Methods for Causal Inference
Lecture 2: Basics of probability

Ava Khamseh
School of Informatics
2023-2024
Causal theory and data

Requires 5 steps:

1. Definition of Causation
2. Clearly formulating causal **assumptions** and creating the **causal model**
3. Linking the structure of causal model to features of data
4. **Estimating**, given the causal model and data
5. **Uncertainty quantification**, e.g., confidence/credible interval
Causal theory and data

Requires 5 steps:

1. Definition of Causation
2. Clearly formulating causal assumptions and creating the causal model
3. Linking the structure of causal model to features of data
4. Estimating, given the causal model and data
5. Uncertainty quantification, e.g., confidence/credible interval

Disclaimer: In this course our focus is on 1-3. We then use simple models to exemplify 4-5 (taking model assumptions as ‘true’), i.e., we do not discuss valid statistical inference.
For causal/statistical inference please refer to the course: Targeted Causal Learning (code: MATH11238).
Causal theory and data

Requires 5 steps:

1. Definition of Causation
2. Clearly formulating causal **assumptions** and creating the **causal model**
3. Linking the structure of causal model to features of data
4. **Estimating**, given the causal model and data
5. **Uncertainty quantification**, e.g., confidence/credible interval

**Defining causation:**
A variable $X$ is a **cause** of a variable $Y$ if $Y$ in any way relies on $X$ for its value. (Intuitively: $X$ is a cause of $Y$ if $Y$ listens to $X$ and decides its value in response to what it hears)

**Pre-requisites:** Elementary concepts from probability theory, statistics, graph theory
Most causal statements are uncertain: “drinking causes liver disease”, does not mean every person who consumes alcohol is certain to have liver disease

Need language and laws of probability.
Basics of probability

Most causal statements are uncertain: “drinking causes liver disease”, does not mean every person who consumes alcohol is certain to have liver disease

Need language and laws of probability.

(Random) variables: Any property or descriptor that can take multiple values, e.g., age (x=40), sex (x’=F), family history of disease (x”=0), ... .
Basics of probability

Most causal statements are uncertain: “drinking causes liver disease”, does not mean every person who consumes alcohol is certain to have liver disease.

Need language and laws of probability.

(Random) variables: Any property or descriptor that can take multiple values, e.g., age \(x=40\), sex \(x'=F\), family history of disease \(x''=0\), ... .

Events: An event is any assignment of a value or set of values to a variable or set of variables.
Basics of probability

Most causal statements are uncertain: "drinking causes liver disease", does not mean every person who consumes alcohol is certain to have liver disease.

Need language and laws of probability.

(Random) variables: Any property or descriptor that can take multiple values, e.g., age ($x=40$), sex ($x'=F$), family history of disease ($x''=0$), ...

Events: An event is any assignment of a value or set of values to a variable or set of variables.

Example: Individual $>40$ and recovered from covid $y=0$, event is $(x>40,y=0)$. So variables are 'age' and 'recovery status' with values $>40$ and $0$.

Can ask what is the probability of an event, e.g., what is $P(x>40,y=0)$?
Basics of probability

Most causal statements are uncertain: “drinking causes liver disease”, does not mean every person who consumes alcohol is certain to have liver disease.

Need language and laws of probability.

(Random) variables: Any property or descriptor that can take multiple values, e.g., age \( (x=40) \), sex \( (x'=F) \), family history of disease \( (x''=0) \), ... .

Events: An event is any assignment of a value or set of values to a variable or set of variables.

Discrete (binary/categorical): Are being treated or not, have a disease or not, ...

Continuous (can take infinite set of values): age, weight, ...

Drug (yes/no) vs dose of drug (categorical). Sun intake (time is continuous)
Basics of probability

For probabilistic modelling (of a random experiment) we need to:

- Describe possible outcomes: sample space
- **Event**: A subset of sample space
- Describe beliefs about likelihood of these events: probability law
Sample space

The sample space is the set of all possible outcomes of the experiment:

Outcomes must be:
- **Mutually Exclusive**: If I tell you, after the experiment, that $A_1$ happened, then it should not be possible that $A_6$ also happened.
- **Collectively Exhaustive**: Collectively, all the outcomes in $\Omega$ exhaust all possibilities.

e.g. Rolling a dice
Probability Axioms

Non-negativity: \( P(A) \geq 0 \)
Normalisation: \( P(\Omega) = 1 \)

For any two mutually exclusive events (i.e. \( A \) and \( B \) cannot co-occur) we have:

\[
P(A \text{ or } B) = P(A) + P(B)
\]
Probability Axioms

Non-negativity: \( P(A) \geq 0 \)
Normalisation: \( P(\Omega) = 1 \)

For any two mutually exclusive events (i.e. A and B cannot co-occur) we have:
\[
P(A \text{ or } B) = P(A) + P(B)
\]

As a consequence, take any two events A and B (they may overlap!), then:
\[
P(A) = P(A \text{ and } B) + P(A \text{ and } 'not B')
\]

Why is the last statement true?
Probability Axioms

- Non-negativity: \( P(A) \geq 0 \)
- Normalisation: \( P(\Omega) = 1 \)

- For any two mutually exclusive events (i.e. \( A \) and \( B \) cannot co-occur) we have:
  \[
P(A \text{ or } B) = P(A) + P(B)
  \]

As a consequence, take any two events \( A \) and \( B \) (they may overlap!), then:
\[
P(A) = P(A \text{ and } B) + P(A \text{ and } ‘not } B’)
\]

Mutually exclusive: If \( A \) is true, either “\( A \) and \( B \)” or “\( A \) and not \( B \)” must be true.
Probability Axioms

- Non-negativity: \( P(A) \geq 0 \)
- Normalisation: \( P(\Omega) = 1 \)
- For any two **mutually exclusive events** (i.e. A and B cannot co-occur) we have:
  \[
P(A \text{ or } B) = P(A) + P(B)
  \]

Corollary: \( B_1, B_2, B_3 \), are exclusive, and together form all of B. Then,

\[
P(A \text{ and } B) = P(A \text{ and } B_1) + P(A \text{ and } B_2) + P(A \text{ and } B_3)
\]

Generalise for (exhaustive, mutually exclusive) **partitions** of B:

\[
P(A \text{ and } B) = \sum_{i=1}^{n} P(A \text{ and } B_i) \quad \text{where } B_i \cap B_j = \emptyset, \bigcup_{i=1}^{n} B_i = B
\]
Probability Axioms

- **Non-negativity**: \( P(A) \geq 0 \)
- **Normalisation**: \( P(\Omega) = 1 \)

For any two **mutually exclusive events** (i.e. A and B cannot co-occur) we have:
\[
P(A \text{ or } B) = P(A) + P(B)
\]

Corollary: Let \( B_i, \ i=1, ..., n \) be mutually exclusive and exhaustive partitions of B, and let A=B (complete overlap). Then,
\[
P(A) = P(A \text{ and } A) = P(A \text{ and } B) = \sum_{i=1}^{n} P(A \text{ and } B_i) \quad \text{where } B_i \cap B_j = \emptyset, \bigcup_{i=1}^{n} B_i = B
\]

See later: “marginalisation”
Intervals

\[ P(\text{age} > 4) = 1 - P(\text{age} \leq 4) = 1 - 0.49 = 0.51 \]

Figure 7.2: Age at adoption, Scotland, 2018

49% of adoptions were of children aged 4 and below.

Total = 471
Intervals

\[ P(\text{age} > 4) = 1 - P(\text{age} \leq 4) = 1 - 0.49 = 0.51 \]

\[ P(4 < \text{age} \leq 12) = \frac{43 + 30 + 34 + 25 + 13 + 14 + 5 + 12}{471} = 0.37 \]

Figure 7.2: Age at adoption, Scotland, 2018

Total = 471
Law of Total probability: Example

Assuming ‘no multi-tasking’, the event:
“Passing the causality exam AND not being on your phone during the lectures” is **mutually exclusive** from
“Passing the causality exam AND being entirely on your phone during the lectures”

\[
P(\text{passing the causality exam}) =\]
\[
P(\text{passing the exam, being entirely on your phone during the lecture}) +
P(\text{passing the exam, fully paying attention during the lecture})\]
Law of Total probability: Example

Assuming ‘no multi-tasking’, the event:
“Passing the causality exam AND not being on your phone during the lectures”
is mutually exclusive from
“Passing the causality exam AND being entirely on your phone during the lectures”

\[
P(\text{passing the causality exam}) = \\
P(\text{passing the exam, being entirely on your phone during the lecture}) + \\
P(\text{passing the exam, fully paying attention during the lecture})
\]
Law of Total probability: Example

Assuming ‘no multi-tasking’, the event:
“Passing the causality exam AND not being on your phone during the lectures”
is mutually exclusive from
“Passing the causality exam AND being entirely on your phone during the lectures”

\[ P(\text{passing the causality exam}) = \]
\[ P(\text{passing the exam, being entirely on your phone during the lecture}) + \]
\[ P(\text{passing the exam, fully paying attention during the lecture}) \]
Conditional Probability

The probability that event $A$ occurs, given that we know some other event $B$ has occurred. (Think of filtering the data based on the value of some variable)

$P(X = x)$ vs $P(X = x | Y = y)$: The probability of $X=x$ can drastically change depending on the knowledge $Y=y$
Conditional Probability

The probability that event A occurs, given that we know some other event B has occurred. (Think of filtering the data based on the value of some variable)

\[ P(X = x) \text{ vs } P(X = x | Y = y) \] : The probability of \( X=x \) can drastically change depending on the knowledge \( Y=y \)

**Example:** \( P(\text{lung cancer} | \text{smoker}) \) vs

\[ P(\text{lung cancer} | \text{smoker, socio-economic status}) \]

Given that the patient is a smoker, does knowing their socio-economic status add further information to the probability of lung cancer?
Conditional Probability

The probability that event A occurs, given that we know some other event B has occurred. (Think of filtering the data based on the value of some variable)

\[ P(X = x) \text{ vs } P(X = x | Y = y) : \text{ The probability of } X=x \text{ can drastically change depending on the knowledge } Y=y \]

**Example:** \( P(\text{lung cancer} | \text{smoker}) \) vs \( P(\text{lung cancer} | \text{smoker, socio-economic status}) \)

Given that the patient is a smoker, does knowing their socio-economic status add further information to the probability of lung cancer?

Relation between “joint”, “conditional”, and “marginal” probabilities:

\[ P(X, Y) = P(X|Y)P(Y) \]
Conditional Law of Total probability: Example

\[ P(\text{passing the causality exam } | \text{ paying attention}) > P(\text{passing the causality exam } | \text{ being on your phone}) \]
**Conditional Law of Total Probability: Example**

\[
P(\text{passing the causality exam} \mid \text{fully paying attention during the lecture}) =
\]
\[
P(\text{passing the exam, attending tutorials} \mid \text{attention in lecture}) +
\]
\[
P(\text{passing the exam, not attending tutorials} \mid \text{attention in lecture})
\]

\[
P(\text{passing the causality exam} \mid \text{being on one’s phone during the lectures}) =
\]
\[
P(\text{passing the exam, attending tutorials} \mid \text{being on phone during lecture}) +
\]
\[
P(\text{passing the exam, not attending tutorials} \mid \text{being on phone lecture})
\]

(\text{‘Divine intervention’ category})
Bayes’ Rule

$A_1, A_2, \ldots, A_n$ are disjoint events forming a partition of the sample space and $P(A_i) > 0, \forall A_i$. Then, for any event $B$, $P(B) > 0$, Bayes’ rule states:

$$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B | A_i)}{P(B)}$$
Bayes’ Rule

$A_1, A_2, \ldots, A_n$ are disjoint events forming a partition of the sample space and $P(A_i) > 0, \forall A_i$. Then, for any event $B, P(B) > 0$, Bayes’ rule states:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)}$$

$$= \frac{P(A_i)P(B|A_i)}{P(A_1 \cap B) + \cdots + P(A_n \cap B)}$$

$$= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)}$$
Bayes’ Rule

$A_1, A_2, \ldots, A_n$ are disjoint events forming a partition of the sample space and $P(A_i) > 0$, $\forall A_i$. Then, for any event $B$, $P(B) > 0$, Bayes’ rule states:

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{P(B)}$$

$$= \frac{P(A_i)P(B \mid A_i)}{P(A_1 \cap B) + \cdots + P(A_n \cap B)}$$

$$= \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n)}$$

Note: For random variables, we often write $P(X, Y)$, instead of $P(X \cap Y)$.
Monte Hall Problem & Application of Bayes’ Rule

\[ X = \text{Door chosen by player} \]
\[ Y = \text{Door hiding the car} \]
\[ Z = \text{Door opened by host} \]
Monte Hall Problem & Application of Bayes’ Rule

\[ X = \text{Door chosen by player} \]
\[ Y = \text{Door hiding the car} \]
\[ Z = \text{Door opened by host} \]

Prove that switching doors improves our chance of winning the car.
Monte Hall Problem & Application of Bayes’ Rule

\[ X = \text{Door chosen by player} \]
\[ Y = \text{Door hiding the car} \]
\[ Z = \text{Door opened by host} \]

Prove that switching doors improves our chance of winning the car.

Note the assumptions:
1. The host will not open the door we have chosen
2. The host will never open a door with a car behind
3. Given a choice of doors, the host will choose at random (whilst 2)
4. Given no info, the car is equally likely to be behind any door
Monte Hall Problem & Application of Bayes’ Rule

\[ X = \text{Door chosen by player} \]
\[ Y = \text{Door hiding the car} \]
\[ Z = \text{Door opened by host} \]

Prove that switching doors improves our chance of winning the car.

Need to show (given the we have selected A and host has shown us C):

\[ P(Y = A | X = A, Z = C) < P(Y = B | X = A, Z = C) \]

Is the car more likely to be behind B than A, i.e. switching improves our chance.
Monte Hall Problem & Application of Bayes’ Rule

\[ P(Y = A | X = A, Z = C) = \frac{P(Z = C | X = A, Y = A)P(Y = A | X = A)}{P(Z = C | X = A)} \]

\( X = \) Door chosen by player
\( Y = \) Door hiding the car
\( Z = \) Door opened by host
Monte Hall Problem & Application of Bayes’ Rule

$X =$ Door chosen by player
$Y =$ Door hiding the car
$Z =$ Door opened by host

$$P(Y = A|X = A, Z = C) = \frac{P(Z = C|X = A, Y = A)P(Y = A|X = A)}{P(Z = C|X = A)}$$

Given we choose A (X=A), and the car is in A (Y=A), then the host is allowed to choose either B or C, as neither has the car behind it. Since the host chooses randomly (assumption 3), we get $1/2$. 
Monte Hall Problem & Application of Bayes’ Rule

\[ X = \text{Door chosen by player} \]
\[ Y = \text{Door hiding the car} \]
\[ Z = \text{Door opened by host} \]

Given we choose A (X=A), what is the probability that the car is behind A? With no further information, this is equal to 1/3.

\[
P(Y = A|X = A, Z = C) = \frac{P(Z = C|X = A, Y = A)P(Y = A|X = A)}{P(Z = C|X = A)}
\]

\[ \frac{1}{3} \]
Monte Hall Problem & Application of Bayes’ Rule

A

B

C

Door chosen by player
Door hiding the car
Door opened by host

\[ P(Y = A|X = A, Z = C) = \frac{P(Z = C|X = A, Y = A)P(Y = A|X = A)}{P(Z = C|X = A)} = \frac{1}{2} \]

Total law of prob
Product rule

\[ P(Z = C|X = A) = \sum_{d=A,B,C} P(Z = C, Y = d|X = A) = \sum_{d=A,B,C} P(Z = C|X = A, Y = d)P(Y = d) \]
Monte Hall Problem & Application of Bayes’ Rule

\[ P(Y = A | X = A, Z = C) = \frac{P(Z = C | X = A, Y = A)P(Y = A | X = A)}{P(Z = C | X = A)} \]

**Total law of prob**

\[ P(Z = C | X = A) = \sum_{d=A,B,C} P(Z = C, Y = d | X = A) = \sum_{d=A,B,C} P(Z = C | X = A, Y = d)P(Y = d) \]

\[ = \frac{1}{3} \left( P(Z = C | X = A, Y = A) + P(Z = C | X = A, Y = B) + P(Z = C | X = A, Y = C) \right) \]

1/2 as above

1: Given we chose A and car is behind B, host is **forced** to choose C (Assumption 2)

0: Given we chose A and car is behind C, the host cannot choose C (Assumption 2)
Monte Hall Problem & Application of Bayes’ Rule

\[ P(Y = A|X = A, Z = C) = \frac{P(Z = C|X = A, Y = A)P(Y = A|X = A)}{P(Z = C|X = A)} \]

- \( X \) = Door chosen by player
- \( Y \) = Door hiding the car
- \( Z \) = Door opened by host
Monte Hall Problem & Application of Bayes’ Rule

Importance: Incorporating knowledge about the process that generated the data. The first step towards causal inference.

‘Host could have opened’, ‘he was forced to open’, ‘randomly opened’, ‘about to open’, ...
Independence

X and Y are independent events: \( P(X,Y) = P(X)P(Y) \)
Equivalently: \( P(X|Y) = P(X) \) (where \( P(Y) \) is non-zero, otherwise \( P(X|Y) \) not defined)

Conditional independence: \( P(X,Y|Z) = P(X|Z)P(Y|Z) \)
Equivalently: \( P(X|Y,Z) = P(X|Z) \) (again, for \( P(Y,Z) \) non-zero)

Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (\( p1, p2 = 0.5 \))
Consider 3 events:
\( H1 = \) first coin is a head
\( H2 = \) second coin is a head
\( J = \) the two tosses have the same results
Independence

X and Y are independent events: $P(X,Y) = P(X)P(Y)$
Equivalently: $P(X|Y) = P(X)$ (where $P(Y)$ is non-zero, otherwise $P(X|Y)$ not defined)

Conditional independence: $P(X,Y|Z) = P(X|Z)P(Y|Z)$
Equivalently: $P(X|Y,Z) = P(X|Z)$ (again, for $P(Y,Z)$ non-zero)

Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses ($p_1, p_2 = 0.5$)
H1 & H2: independent coin tosses
$P(H1,H2) = P(H1|H2)P(H2) = 0.5x0.5 = P(H1)P(H2)$
Independence

X and Y are independent events: $P(X,Y) = P(X)P(Y)$
Equivalently: $P(X|Y) = P(X)$ (where $P(Y)$ is non-zero, otherwise $P(X|Y)$ not defined)

Conditional independence: $P(X,Y|Z) = P(X|Z)P(Y|Z)$
Equivalently: $P(X|Y,Z) = P(X|Z)$ (again, for $P(Y,Z)$ non-zero)

Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses ($p_1, p_2 = 0.5$)
H1 & H2: independent coin tosses
$P(H1,J) = P(J | H1)P(H1) =$
Given H1, what is the probability of J
(i.e second toss also being a head)
So: $P(J | H1) = 0.5$
Independence

X and Y are independent events: \( P(X,Y) = P(X)P(Y) \)
Equivalently: \( P(X|Y) = P(X) \) (where \( P(Y) \) is non-zero, otherwise \( P(X|Y) \) not defined)

Conditional independence: \( P(X,Y|Z) = P(X|Z)P(Y|Z) \)
Equivalently: \( P(X|Y,Z) = P(X|Z) \) (again, for \( P(Y,Z) \) non-zero)

Independence of several events:

**Remark**: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (\( p_1, p_2 = 0.5 \))
H1 & H2: independent coin tosses
\[ P(H1,J) = P(J | H1)P(H1) = 0.5 \times 0.5 = P(J)P(H1) \]
Given H1, what is the probability of J
(i.e second toss also being a head)
So: \( P(J | H1) = 0.5 \)
Independence

X and Y are independent events: \( P(X,Y) = P(X)P(Y) \)
Equivalently: \( P(X|Y) = P(X) \) (where \( P(Y) \) is non-zero, otherwise \( P(X|Y) \) not defined)

Conditional independence: \( P(X,Y|Z) = P(X|Z)P(Y|Z) \)
Equivalently: \( P(X|Y,Z) = P(X|Z) \) (again, for \( P(Y,Z) \) non-zero)

Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (\( p1, p2 = 0.5 \))
H1 & H2: independent coin tosses
\( P(H2,J) = P(J | H2)P(H2) = 0.5 \times 0.5 = P(J)P(H2) \)
So pair-wise independent. BUT ...
Independence

X and Y are independent events: $P(X,Y) = P(X)P(Y)$
Equivalently: $P(X|Y) = P(X)$ (where $P(Y)$ is non-zero, otherwise $P(X|Y)$ not defined)

Conditional independence: $P(X,Y|Z) = P(X|Z)P(Y|Z)$
Equivalently: $P(X|Y,Z) = P(X|Z)$ (again, for $P(Y,Z)$ non-zero)

Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses ($p_1, p_2 = 0.5$)
H1 & H2: independent coin tosses
$P(H1,H2,J) = P(H1 \mid H2,J) P(H2,J) = 1 \times 0.25 = 0.25$
Independence

X and Y are independent events: \( P(X,Y) = P(X)P(Y) \)
Equivalently: \( P(X|Y) = P(X) \) (where \( P(Y) \) is non-zero, otherwise \( P(X|Y) \) not defined)

Conditional independence: \( P(X,Y|Z) = P(X|Z)P(Y|Z) \)
Equivalently: \( P(X|Y,Z) = P(X|Z) \) (again, for \( P(Y,Z) \) non-zero)

Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (p1, p2 = 0.5)
H1 & H2: independent coin tosses
\( P(H1,H2,J) = P(H1 | H2,J) P(H2,J) = 1 \times 0.25 = 0.25 \)
However, \( P(H1)P(H2)P(J)=0.5\times0.5\times0.5=0.125 \neq \)
i.e. not jointly independent
Expected values

The probability distribution of a random variable $X$ provides us with probabilities of all possible values of $X$.

Summarise information, with some loss of information, represented by:

The expected value or mean:

$$\mathbb{E}[X] = \sum_{x} x \cdot P(X = x)$$

For a dice: $(1 \cdot 1/6) + (2 \cdot 1/6) + (3 \cdot 1/6) + (4 \cdot 1/6) + (5 \cdot 1/6) + (6 \cdot 1/6) = 3.5$
Expected values

The probability distribution of a random variable $X$ provides us with probabilities of all possible values of $X$.

Summarise information, with some loss of information, represented by:

The **expected value** or **mean**:

$$
\mathbb{E}[X] = \sum_{x} x \ P(X = x)
$$

For a dice: $(1 \times 1/6) + (2 \times 1/6) + (3 \times 1/6) + (4 \times 1/6) + (5 \times 1/6) + (6 \times 1/6) = 3.5$

The expected value of any function of $X$, e.g. $g(x)$:

$$
\mathbb{E}[g(X)] = \sum_{x} g(x) \ P(X = x)
$$

Dice: $(1 \times 1/6) + (4 \times 1/6) + (9 \times 1/6) + (16 \times 1/6) + (25 \times 1/6) + (36 \times 1/6) = 15.17$
Expected values

The probability distribution of a random variable $X$ provides us with probabilities of all possible values of $X$.

Summarise information, with some loss of information, represented by:

The **expected value** or **mean**:

$$E[X] = \int x \ P(x) \, dx$$

for a continuous variable $X$. 
Variance

The **variance** of a random variable $X$, denoted $\text{Var}(X)$ or $\sigma_X^2$:

$$\text{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

and can be calculated as

$$\text{var}(X) = \sum_x (X - \mathbb{E}[X])^2 p_X(x)$$

(Integral of continuous variables), and measure how “spread out” the values of $X$ in a data set are relative to their mean.

The **standard deviation** $\sigma_X$, (has the same units as $X$).

For a normal distribution, $\sim 2/3$ of the population values of $X$ fall within one $\sigma_X$, 95% fall between $2 \sigma_X$, etc.
Covariance

The degree to which two random variables $X$ and $Y$ co-vary (degree associated):

$$\sigma_{XY} = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

and measures a specific way $X$ and $Y$ co-vary, i.e., linearly. When normalised, it yields the correlation coefficient (Pearson correlation):

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

a dimensionless quantity between -1 and 1.
Covariance

The degree to which two random variables $X$ and $Y$ co-vary (degree associated): 

$$\sigma_{XY} = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

and measures a specific way $X$ and $Y$ co-vary, i.e., linearly. When normalised, it yields the correlation coefficient (Pearson correlation):

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

a dimensionless quantity between -1 and 1.

When $X$ and $Y$ are independent, then $\rho_{XY} = 0$.

The reverse is not true!

(e.g. $\rho_{XY}$ may be zero, but not linear-correlation, hence dependence exists. This requires more complex methods of demonstrating if $P(Y|X) = P(Y)$.)
Anscombe’s Quartet

Group of 4 datasets with nearly identical simple descriptive statistical properties:
- Mean and sample variance of X
- Mean and sample variance of Y
- Correlation between X and Y
- Linear regression line (coefficient the same up to 2 or 3 decimal places)
- $R^2$ coefficient

A note on $R^2$: A measure for goodness-of-fit

$$R^2 = 1 - \frac{\sum_i (y_i - f_i)^2}{\sum_i (y_i - \bar{y})^2} \quad , \quad y_i = f(x_i) \quad , \quad \bar{y} = \frac{1}{n} \sum_i y_i$$

If the fit $y=f(x)$ is a perfect fit, the numerator is zero, $R^2 = 1$, and $R^2 = 0$ implies the fit $f(x)$ is no better than baseline average $\bar{y}$. Negative values corresponds to models worse than the baseline average.
Anscombe’s Quartet

Group of 4 datasets with nearly identical simple descriptive statistical properties:
- Mean and sample variance of X
- Mean and sample variance of Y
- Correlation between X and Y
- Linear regression line (coefficient the same up to 2 or 3 decimal places)
- $R^2$ coefficient

Yet, very different distributions, which can be observed by plotting the graphs

Same Pearson correlation, but, different dependence structure
(X causes Y, but in different ways)

Figure from WikiPedia
Next time

First of the two causal frameworks:
- Potential Outcomes (due to Neyman-Rubin)
- Study our first causal question

Next, we estimate:
- Answer to causal question
- Uncertainty on this answer (under model assumptions)
Methods for Causal Inference
Lecture 2: Basics of probability

Ava Khamseh
School of Informatics
2023-2024