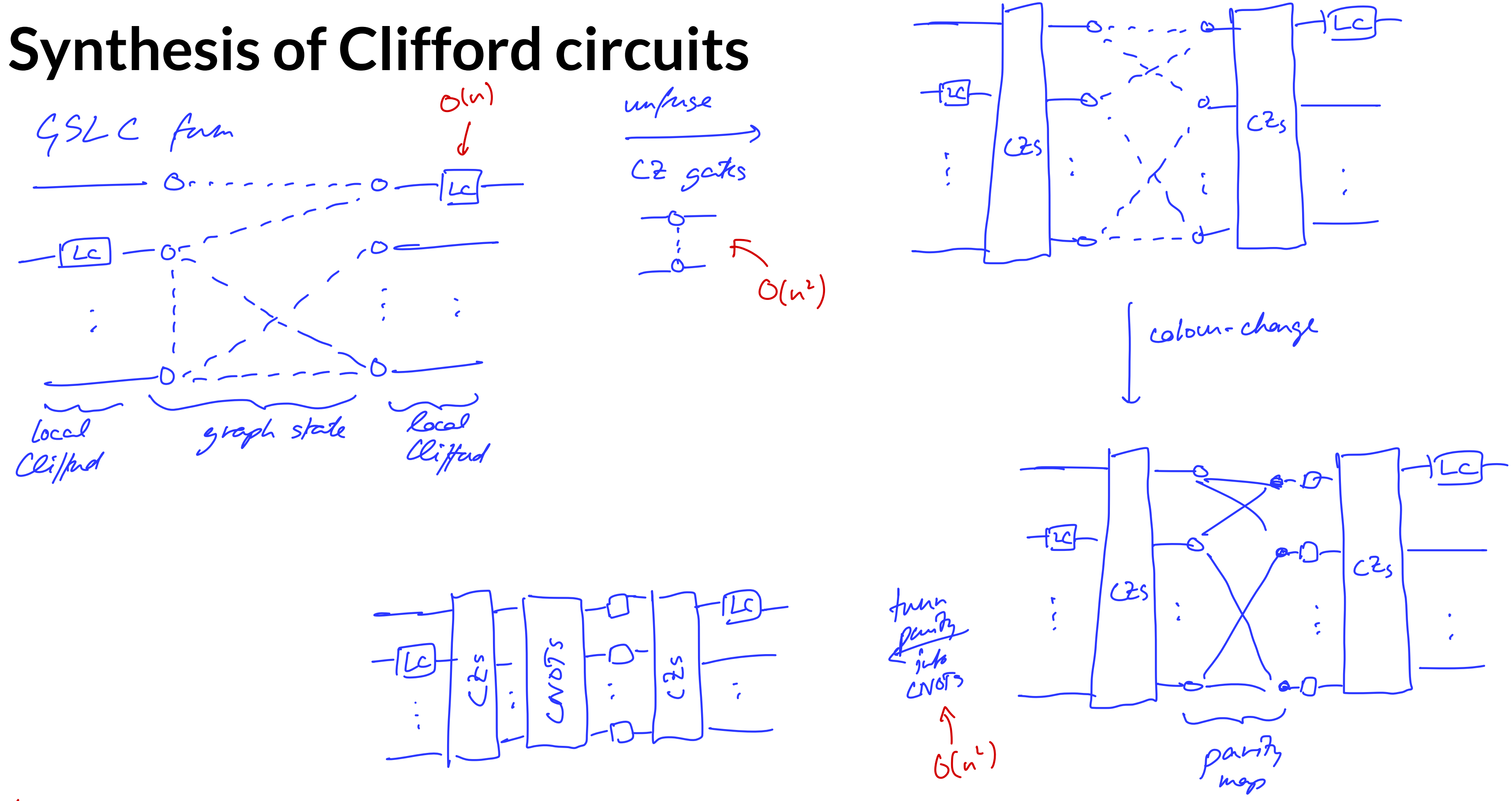
GSLC: no internal spiders







Synthesis of Clifford circuits



Any Clifford circuit can be written with at most $O(n^2)$ gates.

Summary:

- ZX diagrams with phases pi/2 are simulable but interesting
- Correspond to Clifford circuits
- Can bring to surprisingly efficient normal form graphically
- Can simulate efficiently graphically