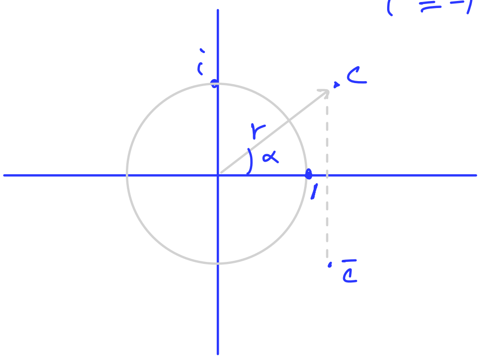
IQPS Lecture 2

States

complex numbers

$i^2 = -1$



cartesian form

$c = a + ib, \quad a, b \in \mathbb{R}$

$\bar{c} = a - ib$

polar form

$= r e^{i\alpha} \quad \alpha$ angle in $[0, 2\pi)$

$r \geq 0$ in $\mathbb{R}$

norm / length

$|c| = r = \sqrt{c \cdot \bar{c}}$

phase = element of $\{ c \in \mathbb{C} \mid |c| = 1 \}$

Def: a (pure) state is a normalised vector $|\psi\rangle$ up to global phase

$\|\psi\| = 1$

in vector space $\mathbb{C}^2$

$|\psi\rangle = e^{i\theta}|\psi\rangle$

because measurement can't physically distinguish the two

Qubit = a state in $\mathbb{C}^2$

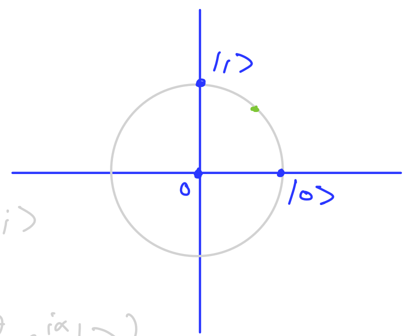
so of the form $|\psi\rangle = re^{i\beta}|0\rangle + se^{i\gamma}|1\rangle$

$r^2 + s^2 = 1$

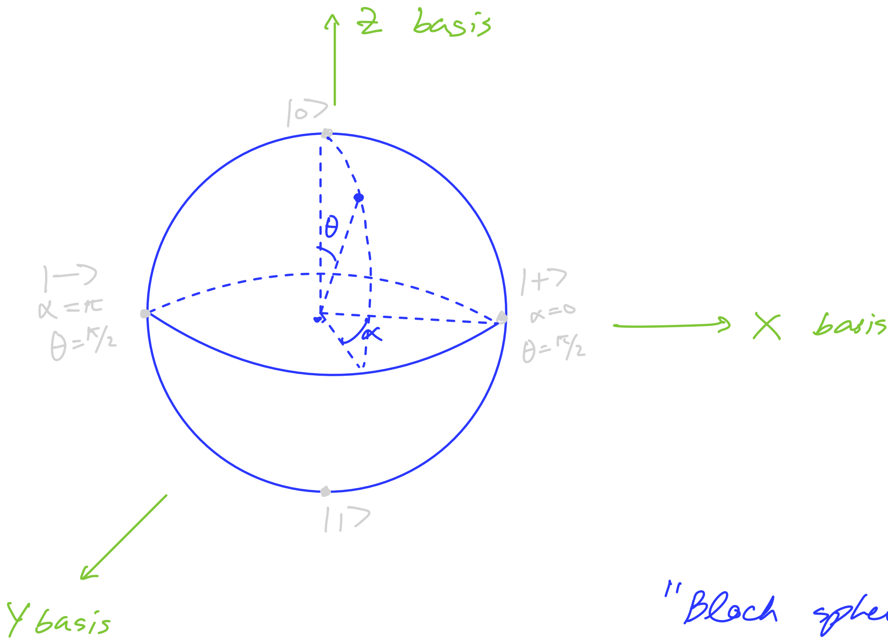
$\Longrightarrow |\psi\rangle = \cos\frac{\theta}{2} e^{i\beta}|0\rangle + \sin\frac{\theta}{2} e^{i\gamma}|1\rangle$

$\Longrightarrow |\psi\rangle = e^{i\beta}\psi\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\alpha}|1\rangle\right)$

for $\alpha = \gamma - \beta$



So qubit is determined by 2 angles, so can plot



"Bloch sphere"

Unitary matrices

time evolution in quantum theory - is linear
- preserves normalisation

so $|\psi\rangle \rightsquigarrow U|\psi\rangle$

so linear map $U: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ (2-by-2 matrix)

such that $U^\dagger U = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \big(\, |\, U|\psi\rangle\, | = |\psi| \,\big)$

called a unitary

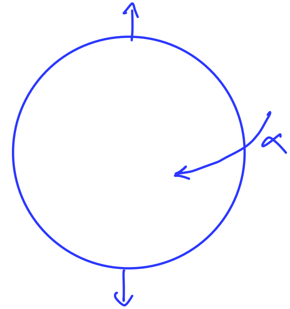
equivalently: Hamiltonian $H: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ s.t. $H^\dagger = H$

$$U = e^{iH} = 1 + \frac{iH}{1!} + \frac{(iH)^2}{2!} + \frac{(iH)^3}{3!} + \cdots$$

e.g. Schrödinger equation

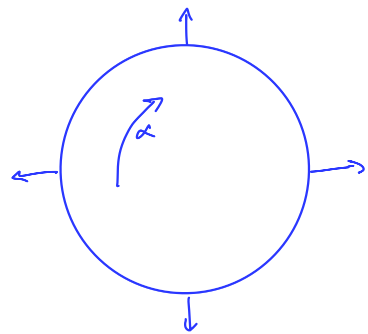
$U: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ correspond to rotations of the Bloch sphere

$Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \sim$



"Z rotation"

$X_\alpha = \begin{pmatrix} \cdot & \cdot \\ - & \cdot \end{pmatrix} \sim$



"X rotation"

Any U can be written as $U = Z_\alpha \cdot X_\beta \cdot Z_\gamma$ for some angles α, β, γ.

"Euler decomposition"

Classical circuits

$$X = \begin{array}{c|c} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \hline & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{array}{c} \to |0\rangle \\ \to |1\rangle \end{array} \end{array}$$

$X|0\rangle = |1\rangle$

$X|1\rangle = |0\rangle$

$$CX = \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{array}{c} \leftarrow |00\rangle \\ \leftarrow |01\rangle \\ \leftarrow |10\rangle \\ \leftarrow |11\rangle \end{array} \end{array}$$

$CX|00\rangle = |00\rangle$

$CX|01\rangle = |01\rangle$

$CX|10\rangle = |11\rangle$

$CX|11\rangle = |10\rangle$

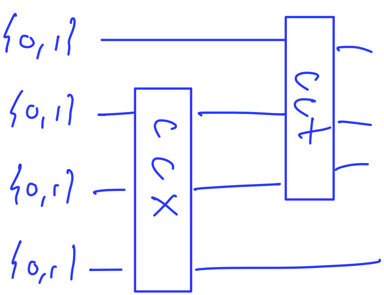
"controlled X gate"

$CCX|011\rangle = |011\rangle$

$|110\rangle = |111\rangle$

"Toffoli gate"

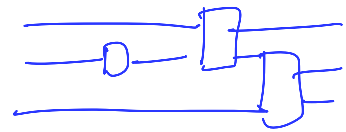
$$CU = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & U \end{array} \right)$$



$\{X, CX, CCX\}$ universal : any bijection

$\{0,1\}^n \longrightarrow \{0,1\}^n$ can be written as a circuit

<u>Quantum circuit</u> is an expression that combines gates from gate set horizontally and vertically, e.g.



X_α, Z_α, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$T = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ "Hadamard gate"

<u>Universality</u>: for any error margin $\varepsilon > 0$, and any target unitary U on n qubits, there is a circuit C such that $| U - [\![C]\!] | \leq \varepsilon$

$$\text{Syntax} \xrightarrow{[\![-]\!]} \text{Semantics}$$

$$\text{Circuits } C \longmapsto \text{unitary } [\![C]\!]$$

e.g. $[\![-\boxed{H}-]\!] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$[\![-\boxed{X}-]\!] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

<u>Thm</u>: $\{X, CX, CCX, H\}$ universal

$\{X, CX, CCX, T\}$ universal

$\{X, CX, Z_\alpha \mid \alpha \in [0, 2\pi)\}$ universal