Introduction to Quantum Programming and Semantics

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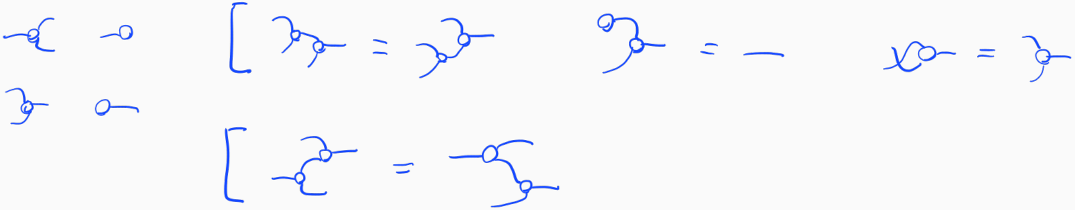
informatics

11th February 2025. Lecture 9: Complementarity

Previously



Overview

- Complementary bases
- Complementary Frobenius structures
- Equivalence
- Bialgebra

Complementary bases

Complementary bases: definition

$\mathbb{C}^2$ $\quad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $\quad \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\}$

Def two bases $\{a_i\}$, $\{b_j\}$ of a Hilbert space $\mathcal{H}$

they are complementary if there exists $s \in \mathbb{R}$ constant

s.t. for all i, j $\quad \langle a_i | b_j \rangle \langle b_j | a_i \rangle = s$.

Complementary bases: examples

$\mathbb{C}^2$ $\quad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $\quad \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\}$ $\quad \left\{ \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \right\}$

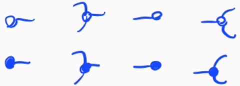
eigenvectors of respectively

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

(Pauli gates)

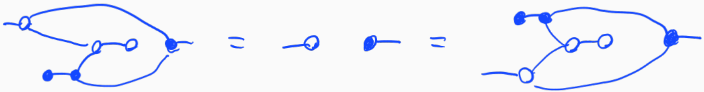
Complementary Frobenius structures

Complementary Frobenius structures: definition



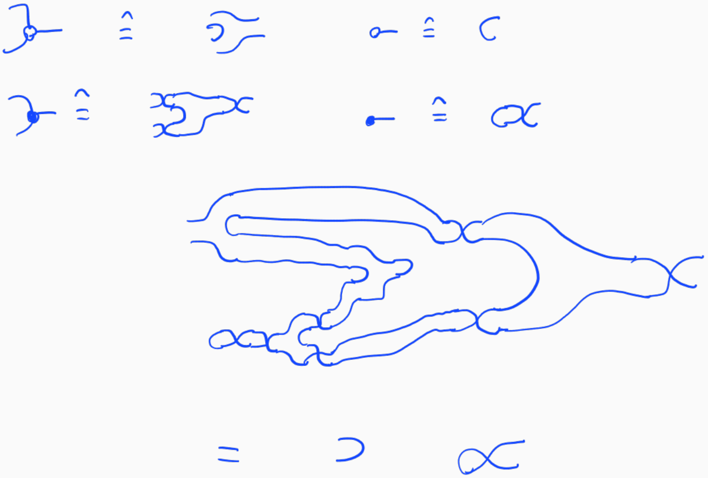
two Frobenius structures.

Def They are complementary if:



Special case:

Complementary Frobenius structures: example

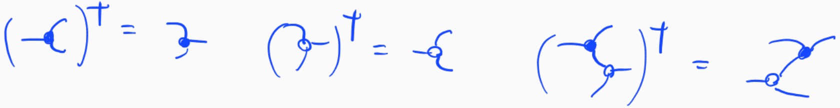


Characterisation with unitaries

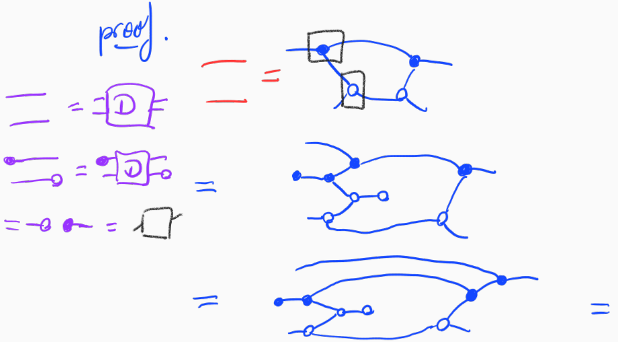
Prop Two Frobenius structures are complementary iff

is unitary.

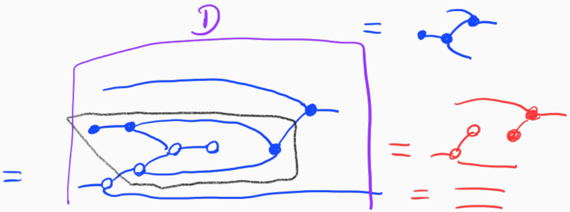
reminder:



proof.



use the following:



Equivalence

Th. In Hilbert spaces,

- compl. bases with $s = 1$
- compl. Frob. struct.

are equivalent.

Proof

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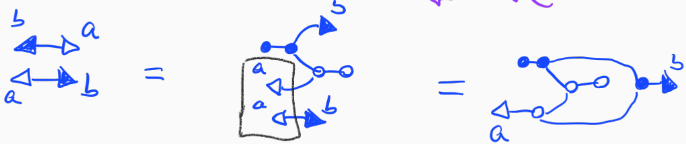
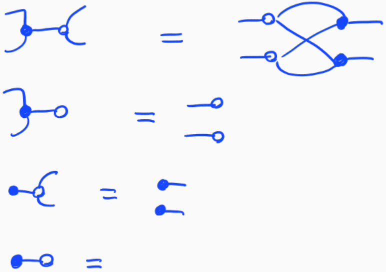
 copy-able for 

$\langle b|a\rangle \langle a|b\rangle =$ 

Further notions

Bialgebra and Hopf algebra

Bialgebra :



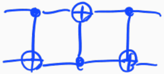
Hopf algebra :

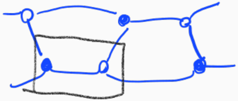
Qubit gates

 $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

 $|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $|-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

CNOT $\hat{=}$ 

 $=$ 



Summary

- Different bases in Hilbert spaces can be complementary
- Equivalent definition through bases or diagrams
- This complementary is necessary to represent quantum operations