Introduction to Quantum Programming and Semantics 2025
Tutorial week 3

Exercise 1

Let $f\colon H \longrightarrow H$ be a linear map such that $f^\dagger f = f f^\dagger$. Then $f = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ for some sets $\{\lambda_j\}_j$ and $\{|\phi_j\rangle\}_j$. Show that:

(a) f is self-adjoint if and only if each $\lambda_j \in \mathbb{R}$;

(b) f is positive if and only if each $\lambda_j \in \mathbb{R}_{\geq 0}$;

(c) f is a projection if and only if each $\lambda_j \in \{0, 1\}$;

(d) f is unitary if and only if each $\lambda_j \in U(1)$;

where $\mathbb{R}_{\geq 0}$ is the set of all nonnegative real numbers and $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$ is the set of all phases.

Exercise 2

Show that $\langle\phi|\psi\rangle = 0$ for the following antipodal states on the Bloch sphere:

$$|\phi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\alpha}|1\rangle$$
$$|\psi\rangle = \cos\frac{\theta+\pi}{2}|0\rangle + \sin\frac{\theta+\pi}{2}e^{i\alpha}|1\rangle$$

Exercise 3

Give the diagrammatic equations of a process $*$ taking two inputs and one output that express the algebraic properties of being:

(a) associative: $x * (y * z) = (x * y) * z$;

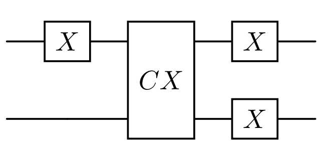
(b) commutative: $x * y = y * x$;

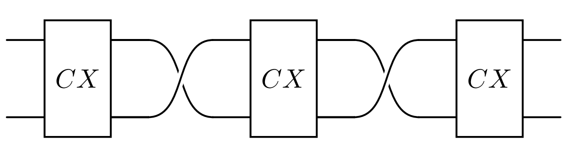
(c) having a unit: there exists a process e (with no inputs) such that $x * e = e * x = x$;

Note: x, y and z should not appear in the diagrams.

Exercise 4

Compute the semantic interpretation of the following circuits either as matrices or as functions on the computational basis states, and then show that they can both be expressed by simpler circuits:

(a) 

(b) 

Exercise 5

Which of these states of $\mathbb{C}^2 \otimes \mathbb{C}^2$ in **Hilb** are entangled and why?

(a) $\frac{1}{2}\big(\,|00\rangle + |01\rangle + |10\rangle + |11\rangle\,\big)$

(b) $\frac{1}{2}\big(\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\,\big)$

(c) $\frac{1}{2}\big(\,|00\rangle + |01\rangle - |10\rangle + |11\rangle\,\big)$

(d) $\frac{1}{2}\big(\,|00\rangle - |01\rangle - |10\rangle + |11\rangle\,\big)$