Introduction to Quantum Programming and Semantics

Lecture 10: Complementarity

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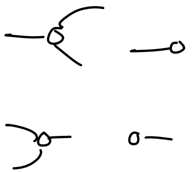
Overview

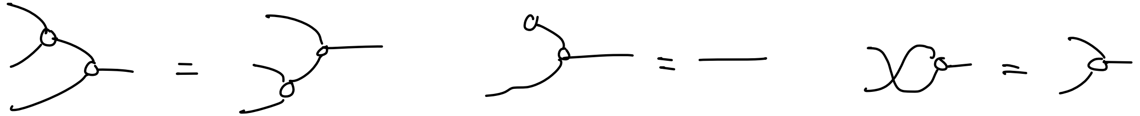
- Complementary bases
- Complementary classical structures
- Equivalence
- Bialgebras

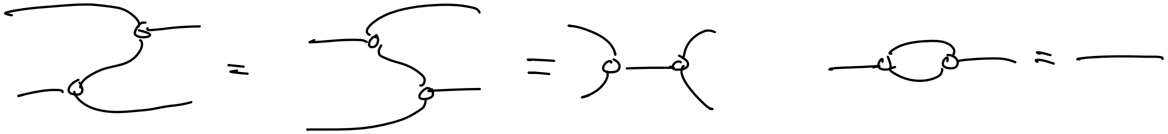
Complementary bases

Recall



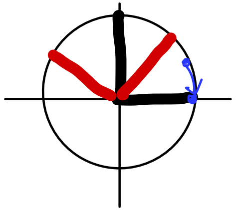
s.t.





Complementary bases: definition

Idea: measure qubit $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ and then in $\left\{ \begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}, \begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2} \right\}$



probability of either outcome is $1/2$

Heisenberg uncertainty: 1^{st} measurement provides no info about 2^{nd}.

<u>Def</u> two basis $\{e_i\}$, $\{d_j\}$ of $\mathbb{C}^n$ are <u>complementary / mutually unbiased</u> if

$$\exists c \ \forall i \forall j: \quad \langle d_j | e_i \rangle \langle e_i | d_j \rangle = c$$

Complementary bases: examples

$\mathbb{C}^2$: $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\}$ $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \right\}$

(eigenvalues of X, Z, Y Pauli gates)

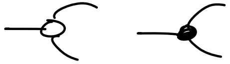
largest family of mutually unbiased bases for $\mathbb{C}^2$

$\mathbb{C}^{p^2}$ for p prime: known how many mut. unbiased bases.

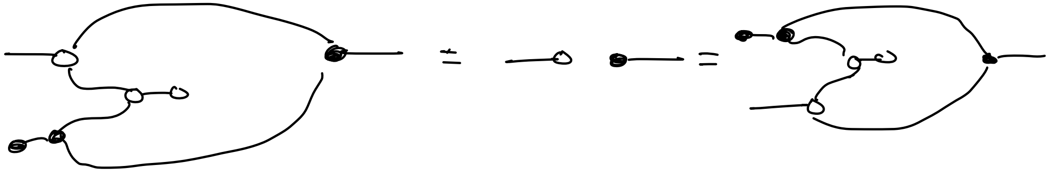
$\mathbb{C}^d$ d other: open question

Complementary classical structures

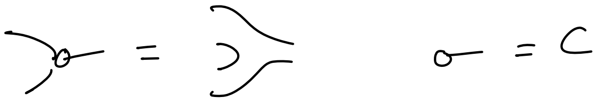
Complementary classical structures: definition

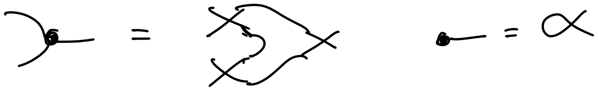
Consider two classical structures 

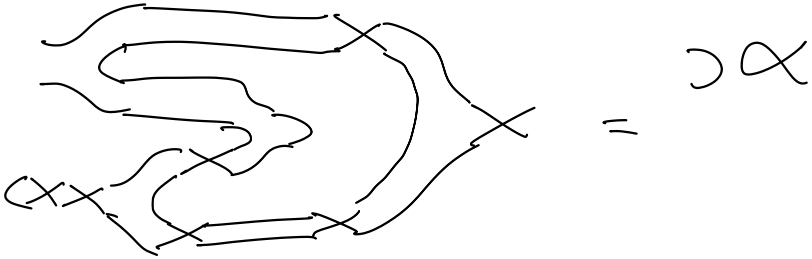
They are <u>complementary</u> if



Complementary classical structures: examples

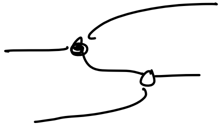






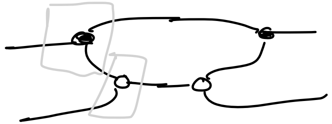
Characterisation with unitaries

Proposition: two classical structures are complementary iff



is unitary

Proof:



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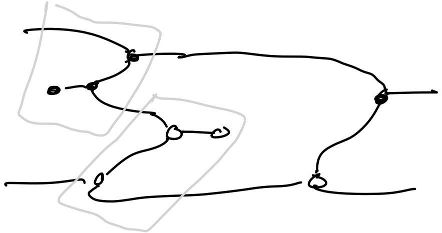

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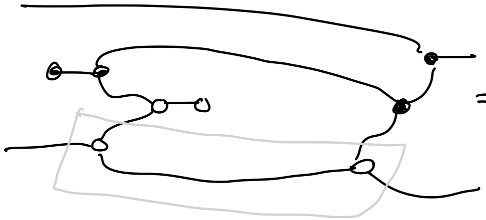

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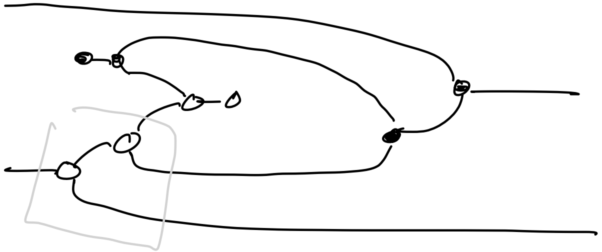

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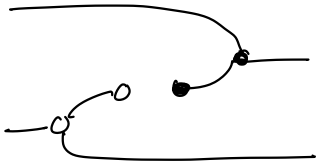

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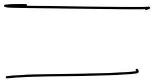


Equivalence

Complementarity: bases vs classical structures

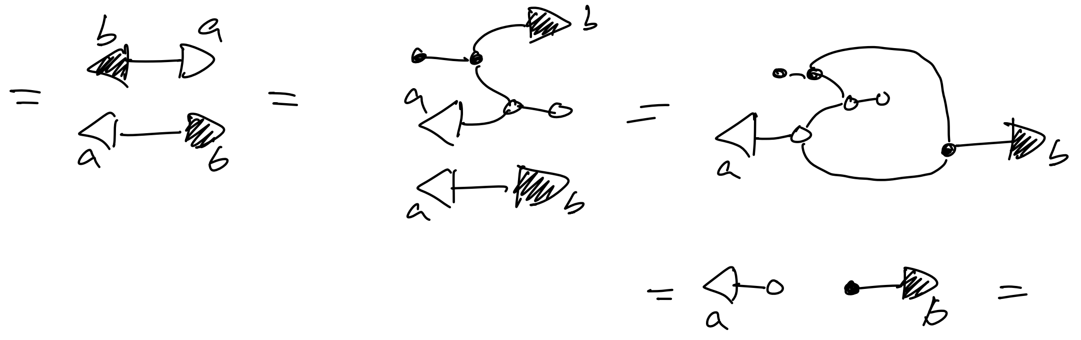
Two classical structures copying orthonormal bases are complementary

$\iff$

the copied bases are mutually unbiased.

Proof:

$\langle b|a\rangle\langle a|b\rangle =$



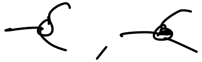
Bialgebras

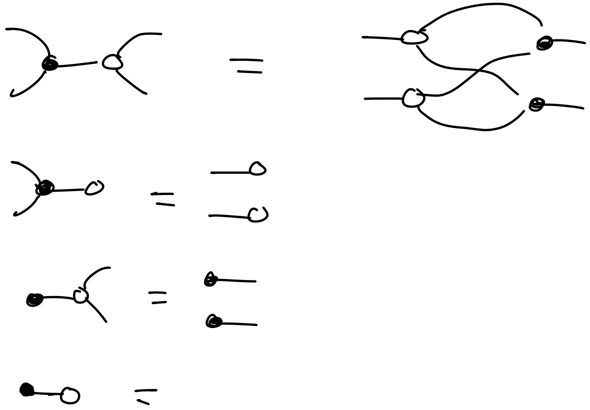
$\mathbb{C}^2$ has bases $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} e^{i\varphi} \\ e^{i\theta} \end{pmatrix}, \begin{pmatrix} e^{i\varphi} \\ -e^{i\theta} \end{pmatrix} \right\}$

that give complementary classical structures,

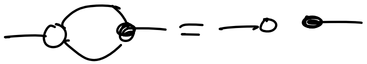
but only mutually unbiased when $\varphi = \theta = 0 \pmod{2\pi}$

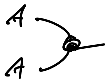
Bialgebras and Hopf algebras

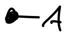
<u>Bialgebra</u> is two classical structures  satisfying



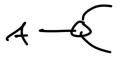
<u>Strongly complementary</u> if complementary and bialgebra.

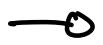
(Hopf algebras: )

If 

, 

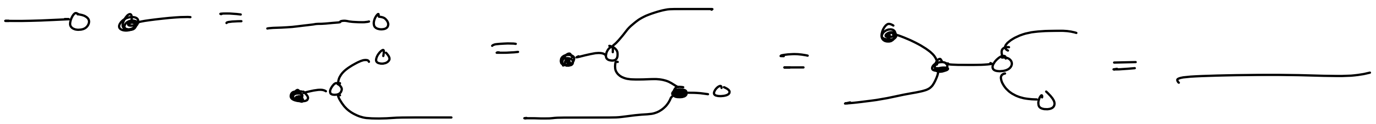
monoid



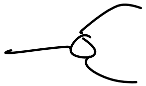
,  comonoid

are both Frobenius and bialgebra, then:

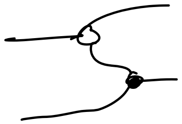
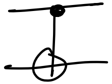
 =

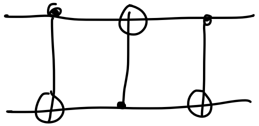
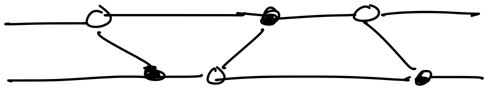


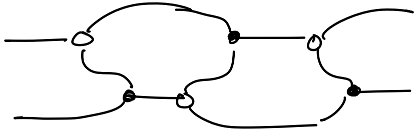
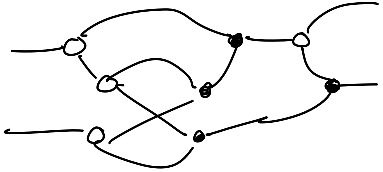
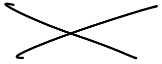
Qubit gates

 copies Z

 copies X

CNOT =  =  = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

 = 

=  =  = ... = 

Summary:

- Different bases in Hilbert spaces can be complementary
- Equivalent definition through bases or diagrams
- Complementarity is necessary to represent quantum operations