1. Given the following TBox, where $A$, $B$, $C$, $D$, $E$, $F$, $H$ are named classes, while $r$ is a named property:

$$\mathcal{T} = \{ C \sqsubseteq D, A \sqsubseteq E, E \sqsubseteq \exists r. F, F \sqsubseteq B, H \sqsubseteq B, F \sqsubseteq H \}$$

Check if the TBox entails $A \sqsubseteq \exists r. B$.

**Solution:**
1. To check $A \sqsubseteq \exists r. B$, we introduce two fresh named classes $A_1$ and $A_2$, and extend the TBox with two axioms: $A_1 \sqsubseteq A$, $\exists r. B \sqsubseteq A_2$. If we can show $A_1 \sqsubseteq A_2$, then we have $A \sqsubseteq \exists r. B$.
2. Axioms in the TBox are already in the normal form
3. $A_1 \sqsubseteq A$, $A \sqsubseteq E \Rightarrow A_1 \sqsubseteq E$ (CR3)
4. $E \sqsubseteq \exists r. F$, $F \sqsubseteq B$, $\exists r. B \sqsubseteq A_2 \Rightarrow E \sqsubseteq A_2$ (CR5)
5. $A_1 \sqsubseteq E$, $E \sqsubseteq A_2 \Rightarrow A_1 \sqsubseteq A_2$ (CR3); thus we have $A \sqsubseteq \exists r. B$.

2. Classify the following TBox:

```
Pericardium ⊑ Tissue ⊓  contained-in.Heart
Pericarditis ⊑ Inflammation ⊓  has-location.Pericardium
Inflammation ⊑ Disease ⊓  has-location.Pericardium
Heartdisease ⊑ Disease ⊓  has-location.Heart
Heartdisease ⊑  has-state.NeedsTreatment
```

**Solution:**
(I) **We need to transfer the axioms into normal form first.**
1. Pericardium ⊑ Tissue ⊓  contained-in.Heart
2. Pericarditis ⊑ Inflammation ⊓  has-location. Pericardium
3. Inflammation ⊑ Disease ⊓  has-location. Tissue
4. Heartdisease ⊑ Disease ⊓  has-location.Heart , Disease ⊓  has-location.Heart ⊑ Heartdisease
5. Heartdisease ⊑  has-state. NeedsTreatment is already normalized

(II) Then we classify the TBox:
1. Heartdisease ⊑ Heartdisease, Disease ⊑ Disease, Heart ⊑ Heart, A1 ⊑ A1,
   NeedsTreatment ⊑ NeedsTreatment, Pericardium ⊑ Pericardium, Tissue ⊑ Tissue,
   Pericarditis ⊑ Pericarditis, Inflammation ⊑ Inflammation, (CR1)
2. Heartdisease ⊑ T, Disease ⊑ T, Heart ⊑ T, A1 ⊑ T, NeedsTreatment ⊑ T, Pericardium ⊑ T, Tissue ⊑ T, Percarditis ⊑ T, Inflammation ⊑ T (CR2)
3. Percarditis ⊑ Inflammation, Inflammation ⊑ Disease => Percarditis ⊑ Disease (CR3)

3. Given the following TBox, where A, B, C, D are named classes, while r is a named property:

\[
\begin{align*}
A & \subseteq B \cap \exists r.C \\
B \cap \exists r.B & \subseteq C \cap D \\
C & \subseteq (\exists r.A) \cap B \\
\exists r.\exists r.B \cap D & \subseteq \exists r.(A \cap B)
\end{align*}
\]

Check whether the following subsumption relations hold w.r.t. the TBox:

(I) We need to transfer the axioms into normal form first.
1. A ⊑ B
2. A ⊑ B \cap \exists r.A
3. B ⊑ B \cap \exists r.A ⊑ \exists r.C

(II) We then do subsumption checking:
\[a. \ A \subseteq B: \text{this is true after normalisation, according to 1.1}\]
\[b. \ A \subseteq \exists r. A, \text{to prove this, we need to add one axiom into the TBox: \{\exists r. A \subseteq A7, \}, and then prove A \subseteq A7. First of all, we need to normalize \exists r. A \subseteq A7.}\]
\[b.1) \exists r. A \subseteq A7, \exists r. A \subseteq A8 \text{ (NF2)}\]
\[b.2) C \subseteq \exists r. A (3.1), A \subseteq A, \exists r. A \subseteq A8 (b.1) => C \subseteq A8 \text{ (CR5)}\]
\[b.3) A \subseteq \exists r. A (1.1), C \subseteq A8 (b.2), \exists r. A \subseteq A7 (b.1) => A \subseteq A7; \text{thus we have } A \subseteq \exists r. A.\]
\[c. \ B \subseteq \exists r. A \subseteq \exists r. C, \text{to prove this, we need to add two axioms into the TBox: \{A8 \subseteq B \cap \exists r. A, \exists r. C \subseteq A9,\}, and the prove A8 \subseteq A9. First of all, we need to normalize } A8 \subseteq B \cap \exists r. A\]
c.1) A₈ ⊑ B, A₈ ⊑ ∃r.A (NF0)
c.2) A ⊑ A, B ⊑ B, C ⊑ C, D ⊑ D, A₁ ⊑ A₁, A₂ ⊑ A₂, A₃ ⊑ A₃, A₄ ⊑ A₄, A₆ ⊑ A₆, A₈ ⊑ A₈, A₉ ⊑ A₉ (CR1), A ⊑ ⊤, B ⊑ ⊤, C ⊑ ⊤, D ⊑ ⊤, A₁ ⊑ ⊤, A₂ ⊑ ⊤, A₃ ⊑ ⊤, A₄ ⊑ ⊤, A₆ ⊑ ⊤, A₈ ⊑ ⊤, A₉ ⊑ ⊤ (CR2)
c.3) A ⊑ ∃r.C (1.1), C ⊑ B (3.1), ∃r.B ⊑ A₂ (2.2) => A ⊑ A₂ (CR5)
c.4) A ⊑ B (1.1), A ⊑ A₂ (C.3), B ∩ A₂ ⊑ A₁ (2.2) => A ⊑ A₁ (CR4)
c.5) A ⊑ A₁ (c.4), A₁ ⊑ C (2.3) => A ⊑ C (CR3)
c.6) A₈ ⊑ ∃r.A (c.1), A ⊑ C (c.5), ∃r.C ⊑ A₉ => A₈ ⊑ A₉, thus we have B ∩ ∃r.A ⊑ ∃r. C.