Introduction to Quantum Programming and Semantics 2026
Tutorial week 3

Exercise 1

(a)

- Assume that f is self adjoint. Therefore we have

$$\begin{aligned}\sum_j \lambda_j |\phi_j\rangle\langle\phi_j| &= \left(\sum_j \lambda_j |\phi_j\rangle\langle\phi_j|\right)^\dagger \\ &= \sum_j (\lambda_j |\phi_j\rangle\langle\phi_j|)^\dagger \\ &= \sum_j \overline{\lambda_j} |\phi_j\rangle\langle\phi_j|\end{aligned}$$

Let us fix a j. If we postcompose the left and right-hand side by $\langle\phi_j|$ and precompose both by $|\phi_j\rangle$ as well, we obtain

$$\begin{aligned}\lambda_j &= \langle\phi_j| (\sum_j \lambda_j |\phi_j\rangle\langle\phi_j|) |\phi_j\rangle \\ &= \langle\phi_j| (\sum_j \overline{\lambda_j} |\phi_j\rangle\langle\phi_j|) |\phi_j\rangle \\ &= \overline{\lambda_j}\end{aligned}$$

and we conclude that $\lambda_j \in \mathbb{R}$.

- Assume that each λ_j is in $\mathbb{R}$. Then we have

$$\begin{aligned}f^\dagger &= (\sum_j \lambda_j |\phi_j\rangle\langle\phi_j|)^\dagger \\ &= \sum_j (\lambda_j |\phi_j\rangle\langle\phi_j|)^\dagger \\ &= \sum_j \overline{\lambda_j} |\phi_j\rangle\langle\phi_j| \\ &= \sum_j \lambda_j |\phi_j\rangle\langle\phi_j| \\ &= f\end{aligned}$$

which concludes.

(b)

- If f is positive, then for all j, then $\langle\phi_j| f |\phi_j\rangle = \lambda_j$ is positive as well.
- If each λ is in $\mathbb{R}_{\geq 0}$, fix a $|\psi\rangle$ such that $\langle\psi|\phi_j\rangle = \alpha_j$, then $\langle\psi| f |\psi\rangle = \sum_j \lambda_j |\alpha_j|^2$, which is positive.

(c)

- If f is a projection, then $ff = f$ and with similar manipulation as in (a), we get for all j that $\lambda_j^2 = \lambda_j$, and thus $\lambda_j \in \{0, 1\}$.
- If all λ_j are in $\{0, 1\}$, then we have

$$\begin{aligned}ff &= \left(\sum_j \lambda_j |\phi_j\rangle\langle\phi_j|\right)\left(\sum_j \lambda_j |\phi_j\rangle\langle\phi_j|\right) = \sum_j \lambda_j^2 |\phi_j\rangle\langle\phi_j| |\phi_j\rangle\langle\phi_j| \\ &= \sum_j \lambda_j^2 |\phi_j\rangle\langle\phi_j| = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j| = f\end{aligned}$$

which concludes.

(d)
- We have $ff^\dagger = \mathrm{id}$, and with similar manipulation as in (a) we get that for all j, $|\lambda_j|^2 = 1$, and thus $\lambda_j \in U(1)$.
- If $\lambda_j \in U(1)$, then $|\lambda_j|^2 = 1$ and

$$ff^\dagger = \left(\sum_j \lambda_j |\phi_j\rangle \langle\phi_j|\right)\left(\sum_j \lambda_j |\phi_j\rangle \langle\phi_j|\right)^\dagger = \sum_j \lambda_j \overline{\lambda_j} |\phi_j\rangle \langle\phi_j| |\phi_j\rangle \langle\phi_j|$$
$$= \sum_j |\lambda_j|^2 |\phi_j\rangle \langle\phi_j| = \sum_j |\phi_j\rangle \langle\phi_j| = \mathrm{id}$$

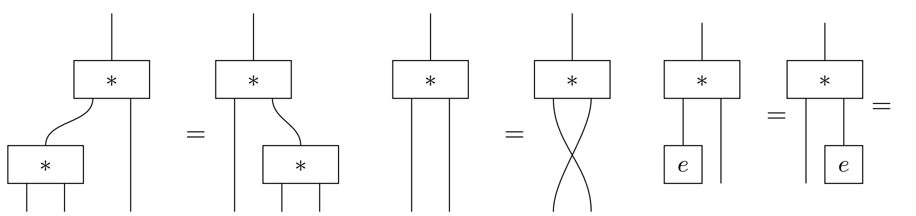
which concludes.

Exercise 2

Use the fact that $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$:

$$\begin{aligned}
\langle\phi|\psi\rangle &= \Big(\cos\frac{\theta}{2}\langle 0| + \sin\frac{\theta}{2}e^{i\alpha}\langle 1|\Big)\Big(\cos\frac{\theta+\pi}{2}|0\rangle + \sin\frac{\theta+\pi}{2}e^{i\alpha}|1\rangle\Big) \\
&= \cos\frac{\theta}{2}\cos\frac{\theta+\pi}{2}\langle 0|0\rangle + \cos\frac{\theta}{2}\sin\frac{\theta+\pi}{2}\langle 0|1\rangle + \sin\frac{\theta}{2}\cos\frac{\theta+\pi}{2}\langle 1|0\rangle + \sin\frac{\theta}{2}\sin\frac{\theta+\pi}{2}\langle 1|1\rangle \\
&= \cos\frac{\theta}{2}\cos\frac{\theta+\pi}{2} + \sin\frac{\theta}{2}\sin\frac{\theta+\pi}{2} \\
&= \cos\Big(\frac{\theta}{2} - \frac{\theta+\pi}{2}\Big) \\
&= \cos\frac{-\pi}{2} \\
&= 0.
\end{aligned}$$

Exercise 3



(1)

Exercise 4

These functions are defined in terms of how they map the elements of the set where they're defined, so it's enough to calculate it by "brute force". For the first one, calculating $(X\otimes X)\circ(CX)\circ(X\otimes Id)$ step by step:

$$\begin{aligned}
(0,0) &\mapsto (1,0) \mapsto (1,1) \mapsto (0,0) \\
(0,1) &\mapsto (1,1) \mapsto (1,0) \mapsto (0,1) \\
(1,0) &\mapsto (0,0) \mapsto (0,0) \mapsto (1,1) \\
(1,1) &\mapsto (0,1) \mapsto (0,1) \mapsto (1,0)
\end{aligned}$$

Which is equal to CX. For the second one, calculating CX,SWAP,CX,SWAP,CX:

$$\begin{aligned}
(0,0) &\mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \\
(0,1) &\mapsto (0,1) \mapsto (1,0) \mapsto (1,1) \mapsto (1,1) \mapsto (1,0) \\
(1,0) &\mapsto (1,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,1) \mapsto (0,1) \\
(1,1) &\mapsto (1,0) \mapsto (0,1) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1)
\end{aligned}$$

which is the SWAP map.

Exercise 5

The first and fourth are product states:

$$\tfrac{1}{2}\big(\,|00\rangle + |01\rangle + |10\rangle + |11\rangle\,\big) = \big(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\big) \otimes \big(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\big),$$
$$\tfrac{1}{2}\big(\,|00\rangle - |01\rangle - |10\rangle + |11\rangle\,\big) = \big(\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\big) \otimes \big(\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\big).$$

In general, because $|0\rangle$ and $|1\rangle$ form an orthonormal basis, any product state takes the form $(a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle) = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle$ for some $a, b, c, d \in \mathbb{C}$. For the second and third states, there is no solution to the ensuing system of linear equations, so these two states are entangled.