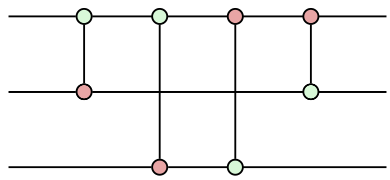
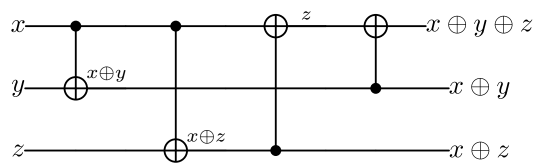
Introduction to Quantum Programming and Semantics 2025
Tutorial week 7

Exercise 1

The phase-free ZX diagram corresponding to the circuit is:



Tracing booleans through the circuit gives:

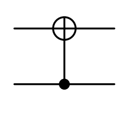


So the parity matrix is

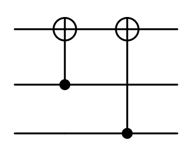
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Executing the algorithm gives the following matrices and circuits:

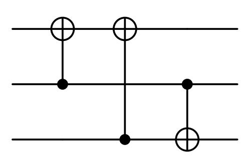
$i = 1$ $j = 2$ $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$



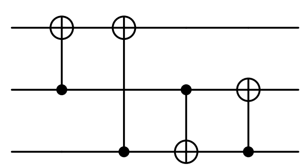
$i = 1$ $j = 3$ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$



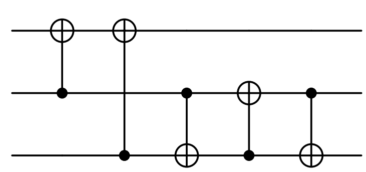
$i = 2$ $k = 3$ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$



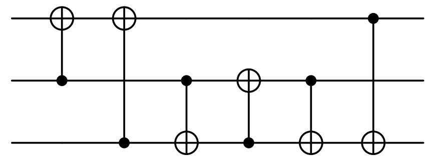
$i = 2$ $j = 3$ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$



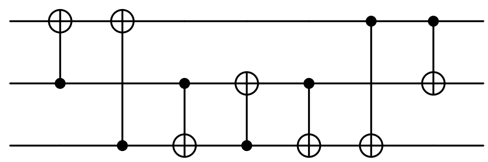
$i = 3$ $j = 2$ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$



$i = 3$ $\quad j = 1$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$i = 2$ $\quad j = 1$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



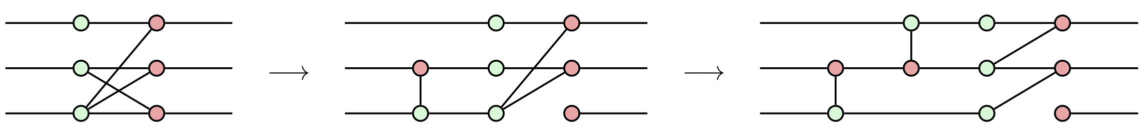
Tracing through this circuit shows that it is indeed equivalent to the original one.

Exercise 2

The matrix is not invertible. Trying to reduce it fails:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{c_3 = c_3 + c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{c_1 = c_1 + c_2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and you end up in a cycle, never reaching the identity matrix. The accompanying ZX diagrams are:



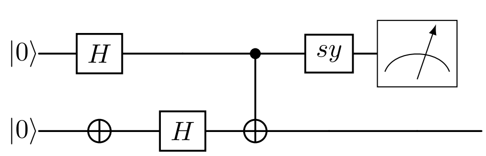
The problem is that output X spiders get disconnected.

Exercise 3

Suppose the ZX diagram in parity normal form is not unitary. If we follow the strategy of [KW Proposition 4.2.12] of applying Gaussian elimination to the biadjacency matrix by introducing CNOTs to the diagram, we would eventually get a biadjacency matrix whose bottom row consists only of 0s. This means that in the ZX-diagram the Z-spider on the bottom qubit is not connected to any X-spider. Such a diagram cannot represent a unitary, as the output of the diagram is then independent of whether we input a $|0\rangle$ or $|1\rangle$ on the bottom qubit, so that it is not injective.

Exercise 4

The left-hand program, as a circuit, is



The state of the two qubits `a` and `b` before measurement is

$$\begin{aligned} &(\mathrm{sy} \otimes \mathrm{id}) \cdot CX \cdot (\mathrm{id} \otimes H) \cdot (H \otimes X) \, |00\rangle \\ &= (\mathrm{sy} \otimes \mathrm{id}) \cdot CX \cdot (\mathrm{id} \otimes H)(|+\rangle \otimes |1\rangle) \\ &= (\mathrm{sy} \otimes \mathrm{id}) \cdot CX \, |+-\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(\mathrm{sy} \otimes \mathrm{id}) \cdot CX((|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)) \\
&= \frac{1}{2}(\mathrm{sy} \otimes \mathrm{id}) \cdot CX(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\
&= \frac{1}{2}(\mathrm{sy} \otimes \mathrm{id})(|00\rangle - |01\rangle + |11\rangle - |10\rangle) \\
&= \frac{1}{2}\left(\begin{pmatrix} 1+i & -1-i \\ 1+i & 1+i \end{pmatrix} \otimes \mathrm{id}\right)\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) \\
&= \frac{1}{2}\left(\begin{pmatrix}1+i\\1+i\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1+i\\1+i\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-1-i\\1+i\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}-1-i\\1+i\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix} 1+i+1+i \\ 1+i-1-i \\ -1-i-1-i \\ -1-i+1+i \end{pmatrix} \\
&= \begin{pmatrix} 1+i \\ 0 \\ -1-i \\ 0 \end{pmatrix} \\
&= \sqrt{2}(1+i)\,|-\rangle
\end{aligned}
$$

So there is a nonzero probability that the outcome of the measurement is 0. The program is therefore not equivalent to the right-hand one.

You can use this knowledge to determine whether or not any phase can propagate through a multiqubit gate, which enables some rewrites. If you know a part of the program has qubits satisfying this, then you can rewrite the program to only act on these inputs. In particular, if all of the gates are 'classical', then you can just run it classically first.