

# Introduction to Quantum Programming and Semantics 2026

## Tutorial week 3

These exercises relate mostly to Lecture 2, and aim to warm you up to linear-algebraic semantics. Only Exercise 3 is about graphical calculus, as in Lecture 4.

### Exercise 1

Let  $f: H \rightarrow H$  be a linear map between finite-dimensional Hilbert spaces such that  $f^\dagger f = f f^\dagger$ . Then we can write  $f = \sum_j \lambda_j |\phi_j\rangle \langle \phi_j|$  for some set  $\{\lambda_j\}_j$  of scalars and some set  $\{|\phi_j\rangle\}_j$  of orthogonal vectors in  $H$ . Show that:

- (a)  $f$  is self-adjoint if and only if each  $\lambda_j \in \mathbb{R}$ ;
- (b)  $f$  is positive if and only if each  $\lambda_j \in \mathbb{R}_{\geq 0}$ ;
- (c)  $f$  is a projection if and only if each  $\lambda_j \in \{0, 1\}$ ;
- (d)  $f$  is unitary if and only if each  $\lambda_j \in U(1)$ ;

where  $\mathbb{R}_{\geq 0}$  is the set of all nonnegative real numbers and  $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$  is the set of all phases.

### Exercise 2

Show that  $\langle \phi | \psi \rangle = 0$  for the following antipodal states on the Bloch sphere:

$$\begin{aligned} |\phi\rangle &= \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\alpha} |1\rangle \\ |\psi\rangle &= \cos \frac{\theta + \pi}{2} |0\rangle + \sin \frac{\theta + \pi}{2} e^{i\alpha} |1\rangle \end{aligned}$$

### Exercise 3

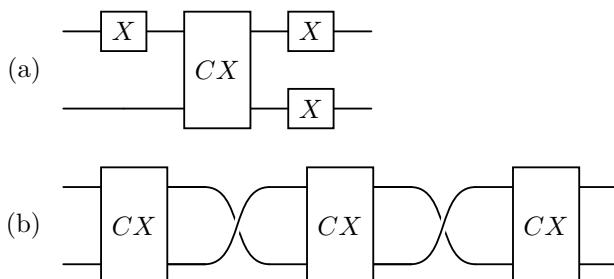
Give the diagrammatic equations of a process  $*$  taking two inputs and one output that express the algebraic properties of being:

- (a) associative:  $x * (y * z) = (x * y) * z$ ;
- (b) commutative:  $x * y = y * x$ ;
- (c) having a unit: there exists a process  $e$  (with no inputs) such that  $x * e = e * x = x$ ;

Note:  $x$ ,  $y$  and  $z$  should not appear in the diagrams.

### Exercise 4

Compute the semantic interpretation of the following circuits either as matrices or as functions on the computational basis states, and then show that they can both be expressed by simpler circuits:



### Exercise 5

Which of these states of  $\mathbb{C}^2 \otimes \mathbb{C}^2$  in **Hilb** are entangled and why?

- (a)  $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
- (b)  $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$
- (c)  $\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$
- (d)  $\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$