Introduction to Quantum Programming and Semantics

Lecture 6: Tensor networks

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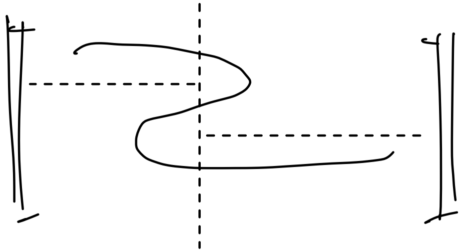


$$= \left([\![\supset]\!] \otimes [\![\sqsubset \ ]\!] \right) \circ \left([\![-]\!] \otimes [\![\subset]\!] \right)$$

$$= \left(\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \circ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

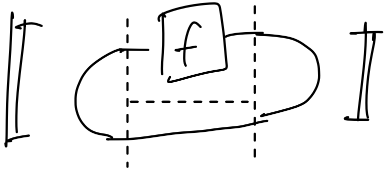
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Overview

- Cups and caps give ability to feedback
- Bases: linear maps vs matrices
- Sums of diagrams
- Tensor networks

Trace and dimension

Feedback



is called the (total) <u>trace</u> of f

$$\parallel$$

$$(1\ 0\ 0\ 1)\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = [\![\supset]\!] \circ (f \otimes id) \circ [\![\subset]\!]$$

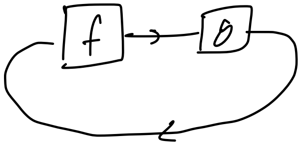
$$\parallel$$

$$(1\ 0\ 0\ 1)\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = (1\ 0\ 0\ 1)\begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix} = a + d$$

if $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

then this equals the linear-algebraic trace of f.

Recall



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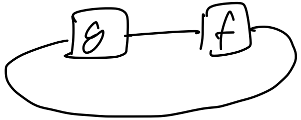
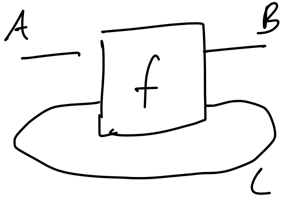
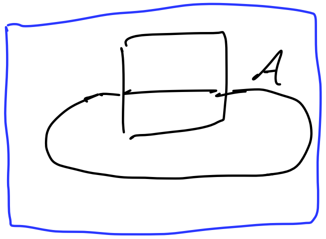


, i.e. $tr(gf) = tr(fg)$



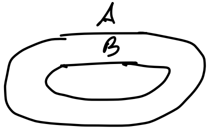
$= tr_C(f) : A \longrightarrow B$ is the <u>partial trace</u> of f

Dimensions

the abstract scalar 

is the <u>dimension</u> of A

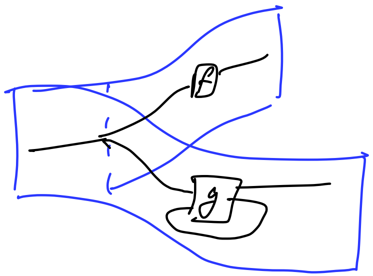
and equals the linear-algebraic dimension

$\dim(A\otimes B) =$ 

$=$ 

$= \dim(A)\cdot\dim(B)$

$\dim(A\oplus B) = \dim(A)+\dim(B)$



$\dim(H)=\infty$

$\Longleftrightarrow$

$H \simeq H\oplus\mathbb{C} \quad \Longrightarrow \quad \dim(H)=\dim(H\oplus\mathbb{C})=\dim(H)+1 \quad ↯$

Bases

Orthonormal bases

linear maps

$f : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$

need to choose bases for $\mathbb{C}^2$ and $\mathbb{C}^3$

3-by-2 matrix

$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

$\langle e_i | e_j \rangle = 0$ if $i \neq j$

$\langle e_i | e_i \rangle = 1$

Sums of diagrams

Superposition

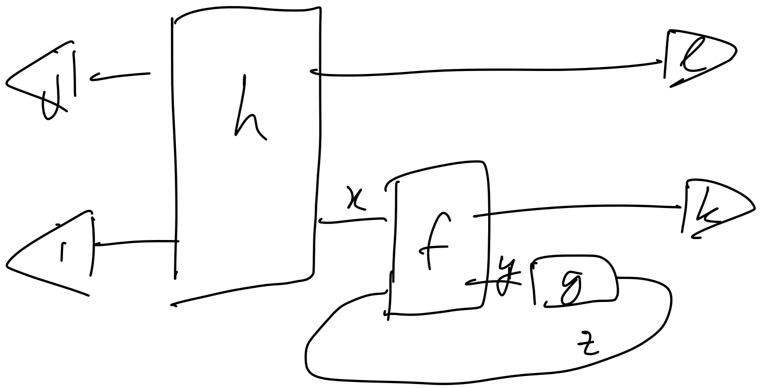
if $\{|e_i\rangle\}_i$ is orthonormal basis, then

$$\underline{\qquad\qquad\qquad} = \sum_i -\!\!-|i\rangle\langle i|-\!\!-$$

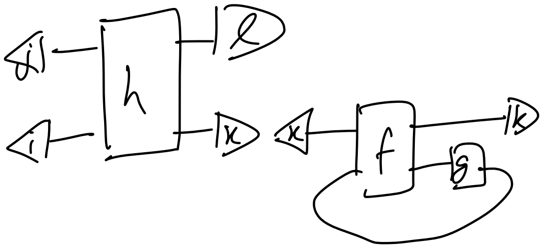
$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ 0 & \ddots & \\ 0 & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & \ddots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \ddots & 0 \\ 0 & & 0\ 1 \end{pmatrix}$$

Tensor contraction

Tensor networks



$$= \sum_{x}$$



$$= \cdots$$

$$= \sum_{x,y,z} h^{l,x}_{i,j} \cdot g^{z}_{y} \cdot f^{y,k}_{z,x}$$

Special cases

tensor product: $(f \otimes g)_{i,j}^{k,l} = f_i^k \, g_j^l$

matrix multiplication: $(g \circ f)_{i,j} = \sum_k f_i^k \, g_k^j$

trace: $tr(f) = \sum_i f_i^i$

Summary:

- Graphical calculus automatically incorporates traces and dimensions
- It pays to be precise about choice of basis
- In computation it is handy to explicitly assume superposition
- Tensor networks are special case of graphical rewriting