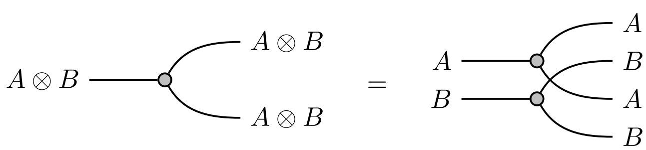
Introduction to Quantum Programming and Semantics 2025
Tutorial week 5

Exercise 1

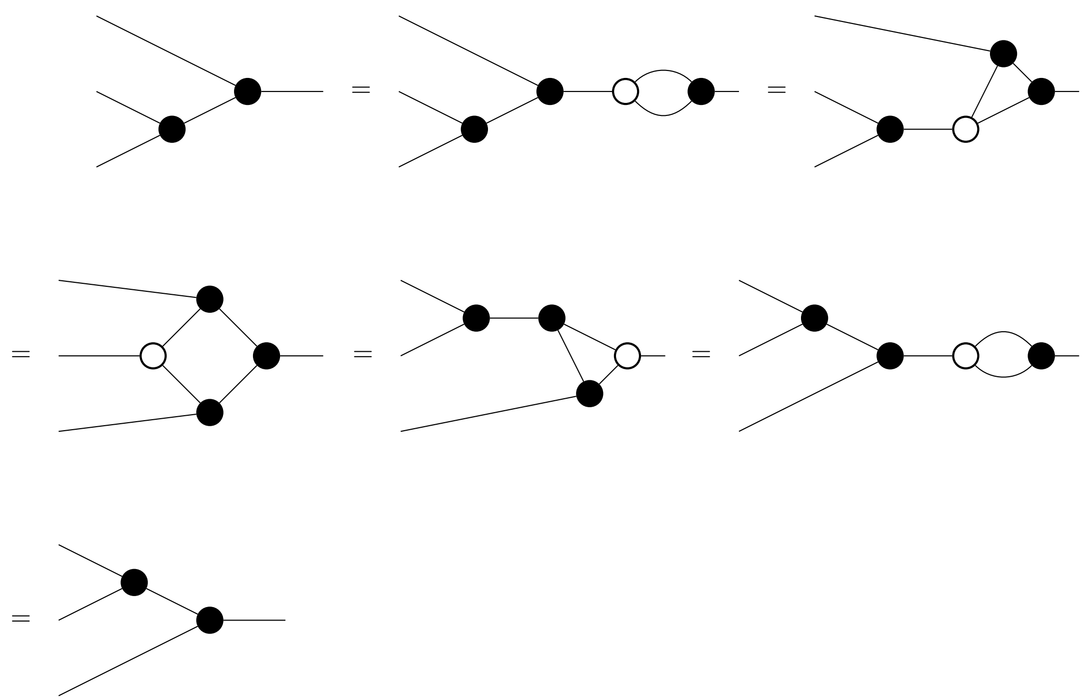
Define the comultiplication for $A \otimes B$ to be the comultiplications of A and B ‘side by side’:



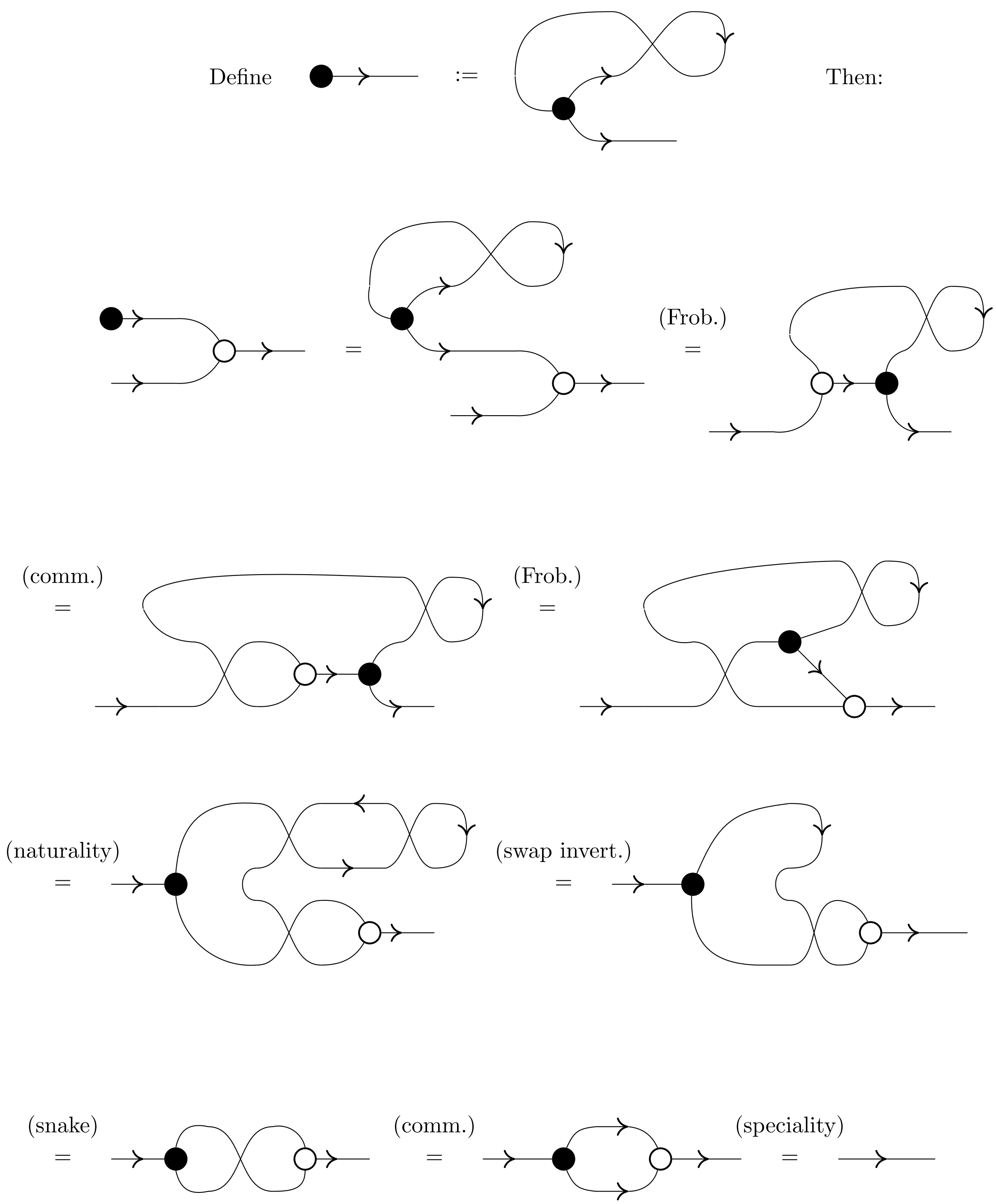
Then by isotopy, the Frobenius law, associativity, and commutativity then exactly when they do for both A and B.

Exercise 2

(a)



(b)



Exercise 3

A Z-spider with m inputs and n outputs is

$$|0\cdots 0\rangle\langle 0\cdots 0| + e^{i\alpha}|1\cdots 1\rangle\langle 1\cdots 1|.$$

If $m > 0 < n$, this is the 2^m-by-2^n matrix of all zeroes except for the top-left and bottom-right:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & e^{i\alpha} \end{pmatrix}$$

It clearly has rank 2.

The special case $m = 0$ gives the vector

$$|0 \cdots 0\rangle + e^{i\alpha} |1 \cdots 1\rangle .$$

The special case $n = 0$ gives the dual vector

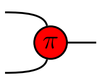
$$\langle 0 \cdots 0| + e^{i\alpha} \langle 1 \cdots 1| .$$

The special case $m = n = 0$ gives the scalar

$$1 + e^{i\alpha}.$$

Exercise 4

The map



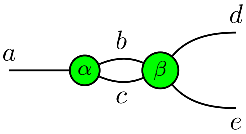
$$\begin{aligned}
&= \frac{1}{2\sqrt{2}} \big(|+\rangle \langle ++| + e^{i\pi} |-\rangle \langle --| \big) \\
&= \frac{1}{2\sqrt{2}} \big(|+\rangle \langle ++| - |-\rangle \langle --| \big) \\
&= \frac{1}{2\sqrt{2}} \big((|0\rangle + |1\rangle)(\langle 00| + \langle 01| + \langle 10| + \langle 11|) - (|0\rangle - |1\rangle)(\langle 00| - \langle 01| - \langle 10| + \langle 11|) \big) \\
&= \frac{1}{2\sqrt{2}} \big(|0\rangle \langle 00| + |0\rangle \langle 01| + |0\rangle \langle 10| + |0\rangle \langle 11| + |1\rangle \langle 00| + |1\rangle \langle 01| + |1\rangle \langle 10| + |1\rangle \langle 11| \\
&\qquad - |0\rangle \langle 00| + |0\rangle \langle 01| + |0\rangle \langle 10| - |0\rangle \langle 11| + |1\rangle \langle 00| - |1\rangle \langle 01| - |1\rangle \langle 10| + |1\rangle \langle 11| \big) \\
&= \frac{1}{\sqrt{2}} \big(|0\rangle \langle 01| + |0\rangle \langle 10| + |1\rangle \langle 00| + |1\rangle \langle 11| \big)
\end{aligned}$$

sends $|00\rangle$ and $|11\rangle$ to $|1\rangle$, and sends $|01\rangle$ and $|10\rangle$ to $|0\rangle$. It is the classical XNOR map.

Exercise 5

$$\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Exercise 6

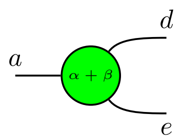


$$= \sum_{b,c} Z[\alpha]_a^{b,c} Z[\beta]_{b,c}^{d,e}$$

$$= \sum_{b,c} \begin{Bmatrix} 1 & \text{if } a = b = c = 0 \\ e^{i\alpha} & \text{if } a = b = c = 1 \\ 0 & \text{otherwise} \end{Bmatrix} \begin{Bmatrix} 1 & \text{if } b = c = d = e = 0 \\ e^{i\beta} & \text{if } b = c = d = e = 1 \\ 0 & \text{otherwise} \end{Bmatrix}$$

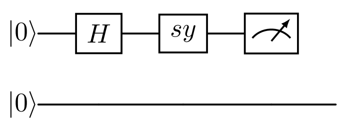
$$= \begin{Bmatrix} 1 & \text{if } a = d = e = 0 \\ e^{i(\alpha+\beta)} & \text{if } a = d = e = 1 \\ 0 & \text{otherwise} \end{Bmatrix}$$

$=$



Exercise 7

The left-hand program, as a circuit, is



The state of the top qubit `a` before measurement is

$$\begin{aligned} \text{sy} \cdot H \cdot |0\rangle &= \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1+i & 1+i \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 \\ 2+2i \end{pmatrix} \end{aligned}$$

So there is a probability of zero that the outcome of the measurement is 0. Disregarding the bottom qubit `b`, the program is therefore equivalent to the right-hand one.