Methods for Causal Inference
Lecture 6: Instrumental variable method

Ava Khamseh
School of Informatics
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So far ...

Causal inference with observed confounders
Overview of the course

- **Lecture 1**: Introduction & Motivation, why do we care about causality? Why deriving causality from observational data is non-trivial.
- **Lecture 2**: Recap of probability theory, variables, events, conditional probabilities, independence, law of total probability, Bayes’ rule
- **Lecture 3**: Recap of regression, multiple regression, graphs, SCM
- **Lecture 4-20:**

![Diagram of causality concepts]

- Causal Effect Estimation
  - Obsv confounders
    - Regression Adjustment
    - Propensity score
  - Unobsv confounders
    - IV
    - Front-door criterion
- Casual Discovery
  - Constraint-based
  - Score-based
  - FCMs
Randomised Controlled Trials (RCTs)

Randomised Control Trials (RCT): Subjects are assigned at random to various groups (treatment or control)

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- Unethical: Asking pregnant women to smoke to observe child birth weight Denying the control subjects a drug, e.g. treatment could have been potentially life saving for cancer patients
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- Randomisation may influence participation and behaviour
Randomising an instrument

Causal inference from studies in which subject have a final choice

Randomisation is confined to an indirect instrument that encourages or discourage participation in treatment or control programmes.

(However, imperfect compliance poses a problem, e.g., subjects that declined taking the drug are precisely those who would have responded adversely. So experiment might conclude the drug is more effective than it actually is. -> more complex methods, e.g. bounds)
Instrumental Variable

Unobserved confounders (U), **violates unconfoundedness**, i.e. conditioning on X alone, would not result in a randomised treatment assignment.

Unconfoundedness is fundamentally unverifiable.

Rubin 1996
Naive regression leads to bias

\[
Y = \tau T + \delta_U U \\
T = \gamma_U U
\]
Naive regression leads to bias

What happens if we naively perform a linear regression of $Y$ on $T$:

$$Y = \tau T + \delta_U U$$

$$T = \gamma_U U$$

$$\frac{\text{Cov}[T, Y]}{\text{Var}[T]} = \frac{\tau \text{Var}[T] + \gamma_U \delta_U \text{Var}[U]}{\text{Var}[T]} = \tau + \frac{\gamma_U \delta_U \text{Var}[U]}{\text{Var}[T]} = \tau + \frac{\delta_U}{\gamma_U}$$

- causal term
- Bias term
Instrumental Variable example

- **Example 1:**
  - T: smoking during pregnancy
  - Y: birthweight
  - X: parity, mother’s age, weight, ...
  - U: Other unmeasured confounders

- Randomise Z (intention-to-treat): either receive encouragement to stop smoking (Z=1), or receive usual care (Z=0)
- Intention-to-treat analysis gives causal effect estimator of encouragement z on outcome y:
  \[
  \mathbb{E}(y|z = 1) - \mathbb{E}(y|z = 0)
  \]

- What can we say about the causal effect of smoking itself?
Instrumental Variable assumptions

- **SUTVA:** Potential outcomes for each individual $i$ are unrelated to the treatment status of other individuals:
  \[
  Y^{(i)}(Z, T) = Y^{(i)}(Z^{(i)}, T^{(i)}) \quad |Z| = |T| = N \text{ individuals}
  \]

- Non-zero average/relevant: Treatment assignment $Z$ associated with the treatment
  \[
  \mathbb{E} \left[ \left( T^{(i)} | z = 1 \right) - \left( T^{(i)} | z = 0 \right) \right]
  \]

- Treatment assignment $Z$ is random ($Z$ and $Y$ do not share a cause).
Instrumental Variable assumptions

- **SUTVA**: Potential outcomes for each individual \( i \) are unrelated to the treatment status of other individuals:

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Y^{(i)}(Z, T) = Y^{(i)}(Z^{(i)}, T^{(i)}) , \quad |Z| = |T| = N \text{ individuals}
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- Non-zero average/relevant: Treatment assignment \( Z \) associated with the treatment

\[
\mathbb{E}\left[\left(T^{(i)}|z = 1\right) - \left(T^{(i)}|z = 0\right)\right]
\]

- Treatment assignment \( Z \) is random (\( Z \) and \( Y \) do not share a cause).

\[
\left(Y^{(i)}|z = 1, t\right) = \left(Y^{(i)}|z = 0, t\right)
\]

- **Exclusion Restriction**: Any effect of \( Z \) on \( Y \) is via an effect of \( Z \) on \( T \), i.e., \( Z \) should not affect \( Y \) when \( T \) is held constant

- **Monotonicity** (increasing encouragement “dose” increases probability of treatment, no defiers):

\[
\left(T^{(i)}|z = 1\right) \geq \left(T^{(i)}|z = 0\right)
\]

Rubin 1996
### Instrumental Variable: Potential values of T

| Population     | T|z=0 | T|z=1 | Description                                      |
|----------------|-----|-----|-----|-------------------------------------------------|
| Never-takers   | 0   | 0   |     | Causal effect of Z on T is zero, since          |
|                |     |     |     | \((T^{(i)}|z = 1) - (T^{(i)}|z = 0) = 0\)      |
| Compliers      | 0   | 1   |     | Causal effect inference: \((Y^{(i)}|T^{(i)} = 1) - (Y^{(i)}|T^{(i)} = 0)\) |
|                |     |     |     | Rule out by **monotonicity**, since             |
|                |     |     |     | \((T^{(i)}|z = 1) - (T^{(i)}|z = 0) = -1\)       |
| Defiers        | 1   | 0   |     | Causal effect of Z on Y is zero, since          |
|                |     |     |     | \((T^{(i)}|z = 1) - (T^{(i)}|z = 0) = 0\)         |

**Notation:** $T=1$ is **not** smoking

(Rubin 1996)
Instrumental Variable: The estimand

Want ATE:

\[ \mathbb{E}[Y_{T=1} - Y_{T=0}] \]

“Almost”

Will estimate:

\[ \tau = \frac{\mathbb{E}[(Y|z=1) - (Y|z=0)]}{\mathbb{E}[(T|z=1) - (T|z=0)]} \]

Rubin 1996
Instrumental Variable: The estimand

Want ATE: \[ \mathbb{E} \left[ \left( Y(i) \middle| t(i) = 1 \right) - \left( Y(i) \middle| t(i) = 0 \right) \right] \]

\[ \tau = \frac{\mathbb{E} [(Y \middle| z = 1) - (Y \middle| z = 0)]}{\mathbb{E} [(T \middle| z = 1) - (T \middle| z = 0)]} \]

Derivation:

\[ \left( Y(i) \middle| T(i) (z = 1) \right) - \left( Y(i) \middle| T(i) (z = 0) \right) = \left( Y(i) \left( t(i) = 1 \right) \cdot (t(i) \middle| z = 1) + Y(i) \left( t(i) = 0 \right) \cdot (1 - (t(i) \middle| z = 1)) \right) \]

\[ - \left[ Y(i) \left( t(i) = 1 \right) \cdot (t(i) \middle| z = 0) + Y(i) \left( t(i) = 0 \right) \cdot (1 - (t(i) \middle| z = 0)) \right] \]

\[ = \left( Y(i) \left( t(i) = 1 \right) - Y(i) \left( t(i) = 0 \right) \right) \cdot \left( (t(i) \middle| z = 1) - (t(i) \middle| z = 0) \right) \]

Hence, the causal effect of Z on Y for individual i, is the product of the causal effect of Z on T, and, the casual effect of T on Y.
**Instrumental Variable: The estimand**

**Want ATE:** \[ \mathbb{E} \left[ \left( Y^{(i)} | t^{(i)} = 1 \right) - \left( Y^{(i)} | t^{(i)} = 0 \right) \right] \]

**Derivation:**
\[
\tau = \frac{\mathbb{E} \left[ (Y | z = 1) - (Y | z = 0) \right]}{\mathbb{E} \left[ (T | z = 1) - (T | z = 0) \right]}
\]

\[
\begin{align*}
&\left( Y^{(i)} | T^{(i)}(z = 1) \right) - \left( Y^{(i)} | T^{(i)}(z = 0) \right) \\
&= \left[ Y^{(i)} \left( t^{(i)} = 1 \right) \cdot \left( t^{(i)} | z = 1 \right) + Y^{(i)} \left( t^{(i)} = 0 \right) \cdot \left( 1 - \left( t^{(i)} | z = 1 \right) \right) \right] \\
&\quad - \left[ Y^{(i)} \left( t^{(i)} = 1 \right) \cdot \left( t^{(i)} | z = 0 \right) + Y^{(i)} \left( t^{(i)} = 0 \right) \cdot \left( 1 - \left( t^{(i)} | z = 0 \right) \right) \right] \\
&= \left( Y^{(i)} \left( t^{(i)} = 1 \right) - Y^{(i)} \left( t^{(i)} = 0 \right) \right) \cdot \left( \left( t^{(i)} | z = 1 \right) - \left( t^{(i)} | z = 0 \right) \right)
\end{align*}
\]

Hence, the causal effect of \( Z \) on \( Y \) for individual \( i \), is the product of the causal effect of \( Z \) on \( T \), and, the causal effect of \( T \) on \( Y \).
Instrumental Variable: The estimand

Want ATE: \[ \mathbb{E} \left[ \left(Y^{(i)}|t^{(i)} = 1\right) - \left(Y^{(i)}|t^{(i)} = 0\right) \right] \]

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Derivation:

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\[ = \left[ Y^{(i)} \left( t^{(i)} = 1 \right) \cdot \left( t^{(i)}|z = 1 \right) + Y^{(i)} \left( t^{(i)} = 0 \right) \cdot \left( 1 - \left( t^{(i)}|z = 1 \right) \right) \right] \]

\[ - \left[ Y^{(i)} \left( t^{(i)} = 1 \right) \cdot \left( t^{(i)}|z = 0 \right) + Y^{(i)} \left( t^{(i)} = 0 \right) \cdot \left( 1 - \left( t^{(i)}|z = 0 \right) \right) \right] \]

\[ = \left( Y^{(i)} \left( t^{(i)} = 1 \right) - Y^{(i)} \left( t^{(i)} = 0 \right) \right) \cdot \left( \left( t^{(i)}|z = 1 \right) - \left( t^{(i)}|z = 0 \right) \right) \]

Hence, the causal effect of Z on Y for individual i, is the product of the causal effect of Z on T, and, the causal effect of T on Y.

Rubin 1996
Instrumental Variable: The estimand

To continue the derivation, we use the fact that:

\[
E[XY] = \int \int xy \, p(x, y) \, dx \, dy = \int dy \, y \, p(y) \int dx \, x \, p(x|y) = \int dy \, y \, p(y)E[x|y]
\]

and write,

\[
E \left[ \left( Y^{(i)} | T^{(i)}(z = 1) \right) - \left( Y^{(i)} | T^{(i)}(z = 0) \right) \right] = E \left[ \left( Y^{(i)} \left( t^{(i)} = 1 \right) - Y^{(i)} \left( t^{(i)} = 0 \right) \right) \cdot \left( \left( t^{(i)} \right| z = 1 \right) - \left( t^{(i)} \right| z = 0 \right) \right]
\]
Instrumental Variable: The estimand

To continue the derivation, we use the fact that:

\[ E[XY] = \int \int x y p(x, y) dx dy = \int dy y p(y) \int dx x p(x|y) = \int dy y p(y) E[x|y] \]

and write,

\[ E \left[ \left( Y^{(i)}|T^{(i)}(z = 1) \right) - \left( Y^{(i)}|T^{(i)}(z = 0) \right) \right] \]

\[ = E \left[ \left( Y^{(i)} \left( t^{(i)} = 1 \right) - Y^{(i)} \left( t^{(i)} = 0 \right) \right) \cdot \left( \left( t^{(i)}|z = 1 \right) - \left( t^{(i)}|z = 0 \right) \right) \right] \]

\[ = E \left[ \left( Y^{(i)} \left( t^{(i)} = 1 \right) - Y^{(i)} \left( t^{(i)} = 0 \right) \right) \right] \cdot \left( \left( t^{(i)}|z = 1 \right) - \left( t^{(i)}|z = 0 \right) \right) = 1 \cdot \]

\[ P \left( \left( t^{(i)}|z = 1 \right) - \left( t^{(i)}|z = 0 \right) = 1 \right) \]

\[ -E \left[ \left( Y^{(i)} \left( t^{(i)} = 1 \right) - Y^{(i)} \left( t^{(i)} = 0 \right) \right) \right] \cdot \left( \left( t^{(i)}|z = 1 \right) - \left( t^{(i)}|z = 0 \right) \right) = -1 \cdot \]

\[ P \left( \left( t^{(i)}|z = 1 \right) - \left( t^{(i)}|z = 0 \right) = -1 \right) \]

0, by monotonicity
Instrumental Variable: The estimand

\[
\frac{\mathbb{E} \left[ (Y^{(i)}|T^{(i)}(z = 1)) - (Y^{(i)}|T^{(i)}(z = 0)) \right]}{\mathbb{E} \left[ (t^{(i)}|z = 1) - (t^{(i)}|z = 0) \right]} = \mathbb{E} \left[ \left( Y^{(i)}(t^{(i)} = 1) - Y^{(i)}(t^{(i)} = 0) \right) \mid \left( (t^{(i)}|z = 1) - (t^{(i)}|z = 0) \right) \right] = 1
\]

i.e. restricting to **compliers**, the average causal effect of Z on Y is proportional to the average causal effect of T on Y.

\[
\tau = \frac{\mathbb{E} \left[ (Y|z = 1) - (Y|z = 0) \right]}{\mathbb{E} \left[ (T|z = 1) - (T|z = 0) \right]}
\]

- In this example, Z was randomly assigned as part of the study
- IV can also be randomised in nature (nature randomiser):
  - Mendelian randomisation
Instrumental Variable: Mendelian Randomisation

Population genetics:
Z = a DNA variant associated with a particular exposure T
T = exposure, e.g. lipid levels in the blood
Y = heart disease
X = population stratification (might affect Z, need to adjust)
U = unobserved variables affecting both lipid levels and disease

Conditional instrument
Instrumental Variable: Economics

How does price of a product casually affect demand?

\[ Z = \text{Market supply} \]
\[ T = \text{Price} \]
\[ Y = \text{Demand} \]
\[ U = \text{Factors confounding influencing price and demand} \]
\[ (\text{e.g. tax imposed}) \]

Exclusion restriction requires that market supply does not affect demand
\[ (\text{e.g. COVID-19 toilet paper fiasco!}) \]
\[ (\text{e.g. Pokemon cards}) \]
Also, individuals may not be independent anymore
The Wald Estimator (for binary variables)

\[
\tau = \frac{\mathbb{E}[(Y|z = 1) - (Y|z = 0)]}{\mathbb{E}[(T|z = 1) - (T|z = 0)]}
\]

\[
\hat{\tau} = \frac{1}{n_{z=1}} \sum_{i \in z=1} Y(i) - \frac{1}{n_{z=0}} \sum_{i \in z=0} Y(i) \\
\frac{1}{n_{z=1}} \sum_{i \in z=1} T(i) - \frac{1}{n_{z=0}} \sum_{i \in z=0} T(i)
\]
IV Estimator: continuous variables case

Linear case:

\[ \tau = \frac{\text{Cov}(Y, Z)}{\text{Cov}(T, Z)} \]

\[ \hat{\tau} = \frac{\hat{\text{Cov}}(Y, Z)}{\hat{\text{Cov}}(T, Z)} \]

Two-Stage Least-squares Estimator
IV Estimator: continuous variables case

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Two-Stage Least-squares Estimator
IV Estimator: continuous variables case

\[ \text{Cov}(Y, Z) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \]

\[ \hat{\tau} = \frac{\hat{\text{Cov}}(Y, Z)}{\hat{\text{Cov}}(T, Z)} \]

\[ Y = \tau T + \delta U U \]
IV Estimator: continuous variables case

\[ \text{Cov}(Y, Z) = \mathbb{E}[YZ] - \mathbb{E}[Y] \mathbb{E}[Z] \]
\[ = \mathbb{E}(\tau T + \delta_u U) Z] - \mathbb{E}[\tau T + \delta_u U] \mathbb{E}[Z] \]

By linearity and exclusion restriction

\[ \hat{\tau} = \frac{\text{Cov}(Y, Z)}{\text{Cov}(T, Z)} \]

\[ Y = \tau T + \delta_U U \]
IV Estimator: continuous variables case

\[ \text{Cov}(Y, Z) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \]

\[ = \mathbb{E}(\tau T + \delta_u U)Z] - \mathbb{E}[\tau T + \delta_u U]\mathbb{E}[Z] \]

\[ = \tau \mathbb{E}[TZ] + \delta_u \mathbb{E}[UZ] - \tau \mathbb{E}[T]\mathbb{E}[Z] - \delta_u \mathbb{E}[U]\mathbb{E}[Z] \]

\[ \hat{\tau} = \frac{\hat{\text{Cov}}(Y, Z)}{\hat{\text{Cov}}(T, Z)} \]

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IV Estimator: continuous variables case

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\[ = \mathbb{E}(\tau T + \delta_u U)Z - \mathbb{E}[\tau T + \delta_u U]\mathbb{E}[Z] \]

\[ = \tau \mathbb{E}[TZ] + \delta_u \mathbb{E}[UZ] - \tau \mathbb{E}[T]\mathbb{E}[Z] - \delta_u \mathbb{E}[U]\mathbb{E}[Z] \]

\[ = \tau \text{Cov}(T, Z) + \delta_U \text{Cov}(U, Z) \]

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**IV Estimator: continuous variables case**

\[ \text{Cov}(Y, Z) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \]

\[ = \mathbb{E}(\tau T + \delta_u U)Z] - \mathbb{E}[\tau T + \delta_u U]\mathbb{E}[Z] \]

\[ = \tau \mathbb{E}[TZ] + \delta_u \mathbb{E}[UZ] - \tau \mathbb{E}[T]\mathbb{E}[Z] - \delta_u \mathbb{E}[U]\mathbb{E}[Z] \]

\[ = \tau \text{Cov}(T, Z) + \delta_u \text{Cov}(U, Z) \quad \text{Instrument is not} \]

\[ = \tau \text{Cov}(T, Z) \quad \text{confounded by } U \]

\[ \hat{\tau} = \frac{\hat{\text{Cov}}(Y, Z)}{\hat{\text{Cov}}(T, Z)} \]

\[ Y = \tau T + \delta_U U \]
IV Estimator: continuous variables case

Two-Stage Least Squares Estimator (linear regression):

1. Estimate $\mathbb{E}[T|Z]$, to obtain $\hat{T}$ in subspace

2. Estimate $\mathbb{E}[Y|\hat{T}]$, to obtain $\hat{\tau}$, which is the fitted coefficient in front of $\hat{T}$ in this regression.
IV Estimator: continuous variables case

Two-Stage Least Squares Estimator (linear regression):

1. Estimate $\mathbb{E}[T|Z]$, to obtain $\hat{T}$ in subspace

2. Estimate $\mathbb{E}[Y|\hat{T}]$, to obtain $\hat{\tau}$, which is the fitted coefficient in front of $\hat{T}$ in this regression.
Other remarks

Double-blind studies:
To ensure exclusion restriction, investigators withhold knowledge of the assigned treatment $Z$ from participants and doctors

Example: Those randomly assigned $z=1$, receive aspirin, but those assigned $z=0$ receive placebo, do not. The pills look identical. Neither doctor nor patient knows which is which, “double-blind placebo-controlled” randomised experiment.

Often no feasible, e.g. heart surgery, has no convincing placebo!
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Causality

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