Introduction to Quantum Programming and Semantics 2025
Tutorial week 8

Exercise 1

In this exercise you'll show that the eigenvectors of the Hadamard gate can be represented by a Clifford+T circuit.

(a) Show that $|H\rangle = |0\rangle + (\sqrt{2} - 1)\,|1\rangle$ is an eigenvector of the Hadamard gate.
What is its eigenvalue?

(b) Find an eigenvector with a different eigenvalue.

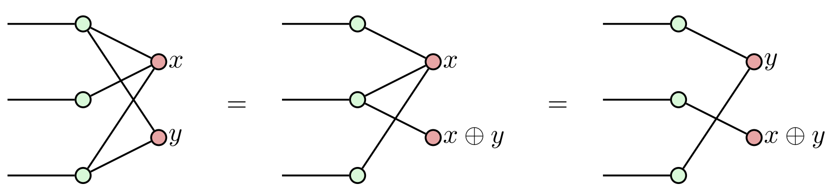
(c) Write $|H'\rangle = 2\cos(\frac{\pi}{8})\,|H\rangle$ as $a\,|0\rangle + b\,|1\rangle$, with a and b in terms of $\pm e^{\pm i\frac{\pi}{8}}$ factors.
(Hint: $\tan(\frac{\pi}{8} = \sqrt{2} - 1$.)

(d) Write $e^{-\frac{\pi}{8}} HS\,|H'\rangle$ as a superposition of $|0\rangle$ and $|1\rangle$.

(e) Give a sequence of Clifford+T gates $G_1, \ldots, G_k$ such that $G_k \cdots G_1\,|+\rangle = |H\rangle$ (up to phase).

Exercise 2

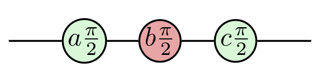
Prove the following two equations for $x, y \in \{0, 1\}$.



(Hint: use strong complementarity.)

Exercise 3

A single-qubit Clifford circuit is constructed out of just Hadamard and S gates. Show that any single-qubit Clifford circuit can be rewritten to the form



for integers a, b, c. (Hint: we know that a single S or H can be brought to this form, so it suficces to show that when you compose this normal form with an additional S or H the result can also be brought to this normal form. You probably will want to make a case distinction on b.)

Exercise 4

A Clifford state is the result of applying a Clifford circuit to the state $|0 \cdots 0\rangle$. Show that the following are Clifford states.

(a) $|1\rangle$

(b) $|+\rangle$

(c) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

(d) $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

(e) $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$