Introduction to Quantum Programming and Semantics

Lecture 5: Bending space and time

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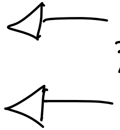
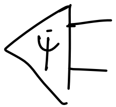
Overview

- Cups and caps
- Quantum teleportation
- Map-state duality
- Graphical symmetries

Cups and caps

Bell states

$$\begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} =$$  $\neq$  $$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$$ is entangled,

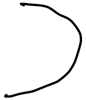
in fact maximally so

called "Bell state"

special notation: 

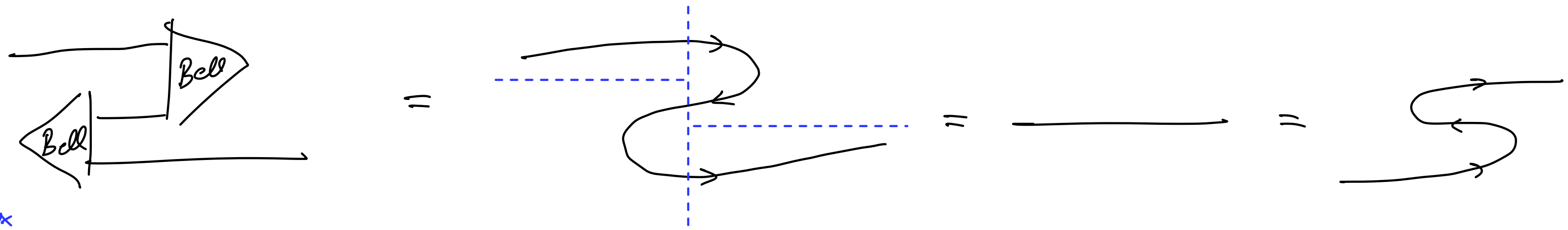
also called "cup"

transpose effect $(1\ 0\ 0\ 1) : \mathbb{C}^4 \longrightarrow \mathbb{C}^1$

special notation 

"cap"

Snake equation



↑ syntax

↓ semantics

$$\|$$

$$(Bell^T \otimes id) \circ (id \otimes Bell)$$

$$\mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^1 \xrightarrow{id \otimes Bell} \mathbb{C}^1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow{Bell^T \otimes id} \mathbb{C}^1 \otimes \mathbb{C}^2 \simeq \mathbb{C}^2$$

Graphical calculus of string diagrams including cups and caps

is still sound and complete

with nondirected isotopy

graphs G, H embedded in $\mathbb{R}^4$ isotopic

$\Longleftrightarrow$

$[0,1] \times \mathbb{R}^4 \xrightarrow{\varphi} \mathbb{R}^4$ s.t. $\varphi(0) = G$

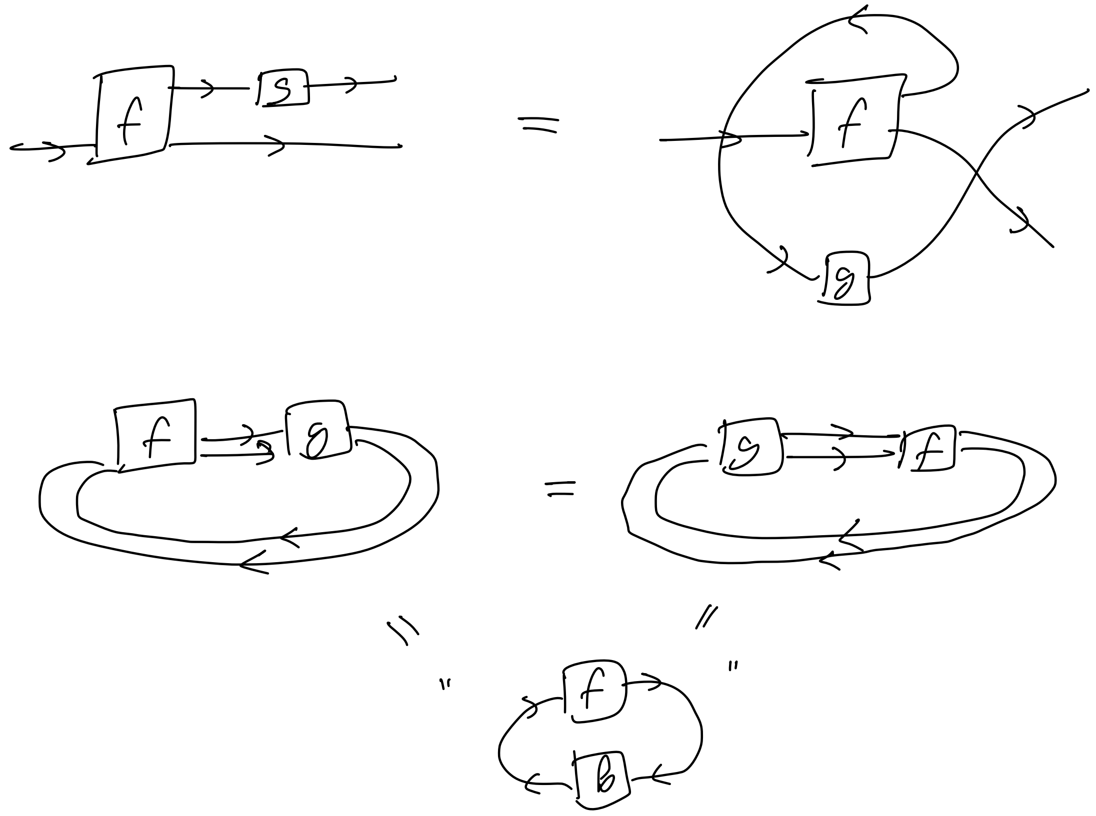
$\varphi(1) = H$

φ smooth

$\forall t \in [0,1]: \quad \varphi(\text{vertex } v) = v$

~~$\forall t \in [0,1]: \quad s \leq s' \implies \varphi(t)(s,x,y,z) \leq \varphi(t)(s',x,y,z)$~~

So only connectivity matters



Teleportation

Quantum teleportation

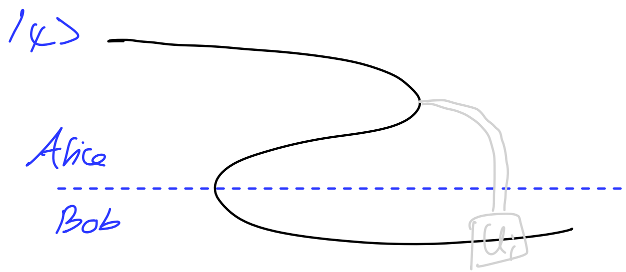
Alice and Bob share Bell state

Alice gets unknown input qubit $|\psi\rangle$

Goal: Bob wants $|\psi\rangle$

Method:

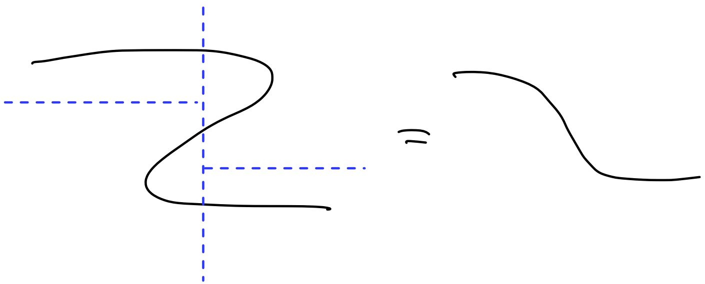
1. Alice measures her half and $|\psi\rangle$
2. Alice sends outcome to Bob (2 bits)
3. Bob applies unitary to his half



Teleportation in OpenQASM

```
gate H a { U ( pi /2 , 0 , pi ) a ; }
gate X a { U ( pi , 0 , pi ) a ; }
gate Z a { U (0 , 0 , pi ) a ; }
gate CX a , b { ctrl @ U ( pi , 0 , pi ) a , b ; }

qubit input_state ;
reset input_state ;

// Create a Bell state
qubit [2] bell_state ;
reset bell_state ;
H bell_state [0];
CX bell_state [0] , bell_state [1];

// Entangle the input state with Bell state
CX input_state , bell_state [0];

// Measure and correct
bit m1 = measure input_state ;
bit m2 = measure bell_state [0];
if ( m2 == true ) { X bell_state [1]; }
if ( m1 == true ) { Z bell_state [1]; }
```

"Classical telepotation"



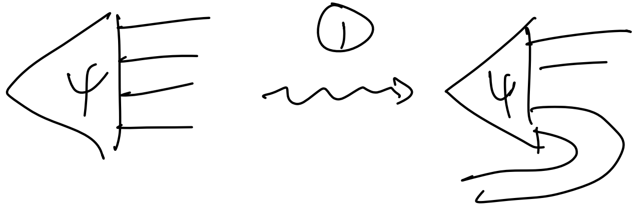
cup = one-time pad

cap = XOR

protocol means correctness of classical one-time pad encryption

Taking names

Names



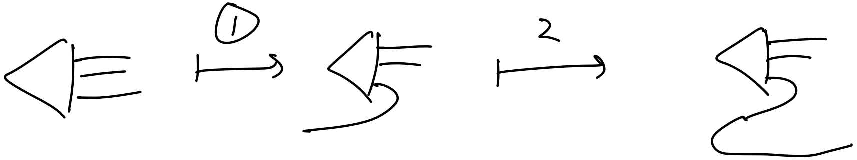
'state'

'gate'

"map-state duality"

"Choi-Jamiołkowski duality"

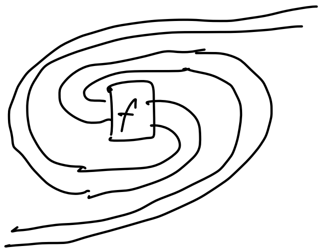
1-to-1 correspondence

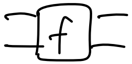


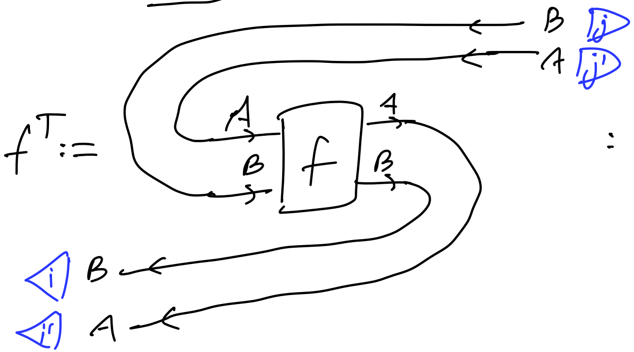
Map-state duality

Transpose and adjoint

Transpose

$f^{TT} =$ 

$=$ 

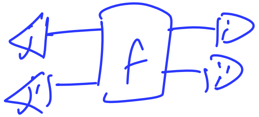
$f : A \otimes B \longrightarrow A \otimes B \implies f^T :=$ 

$: B^* \otimes A^* \longrightarrow B^* \otimes A^*$

dual vector space

$A = B = \mathbb{C}^2$

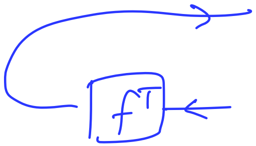
$f : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$



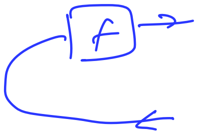
$\langle jj' | f^T | ii' \rangle = \langle ii' | f | jj' \rangle$

So f^T is transpose matrix of f

Claim:

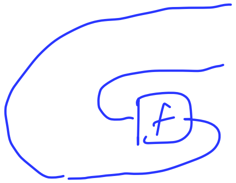


$=$

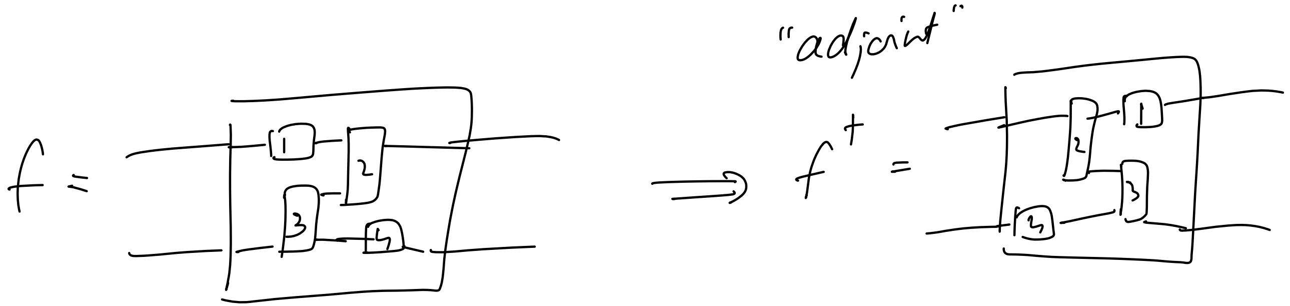


|| (by def)

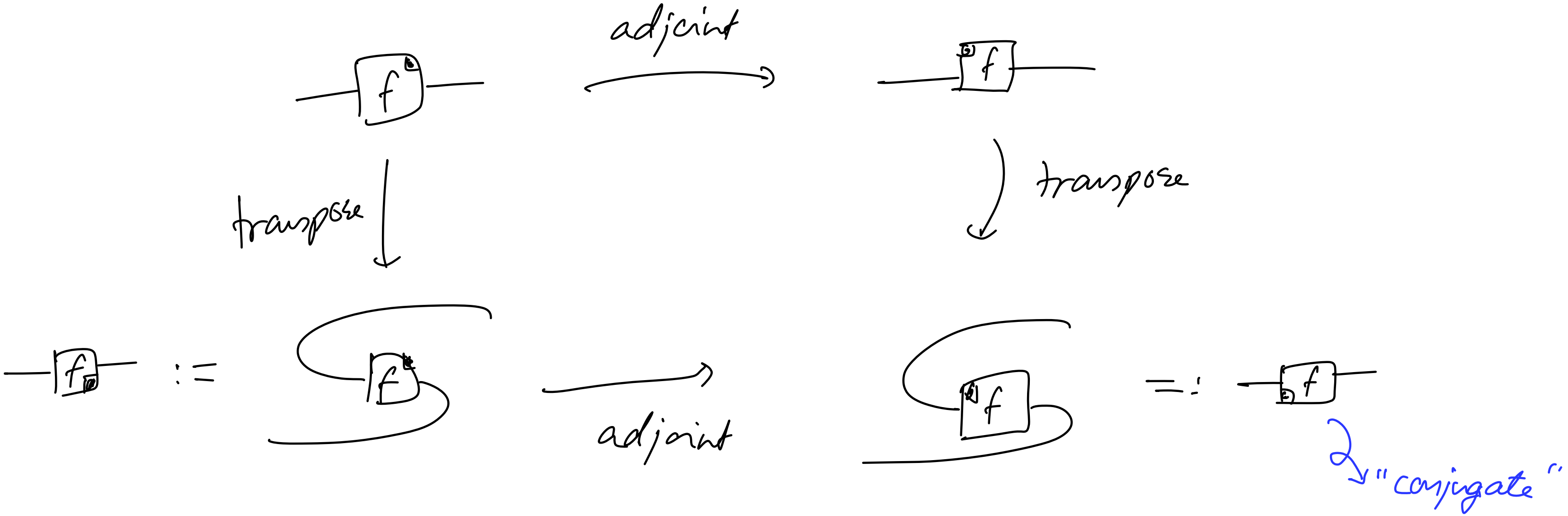
Proof:



Adjoint



as matrix: adjoint = complex conjugate transpose



Summary:

- Can trade time for space in string diagrams
- Models (postselected) quantum teleportation
- Can trade maps for states in string diagrams: gate injection
- Graphical calculus has meaningful symmetries