# Introduction to Quantum Programming and Semantics

## Lecture 6: Tensor networks

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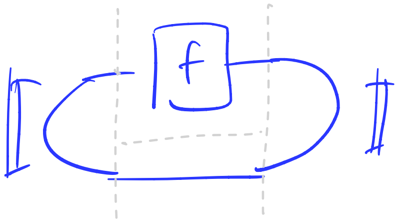
# Overview

- Cups and caps give ability to feedback
- Bases: linear maps vs matrices
- Sums of diagrams
- Tensor networks

# Trace and dimension

# Feedback

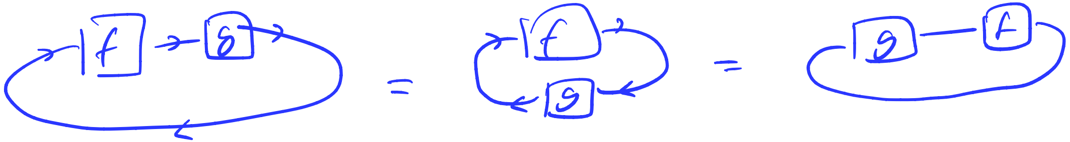


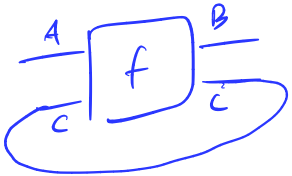

is called the (total) trace of $f$

if $f : \mathbb{C}^2 \to \mathbb{C}^2$ then $tr(f) \in \mathbb{C}$ is the linear-algebraic trace of $f$

i.e. $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $tr(f) = a+d$

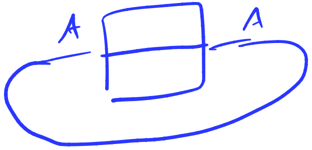
$$a+d = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = [\![ \supset ]\!] \circ (f \otimes id) \circ [\![ \subset ]\!]$$



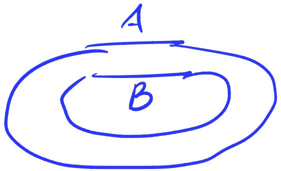

$= tr_C(f) : A \to B$ is the partial trace of $f$

# Dimensions




$= \text{dimension of } A$

$\dim(A \otimes B) =$




$= \bigcirc^A \quad \bigcirc^B = \dim(A) \cdot \dim(B)$

$$\dim(A \oplus B) = \dim(A) + \dim(B)$$

$$\dim(H) = \infty \implies \dim(H) + 1 = \dim(H) + \dim(\mathbb{C}) = \dim(H \oplus \mathbb{C}) = \dim(H) \quad ↯$$

$$\Longleftrightarrow$$

$$H \simeq H \oplus \mathbb{C}$$

# Bases

# Orthonormal bases

need to choose bases for $\mathbb{C}^2$ and $\mathbb{C}^3$

linear map $\rightsquigarrow$ 3-by-2 matrix

$f: \mathbb{C}^2 \longrightarrow \mathbb{C}^3$

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$\langle e_i | e_j \rangle = 0$ if $i \neq j$

$\langle e_i | e_i \rangle = 1$

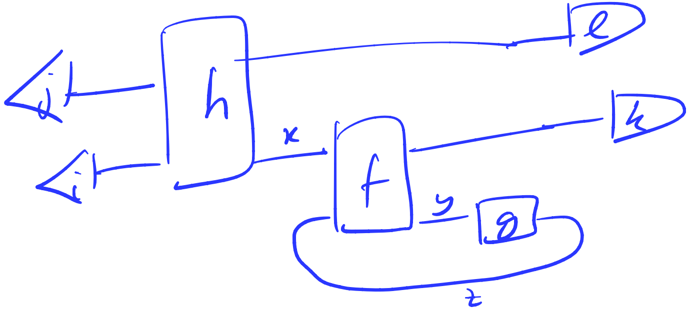
# Sums of diagrams

# Superposition

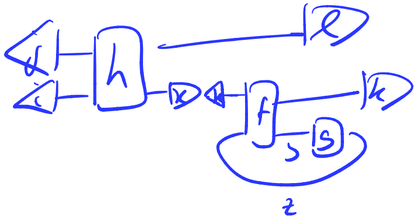
$$\sum_i \begin{array}{cc} \begin{pmatrix} 0 \cdots 0 & & 0 \\ & 1 & \\ 0 & & 0 \cdots 0 \end{pmatrix} & \leftarrow i \\ \uparrow i & \end{array} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

# Tensor contraction

# Tensor networks




$= \sum_x$




$= \dots$

$$= \sum_{x,y,z} h_{ij}^{x\ell}\, g_y^{z}\, f_{zx}^{yk}$$

# Special cases

tensor product: $(f \otimes g)^{kl}_{ij} = f^k_i \, g^l_j$

matrix multiplication: $(g \circ f)_{ij} = \sum_k f^k_i \, g^j_k$

trace $\mathrm{tr}(f) = \sum_i f^i_i$

# Summary:

- Graphical calculus automatically incorporates traces and dimensions
- It pays to be precise about choice of basis
- In computation it is handy to explicitly assume superposition
- Tensor networks are special case of graphical rewriting