Introduction to Quantum Programming and Semantics

Lecture 13: Clifford circuits

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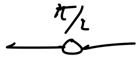
Overview

- Clifford circuits and Clifford diagrams
- Graph states with local Clifford operations
- Local complementation, pivoting
- Strong simulation of Clifford circuits
- Synthesis of Clifford circuits

Clifford circuits and Clifford diagrams

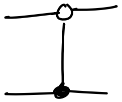
Clifford circuits *are made from:*

$S =$ 

$= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

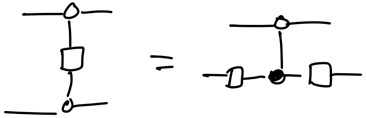
$H =$ 

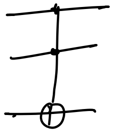
$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

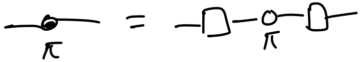
$CX =$  $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

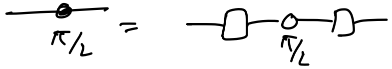
e.g.: $Z =$ 

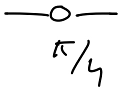
$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$CZ =$ 

$CCX =$ 

$X =$ 

$\sqrt{X} = V =$ 

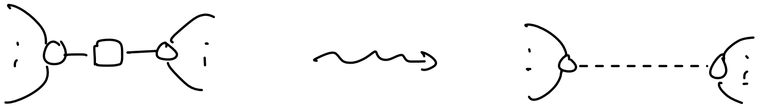
but not: $T =$ 

Clifford diagrams

is a ZX diagram made of "Clifford spiders"

for $k \in \mathbb{Z}$

notation:



"Hadamard edge"

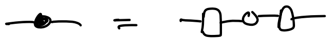
a ZX diagram is graph-like if:

- all spiders are Z spiders
- all edges are Hadamard edges
- no parallel edges or self-loops
- every input/output is connected to a spider

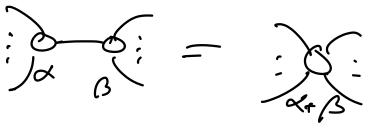
Clifford diagrams

Proposition: every ZX diagram is equal to a graph-like one.

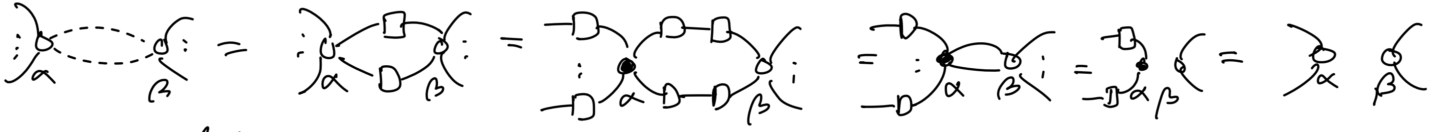
Proof: 1. use colour change rule to eliminate all X spiders



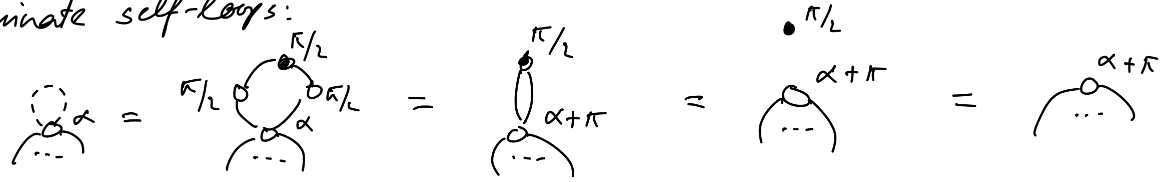
2. use spider fusion to eliminate all non-Hadamard edges



3. eliminate parallel edges



eliminate self-loops:



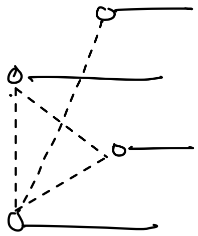
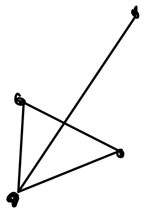
4. use

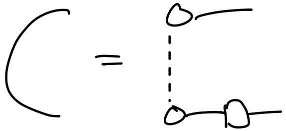
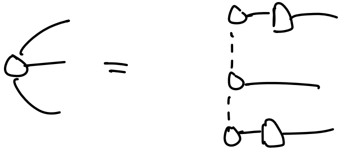
Graph states with local Clifford operations

Graph states

are graph-like ZX diagram with no input

e.g.

$|G\rangle =$  $G =$ 

but not:  

def: a graph state with local Cliffords (GSLC) is a state of form $(U_1 \otimes \ldots \otimes U_n)|G\rangle$ for some graph G and 1-qubit Clifford gates U_i.

Normal form

Def: a Clifford state is a vector of form $C|0\cdots0\rangle$ for some some Clifford circuit C.

Thm: any Clifford state is equal to a GSLC.

Local complementation, pivoting

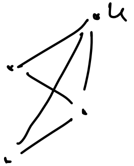
Local complementation of a graph G about vertex u

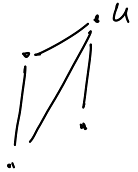
is a new graph $G * u = (V, E')$ where

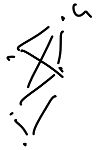
$\forall v, w \in N_G(u)$: $\quad (v,w) \in E' \iff (v,w) \notin E$

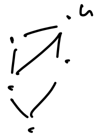
neighbourhood of u

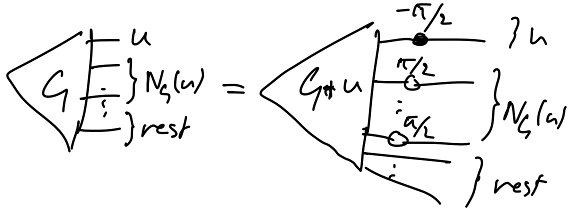
and $\quad E' \cap (V \setminus N_G(u)) \times V = E \cap (V \setminus N_G(u)) \times V$

e.g.: $G =$ 

$G * u =$ 

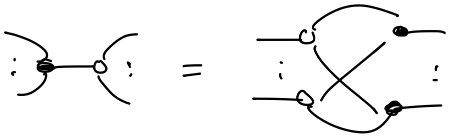
$G =$ 

$G * u =$ 

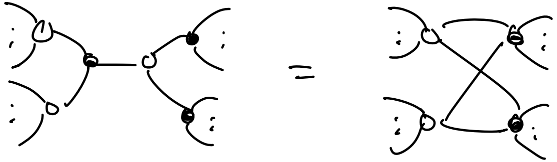
Proposition: 

Pivoting

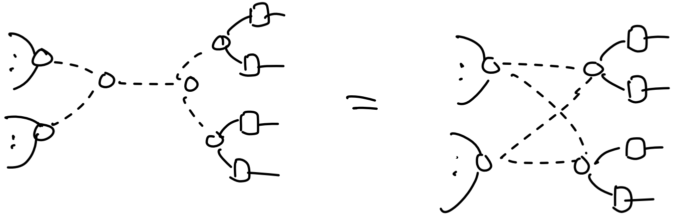
Consider strong complementarity:



add context

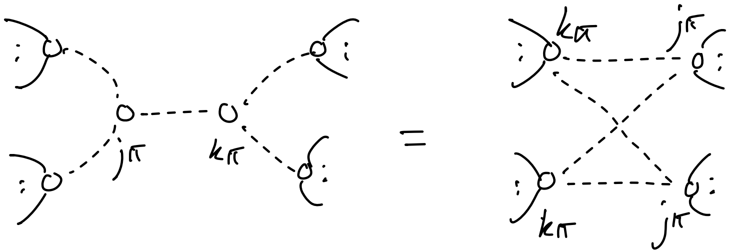


deletes spiders



deletes adjacent phase-free spiders

pivot rule

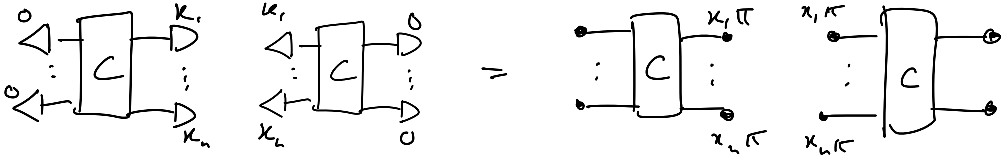


Simulating Clifford circuits

Simulation of Clifford circuits

given Clifford circuit C, compute $\mathrm{Prob}(x_1, \ldots, x_n \mid |\psi\rangle)$, where $|\psi\rangle := C|000\cdots0\rangle$

use Born rule: $\mathrm{Prob}(x_1, \ldots, x_n \mid |\psi\rangle) = |\langle x_1 \cdots x_n | C | 0 \cdots 0\rangle|^2$

$=$ 

$= D$

algorithm ("Clifford simplification rewrite strat"):

1. convert to graph-like diagram
2. repeat until impossible: local complementation and pivoting
3. remove all isolated $\{0,\pi\}$-spiders

<u>prop</u>: this terminates

Simulation of Clifford circuits

Prop: algorithm terminates in polynomial time (in #qubits, and #gates in C)
$\quad$

↳ n ↳ k

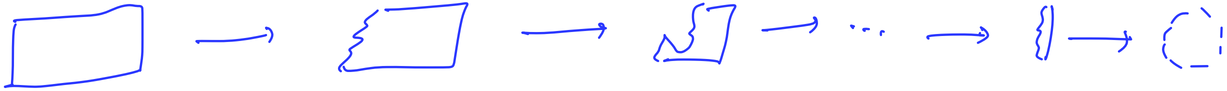
$O((n+k)^3)$

but this is not optimal: doing step 2 optimally $O(n^2k)$ steps

if $k \gg n$, this makes big difference

idea: 1. avoid big spiders:

2. apply local complementation and pivoting left-to-right



each step involves at most $O(n)$ spiders (hence $O(n^2)$ wires)

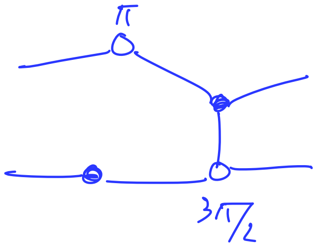
Synthesising Clifford circuits

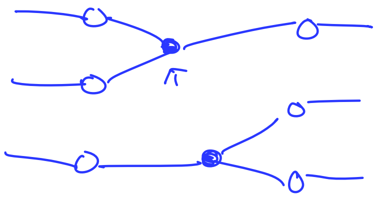
Synthesis of Clifford circuits

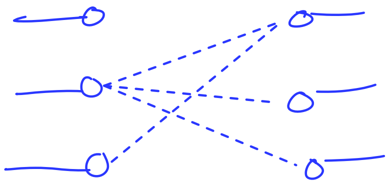
Clifford diagram $\longrightarrow$ "AP-form" $\longrightarrow$ GSLC

"AP-form": all interior spider
- have phase $0, \pi$
- only connected to boundary spiders

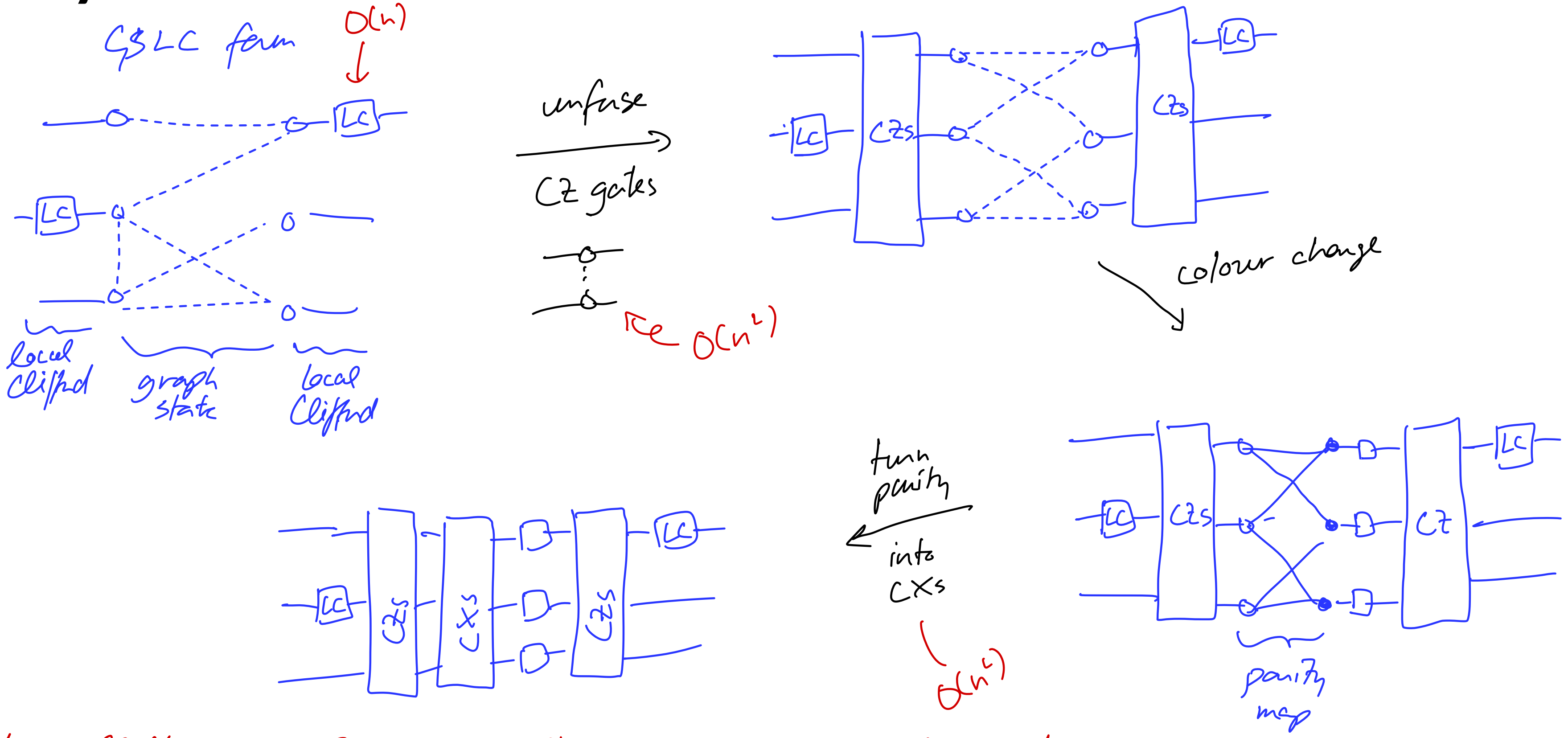
GSLC: no internal spiders







Synthesis of Clifford circuits



Any Clifford circuit can be written with at most $O(n^2)$ gates.

Summary:

- ZX diagrams with phases pi/2 are simulable but interesting
- Correspond to Clifford circuits
- Can bring to surprisingly efficient normal form graphically
- Can simulate efficiently graphically