Introduction to Quantum Programming and Semantics

Lecture 8: Classical data

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Overview

- Qiskit
- Frobenius algebras
- Spiders
- Phases

Frobenius algebras

Copying orthonormal bases

$\{|0\rangle, |1\rangle\}$ basis for $\mathbb{C}^2$ $\rightsquigarrow$ copying machine $-\!\!\circ\!\!\prec \,:\, \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$

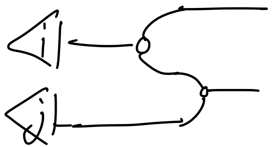
$$|0\rangle \longmapsto |00\rangle$$
$$|1\rangle \longmapsto |11\rangle$$

what is the adjoint ? $\succ\!\!\circ\!\!- \,:\, \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

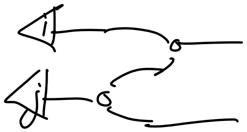
$$|ij\rangle \longmapsto \begin{cases} |i\rangle & \text{if } i=j \\ 0 = \begin{pmatrix}0\\0\end{pmatrix} & \text{if } i \neq j \end{cases}$$

$$\langle -\!\!\circ\!\!\prec(|i\rangle) \,|\, jk\rangle = \langle i \,|\, \succ\!\!\circ\!\!-(|jk\rangle)\rangle$$
$$\parallel$$
$$\langle ii \,|\, jk\rangle$$
$$\parallel$$
$$\begin{cases} 1 & \text{if } i=j=k \\ 0 & \text{otherwise} \end{cases}$$

Idea:



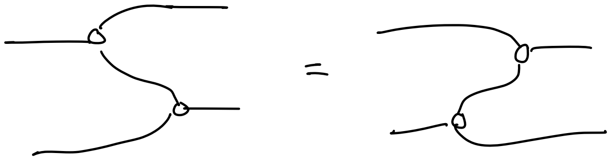
$$= \begin{bmatrix} |ii\rangle & \text{if } i=j \\ 0 & \text{if } i\neq j \end{bmatrix} =$$



Interacting monoid and comonoid

in case of copying basis:

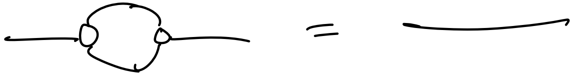
"Frobenius law":



basis is orthogonal

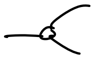
$\langle i|j\rangle = \begin{cases} 0 & \\ 1 & \text{if } i=j \end{cases}$

"speciality":

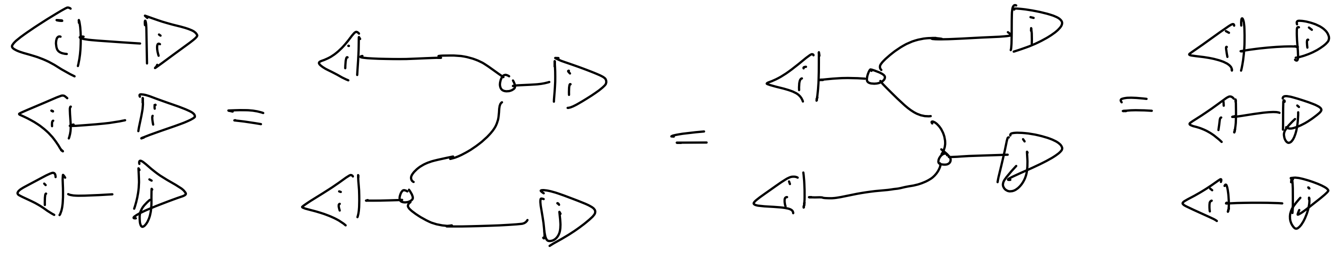


basis is normalised

$\langle i|i\rangle = 1$

Claim: if  copies basis, and satisfies Frobenius law, then basis vectors are orthogonal.

Proof:



if $\langle i|j\rangle = 0$ then done.

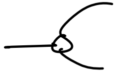
if $\langle i|j\rangle \neq 0$, then ~~$\langle i|i\rangle$~~ $\langle i|i\rangle =$ ~~$\langle i|i\rangle$~~ $\langle i|j\rangle$, similarly $\langle j|i\rangle = \langle j|j\rangle$

so $\langle i-j|i-j\rangle = \langle i|i\rangle - \langle i|j\rangle - \langle j|i\rangle + \langle j|j\rangle$

$= 0$

so $|i-j\rangle = 0$ so $i=j$

Classical structure is :

a map  : $A \longrightarrow A \otimes A$ such that

- associative
- commutative
- unital
- special
- satisfies Frobenius law

<u>Thm</u>: one-to-one correspondence $\{$classical structures $\mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n\}$

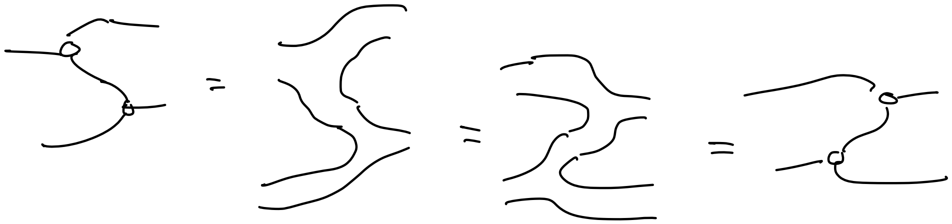
$\simeq \{$ orthonormal basis for $\mathbb{C}^n\}$

Examples:

- $|i\rangle \mapsto |ii\rangle$ for orthonormal basis $\{|i\rangle\}$

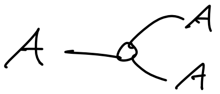
- pair of pants: 
  (i.e. n-by-n matrices)

  satisfies Frobenius law

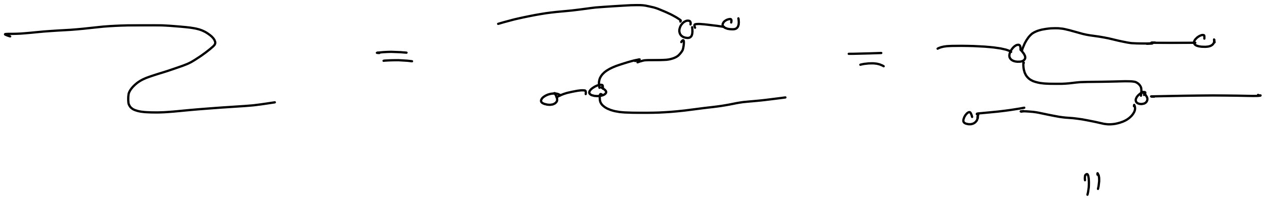
  but is not commutative

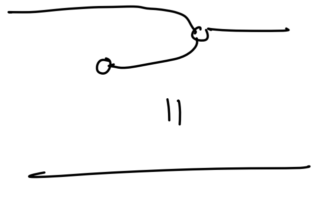
Self-duality

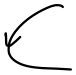
suppose  is a classical structure.

set

then



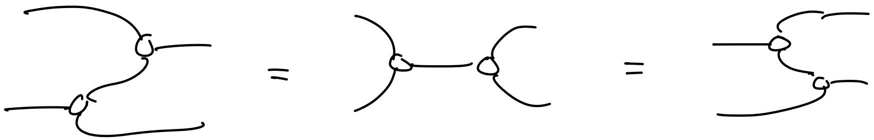


so don't need decorating arrows 
for bits (instead of qubits)

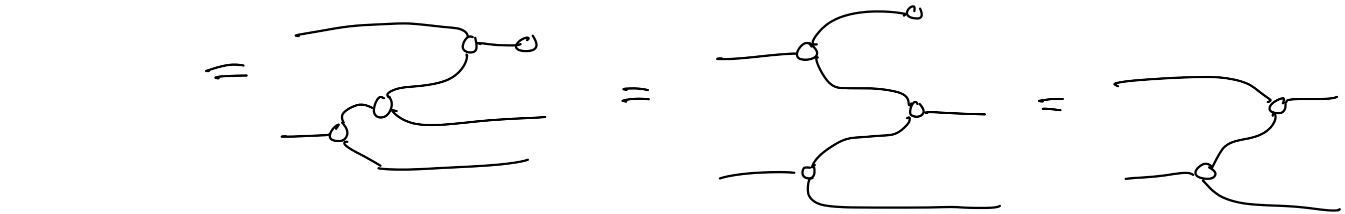
Spiders

Normal form

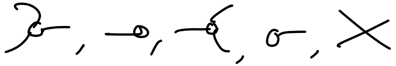
e.g.:

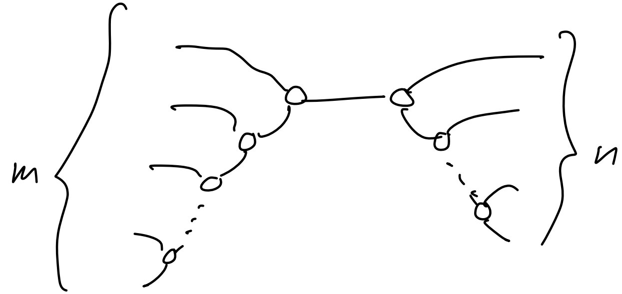


proof:

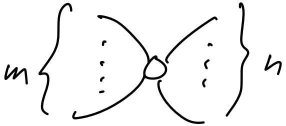


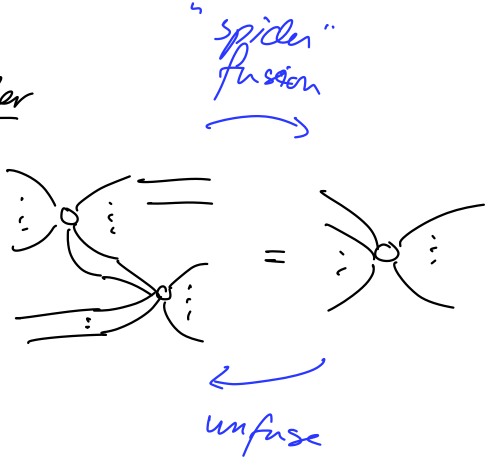
Spiders

theorem: any connected diagram built from 
is equal to:

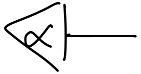
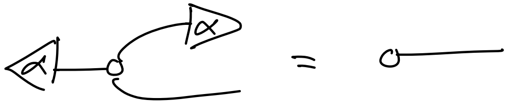


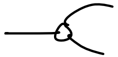
So: instead of a copying machine, can have a spider

i.e. a family  of maps such that

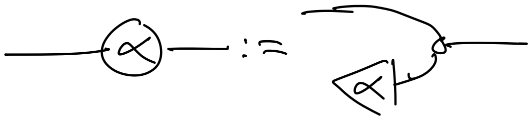


Phases

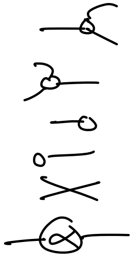
Phases are states  such that 

e.g.: if  copies $\{|i\rangle\}$, then phases are vectors $\alpha = \sum_i \alpha_i |i\rangle$

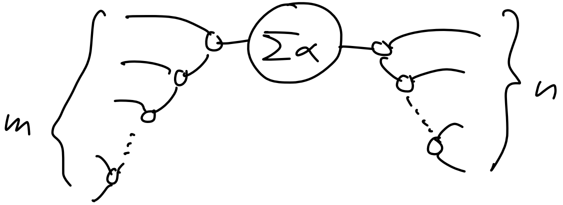
with $\sum_i |\alpha_i|^2 = 1$

"phase shift" is a map 

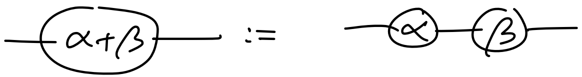
Phased spider theorem: any connected diagram built from



is equal to:

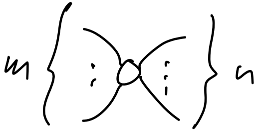


where

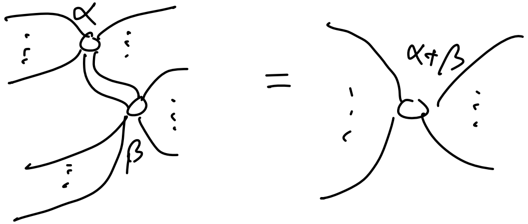


So instead of copying map, can have <u>spider</u>

i.e. family



such that



Summary:

- Qiskit is fairly low-level circuit description language
- Frobenius law is extreme form of ‘only connectivity matters’
- Can equivalently think of classical data as spiders
- Spiders can carry phases around