THE UNIVERSITY *of* EDINBURGH

Methods for Causal Inference
Lecture 13: Do-Calculus

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School of Informatics

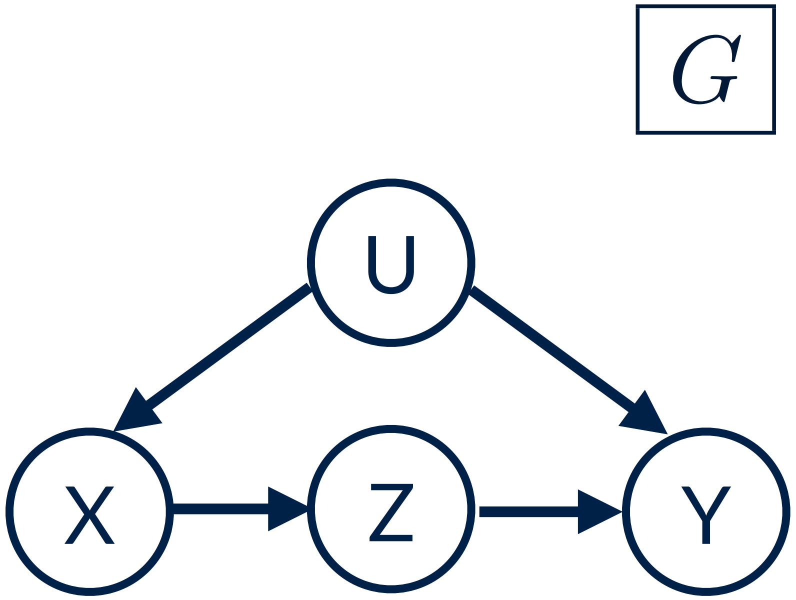
2025-2026

Do-Calculus

Not all causal quantities are identifiable
(this depends on the structure of the graph)

Here, we generalise the rules of front/back-door criteria: **do-calculus**

Let X, Y, Z be arbitrary disjoint sets of nodes in a DAG G.



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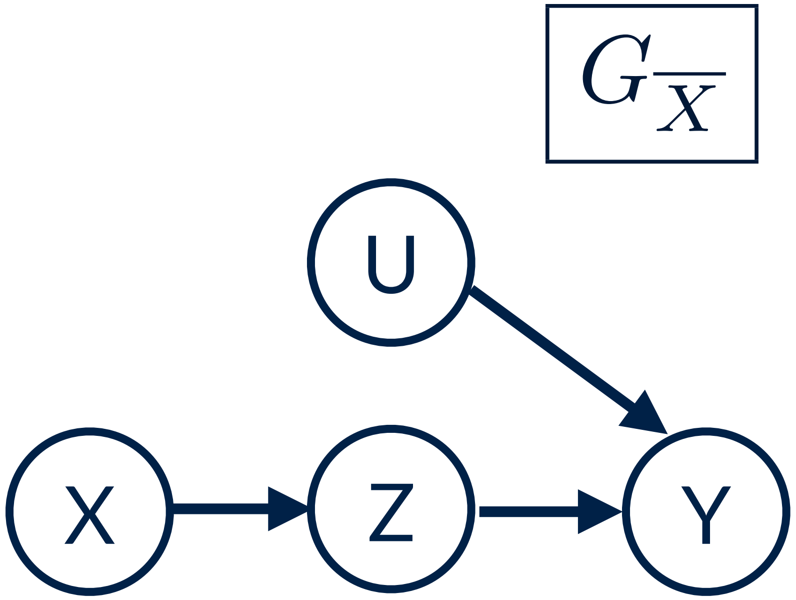
Notation

$G_{\overline{X}}$: The graph obtained by deleting all arrows pointing to nodes in X



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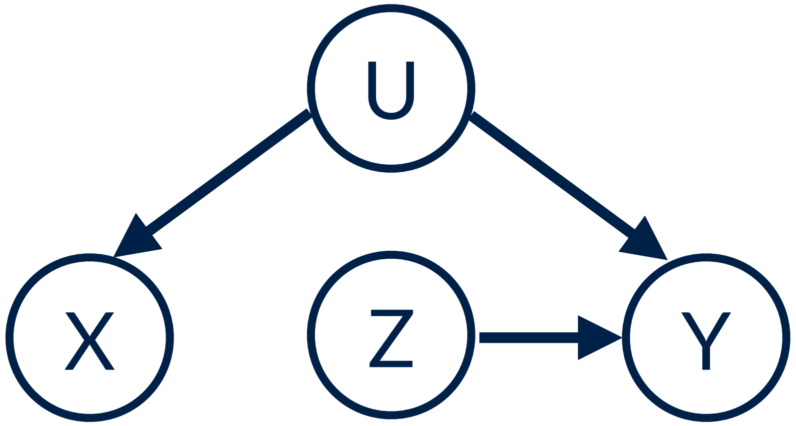
Note for example: $G_{\underline{X}} = G_{\overline{Z}}$

$\boxed{G_{\underline{X}}}$



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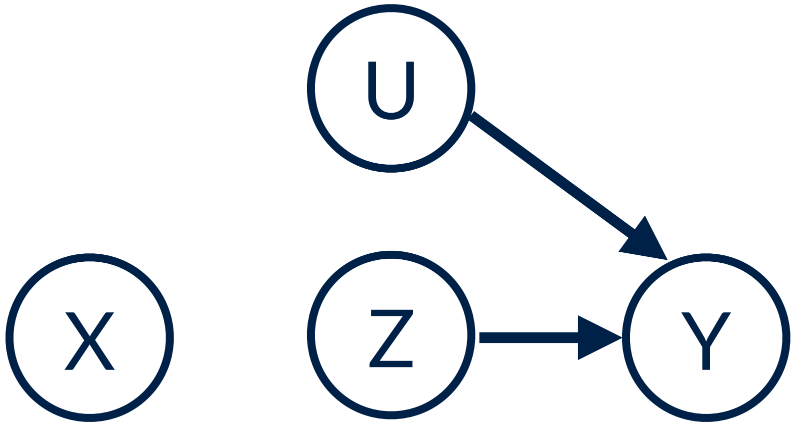
More examples: $G_{\overline{XZ}}$



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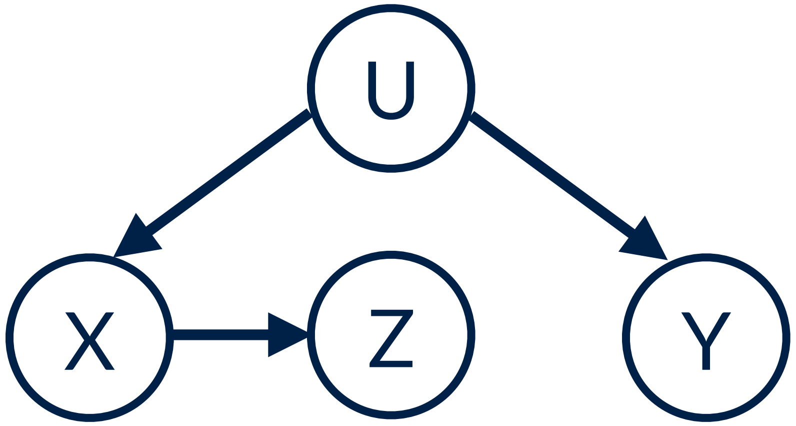
More examples: $G_{\underline{Z}}$



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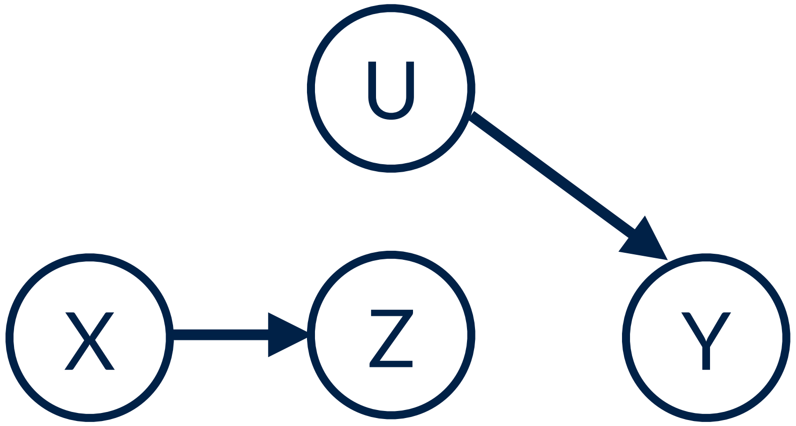
More examples: $G_{\overline{X}\underline{Z}}$



Do-Calculus Rules

Let X, Y, Z, W be arbitrary disjoint sets of nodes in a DAG G

Rule 1 (insertion/deletion of observations):

$$p(Y|do(X=x),Z,W) = p(Y|do(X=x),W) \text{ if } (Y \perp\!\!\!\perp Z)|X,W \text{ in } G_{\overline{X}}$$

i.e. if Y and Z are d-separated by X, W in a graph where incoming edges in X have been removed.

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In the special case where $X = \emptyset$ the above states:

$$p(Y|Z, W) = p(Y|W) \text{ if } (Y \perp\!\!\!\perp Z)|W$$

Which is simply d-separation. So the above is the **generalisation of d-separation in the presence of an intervention do(X=x)**

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Rule 2 (Action/observation exchange):

$$p(Y|do(X=x), do(Z=z), W) = p(Y|do(X=x), z, W) \text{ if } (Y \perp\!\!\!\perp Z)|X, W \text{ in } G_{\overline{X}\underline{Z}}$$

i.e. if Y and Z are d-separated by X, W in a graph where incoming edges in X and outgoing edges from Z have been removed.

This rules provides a condition for an external intervention do(Z=z) to have the same effect on Y as the passive observation Z=z.

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$$p(Y|do(Z=z),W) = p(Y|z,W) \text{ if } (Y \perp\!\!\!\perp Z)|W \text{ in } G_{\underline{Z}}$$

Which is the generalisation of backdoor criterion (adjustment formula).

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where Z(W) is the set of Z-nodes that are not ancestors of any W-node in $G_{\overline{X}}$

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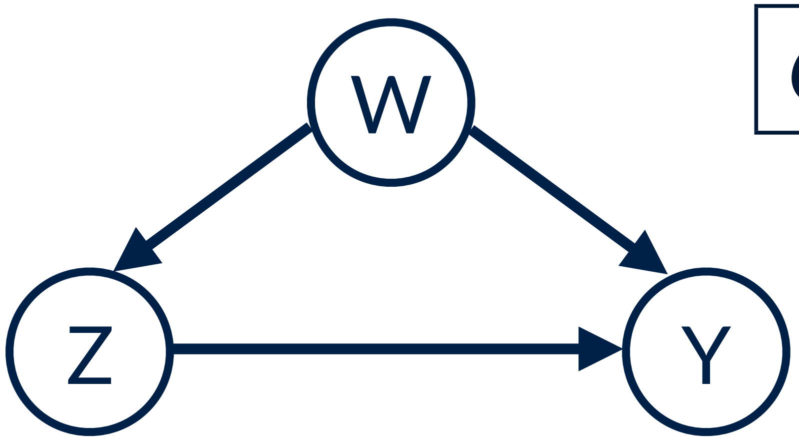
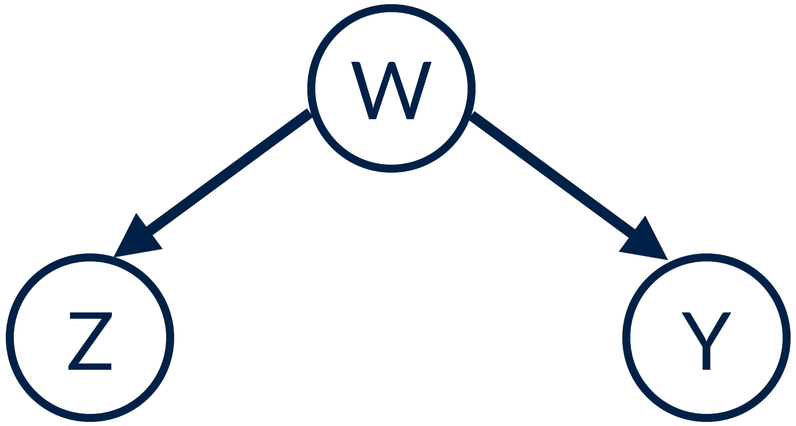
Provides conditions for introducing/deleting an external intervention without affecting the conditional probability of Y.

Proof in Pearl 1995a

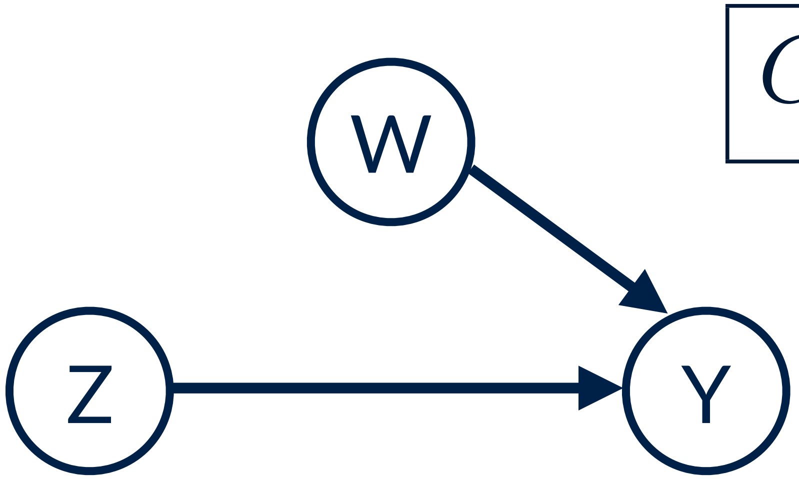
Graph examples



$\boxed{G}$



$\boxed{G_{\underline{Z}}}$



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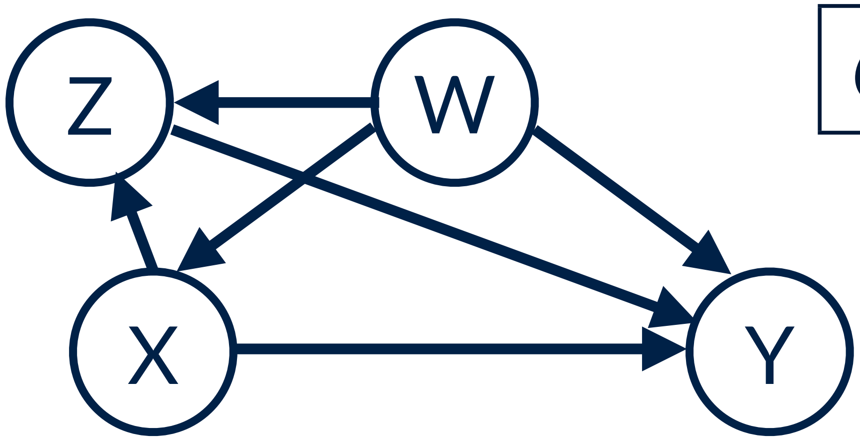
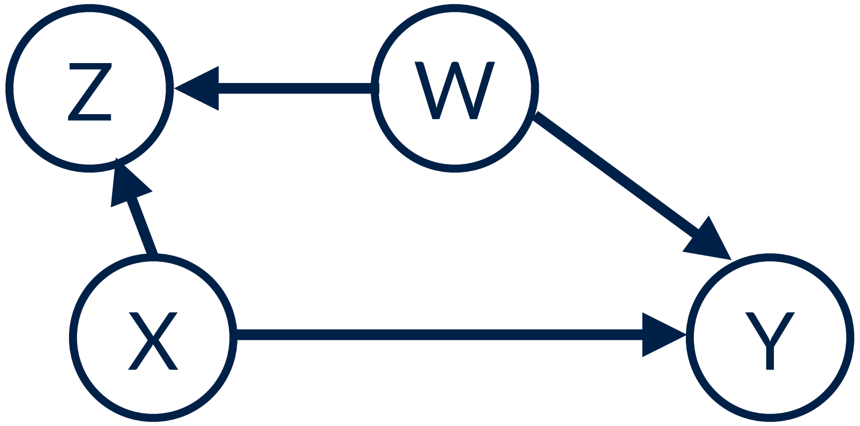
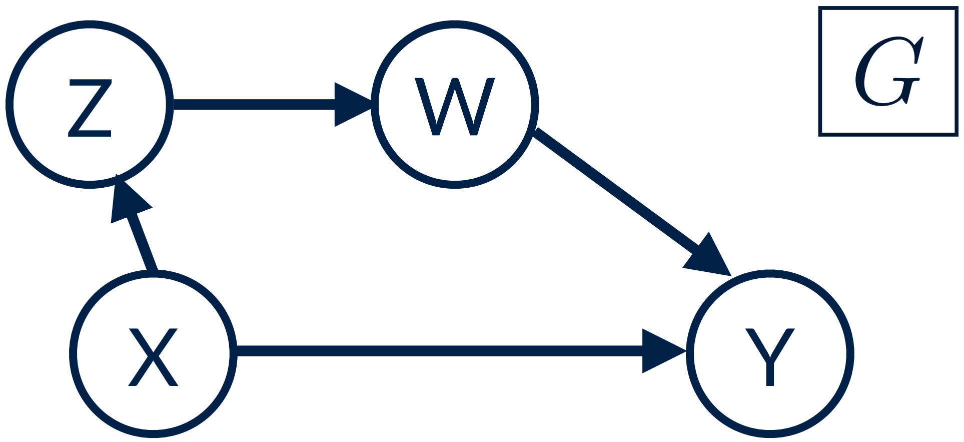


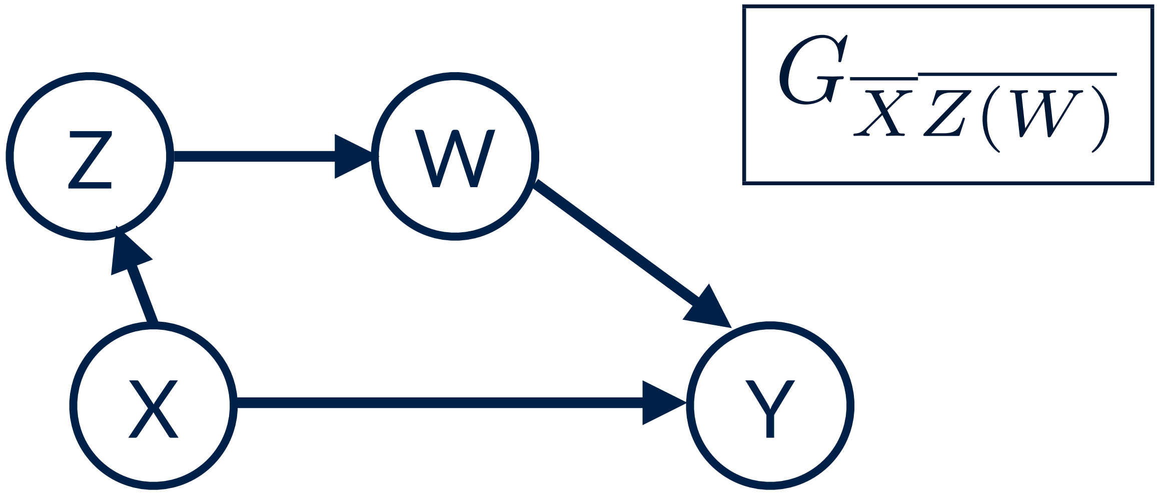
G



$G_{\overline{X}\underline{Z}}$

Graph examples





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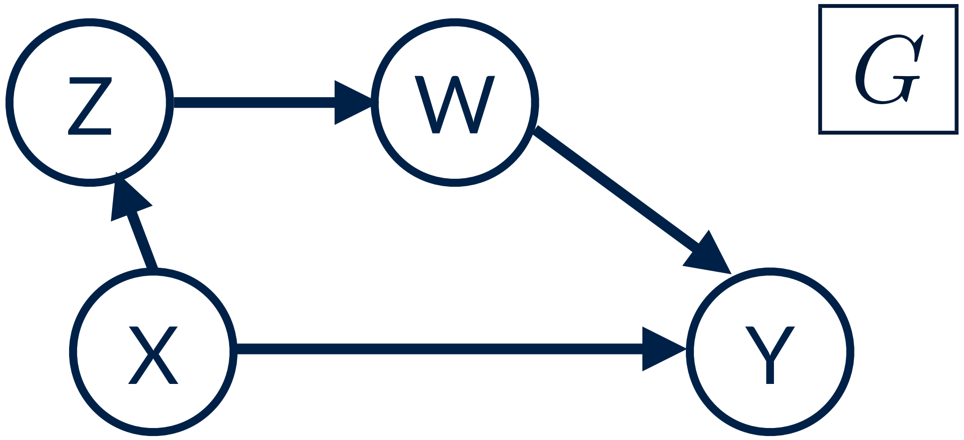
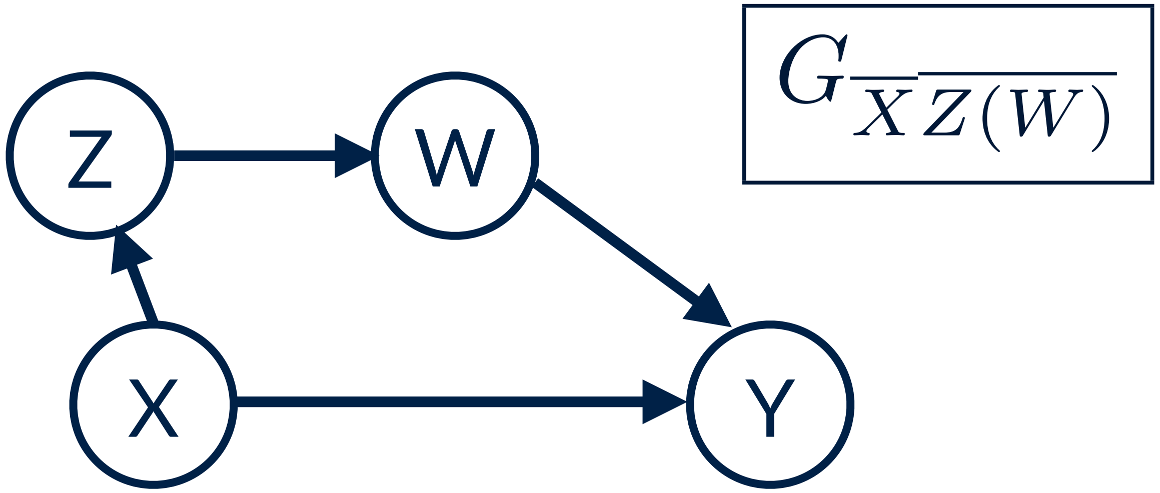
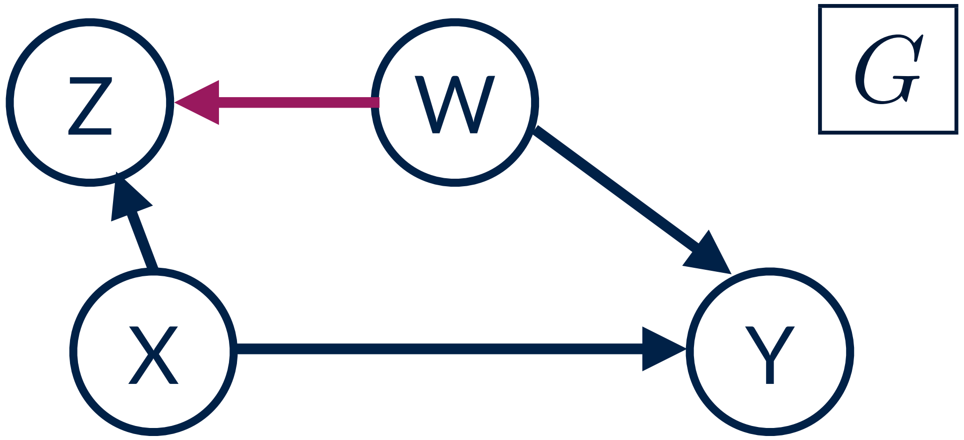
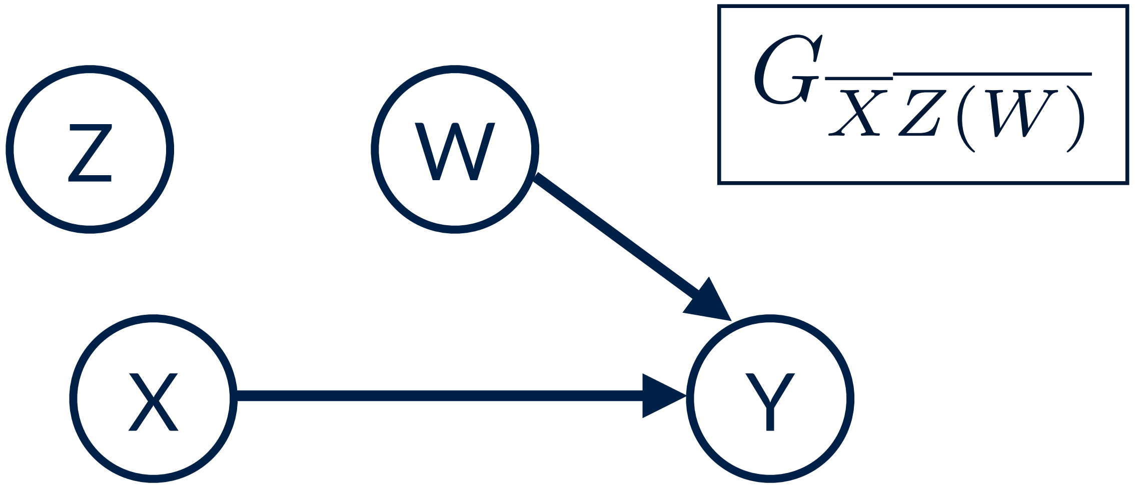








where Z(W) is the set of Z-nodes that are not ancestors of any W-node in $G_{\overline{X}}$

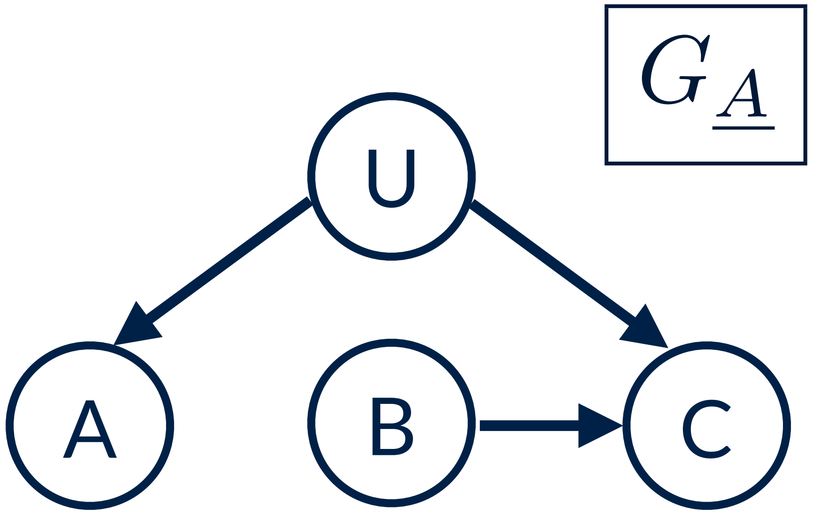
Derivation of front-door criterion using do-calculus

Task 1: Compute $p(B|do(A=a))$

We need to write this in a format without the 'do'. Rule 2 is useful here.
We use Rule 2, special case:

$$p(B|do(A=a)) = p(B|a) \text{ if } (B \perp\!\!\!\perp A) \text{ in } G_{\underline{A}}$$

And the condition is satisfied because the path $A \leftarrow U \rightarrow C \leftarrow B$ is blocked by C, so B and A are d-separated in this graph.

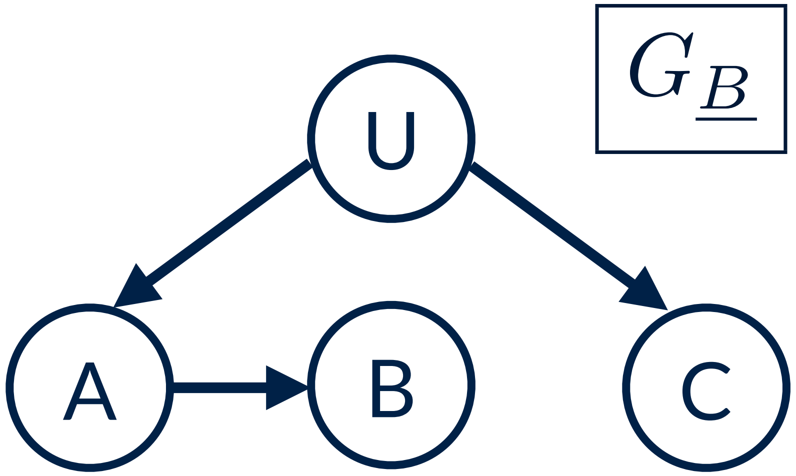


Pearl's 2009 book, page 87

Derivation of front-door criterion using do-calculus

Task 2: Compute $p(C|do(B = b))$

We cannot apply rule 2 to replace $do(B = b)$ with b because $G_{\underline{B}}$ contains a back-door path from B to C: $B \leftarrow A \leftarrow U \rightarrow C$



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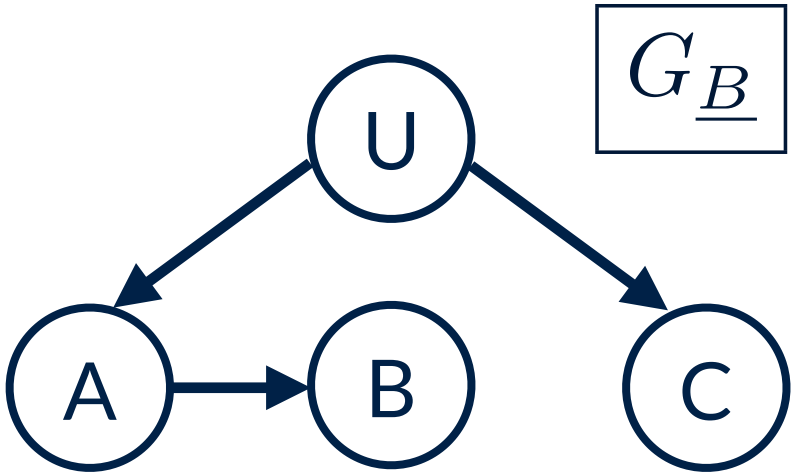
BUT, we can use block this path by measuring A. So marginalising gives:

$$p(C|do(B=b)) = \sum_A p(A,C|do(B=b)) = \sum_A p(C|A,do(B=b))p(A|do(B=b))$$



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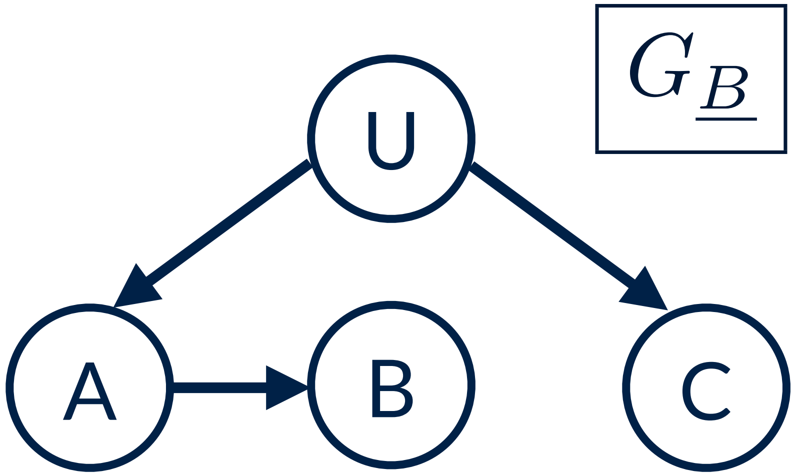
$$p(A|do(B=b)) = p(A) \qquad (A \perp\!\!\!\perp B) \text{ in } G_{\overline{B}}$$

Immediate via do-operation/graph manipulation (with B being a descendent of A in G), or, Rule 3: Due to d-separation of A and B (conditional on nothing) in graph $\boxed{G_{\overline{B}}}$



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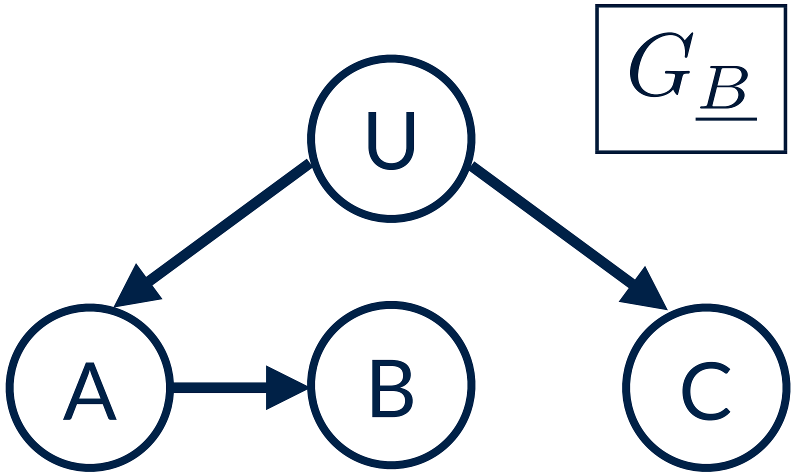
$$p(C|A,do(B=b)) = p(C|A,b) \quad (C \perp\!\!\!\perp B|A) \text{ in } G_{\underline{B}}$$

Which uses Rule 2, with C and B d-separated given A.

Therefore,

$$p(C|do(B=b)) = \sum_A p(C|A,b)p(A)$$



Pearl's 2009 book, page 87

Derivation of front-door criterion using do-calculus

Task 3: Compute $p(C|do(A=a))$. Marginalising over B gives:

$$p(C|do(A=a)) = \sum_B p(C|B, do(A=a)) \boxed{p(B|do(A=a))}$$

Second term already done. First term, no rule can be applied to eliminate do(A).

Pearl's 2009 book, page 87

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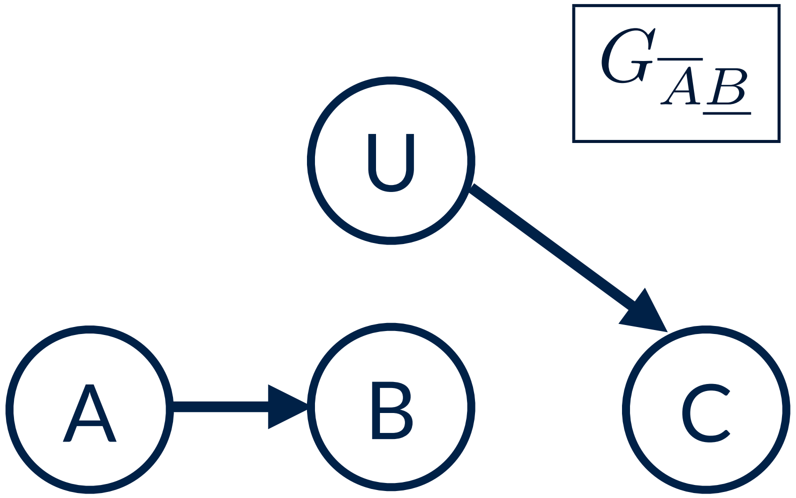
$$p(C|B, do(A=a)) = p(C|do(B=b), do(A=a))$$

since, $(C \perp\!\!\!\perp B|A)$ in $G_{\overline{A}\underline{B}}$



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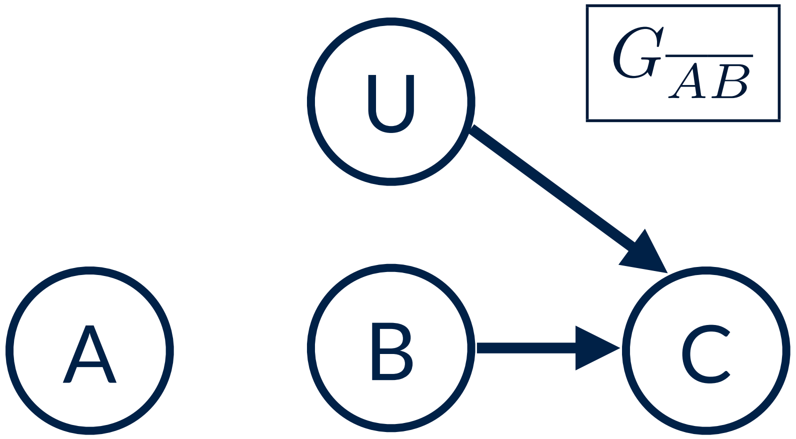
since, $(C \perp\!\!\!\perp B|A) \text{ in } G_{\overline{A}\underline{B}}$

Then, we use Rule 3, to delete do(A):

$$p(C|B, do(A=a)) = p(C|do(B=b))$$

since, $(C \perp\!\!\!\perp A|B) \text{ in } G_{\overline{AB}}$

which again, we have competed before.



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$$p(C|do(A=a)) = \sum_B \boxed{p(C|B, do(A=a))p(B|do(A=a))}$$

Putting all terms together:

$$p(C|do(A=a)) = \sum_B p(B|a) \sum_{A'} p(C|A', B)p(A')$$

Front-door criterion!

Pearl’s 2009 book, page 87

A statement about estimation

Recall: The Backdoor Criterion

Backdoor Criterion: Given an ordered pair of variables (T,Y) in a DAG G, a set of variables X satisfies the backdoor criterion relative to (T,Y) if:

(i) no node in X is a descendent of T

(ii) X block every path between T and Y that contains an arrow into T

If X satisfies the backdoor criterion then the causal effect of T on Y is given by:

$$p(Y = y|do(T = t)) = \sum_x p(Y = y|T = t, X = x)p(X = x)$$

In other words, condition on a set of nodes X such that:

(i) We block all spurious paths between T and Y

(ii) We leave all direct paths from T to Y unperturbed

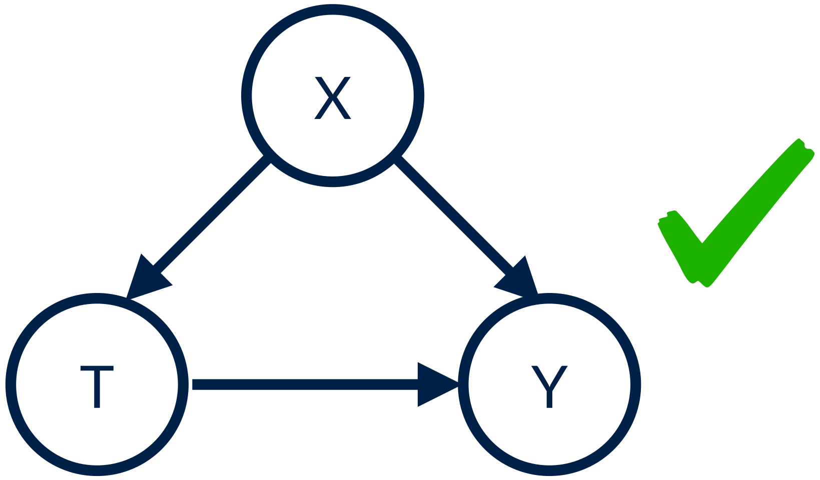
(iii) We create no new spurious paths (do not unblock any new paths)

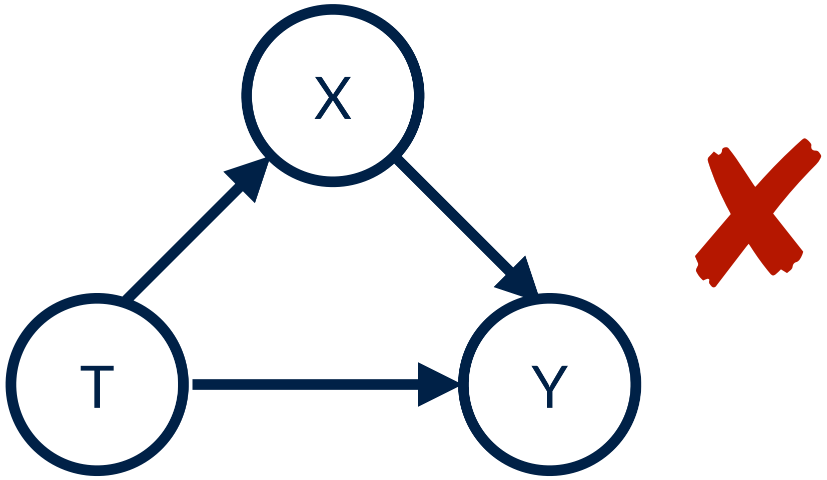
Any set X that satisfies the backdoor criterion (hence can be used in the adjustment formula) is called an **Adjustment Set**

Pearl, Causal Inference
in Statistics (2016)

Pearl: To adjust or not to adjust

Pearls algorithmic approach (*do*-calculus) tells us to adjust or not.





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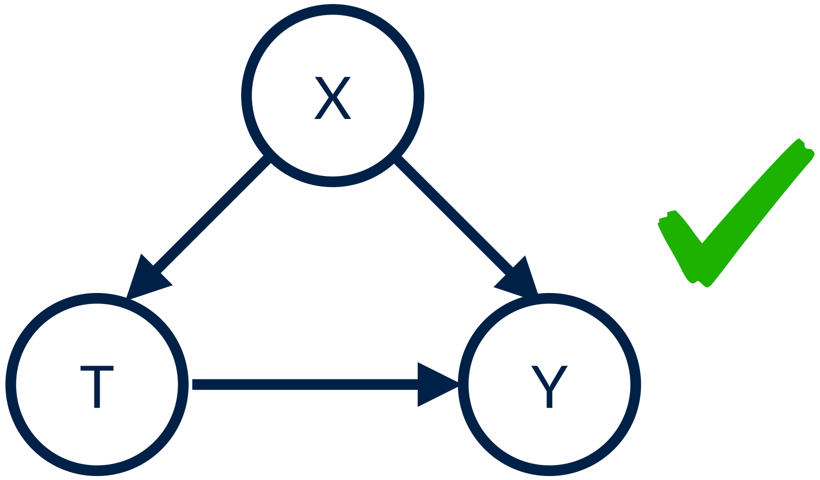
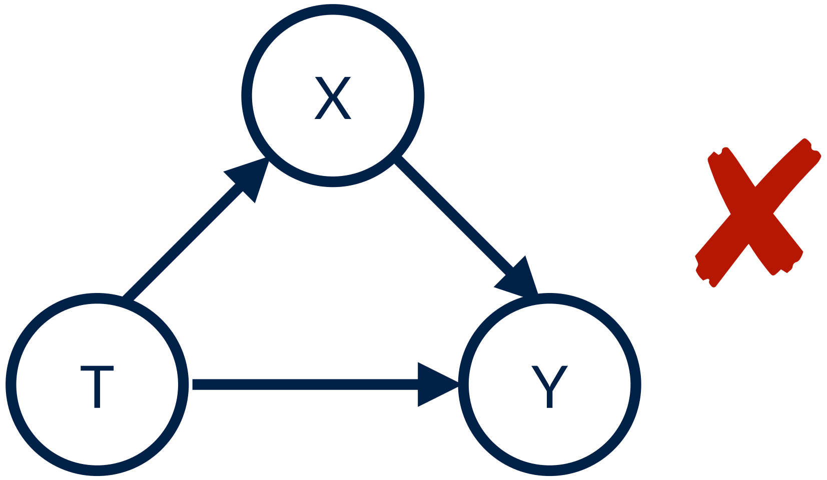




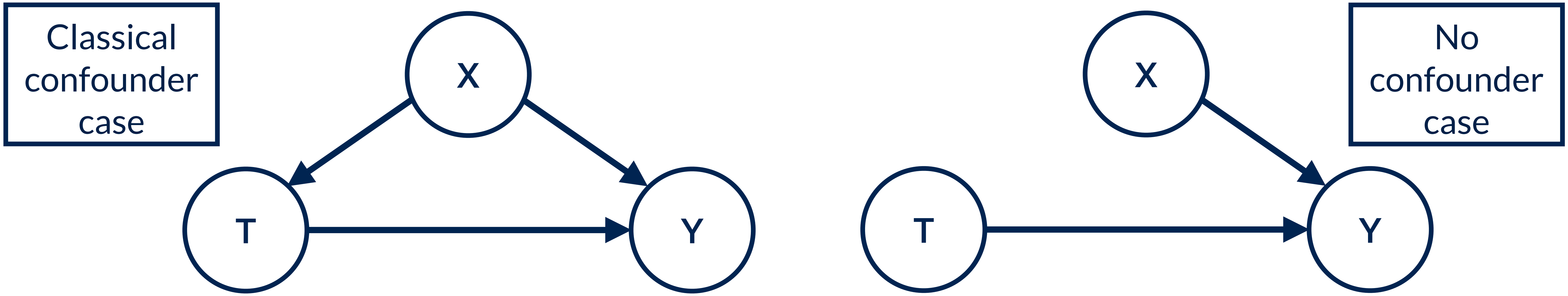
The Causal Effect Rule: Given a graph G in which a set of variables PA are designated as the parents of T, the causal effect of T on Y is given by:

$$\boxed{p(Y=y|do(T=t)) = \sum_x p(Y=y|T=t, PA=X)p(PA=X)}$$

Conclusion: The set of **parents of T** is always an adjustment set for the causal effect of T on Y, i.e., to identify

Pearl, Causal Inference in Statistics (2016)

Confounder vs not a confounder



Classical confounder case

No confounder case

$$
\begin{aligned}
\mathbb{E}_X\left[\mathbb{E}_Y\left[Y|X,T\right]\right] &= \int dx\ p(x) \int dy\ y\ p(y|x,t) \\
&= \int dx\ p(x) \int dy\ y\ \frac{p(y,x|t)}{p(x|t)} \\
&= \int dx\ p(x) \int dy\ y\ \frac{p(y,x|t)}{p(x)} \\
&= \int dy\ y\ p(y|t) = \mathbb{E}_Y\left[Y|T\right],
\end{aligned}
$$

Independence of X and W on the RHS graph

Optimal adjustment sets

Question: Suppose we identify multiple adjustment sets, which do we choose?

Idea: We aim to estimate a causal effect, e.g., $p(Y = y|do(X) = x)$ but we do so from *observational data*. Thus, there will be some error due to finite data and the *smaller* this error, the *better* our estimate of the causal effect.

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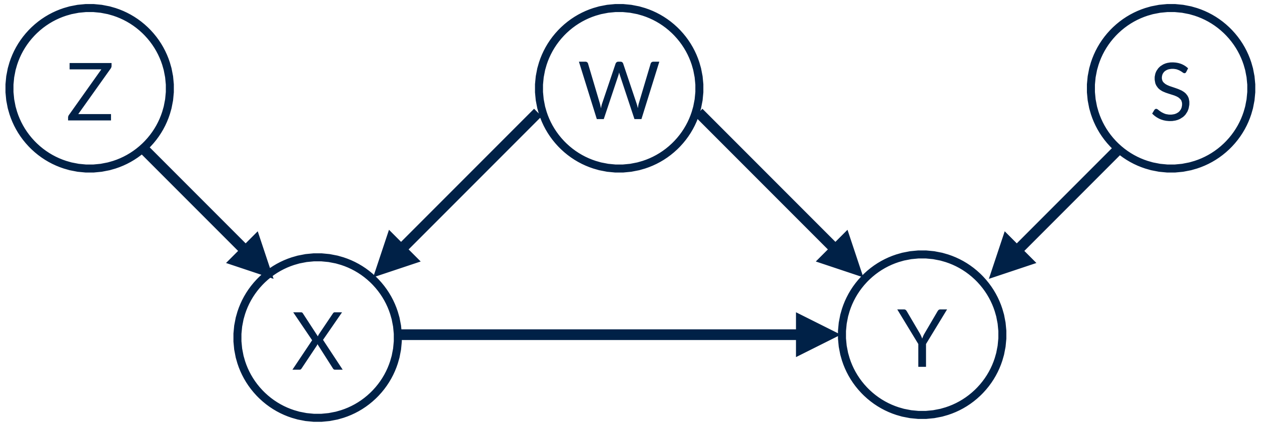
Initial guess: The more variables we condition on, the harder it is to estimate a conditional probability or conditional expectation value …

… so the *smallest* adjustment set should be the *optimal* adjustment set!

Adjustment sets:
{W}, {W, Z}, {W, S}, and {W, S, Z}



… so it should be {W}?

Simulation

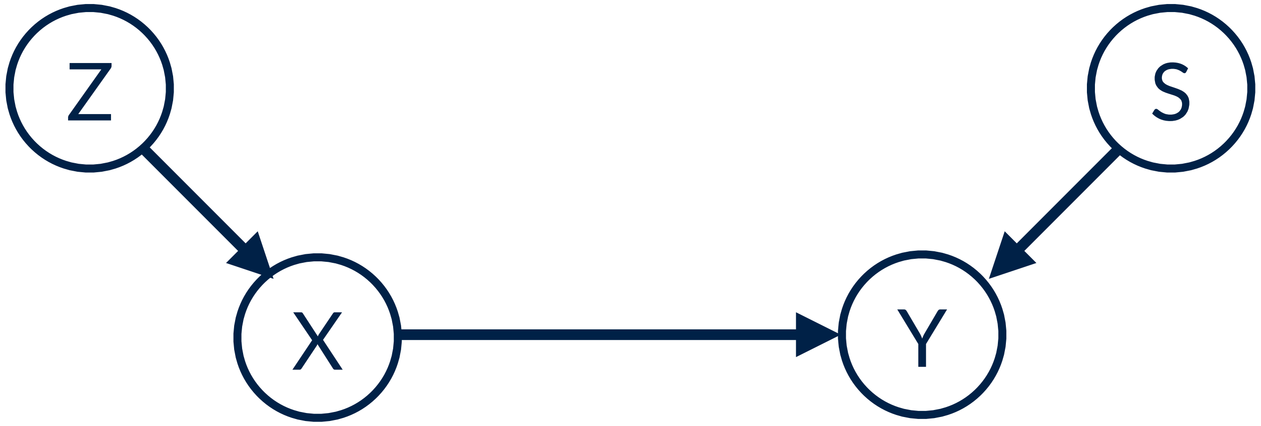
W is a confounder, so always needs to be adjusted for

For simplicity, we simulate a linear model from the causal graph below, and consider different noise on source (τ^2), and noise on target (σ^2)

$$\begin{cases} X \sim \mathcal{N}(0, \tau^2) \\ Y = X + \epsilon, \qquad \text{with } \epsilon \sim \mathcal{N}(0, \sigma^2) \end{cases}$$

Idea: Lowering/Increasing noise on X corresponds to conditioning on Z

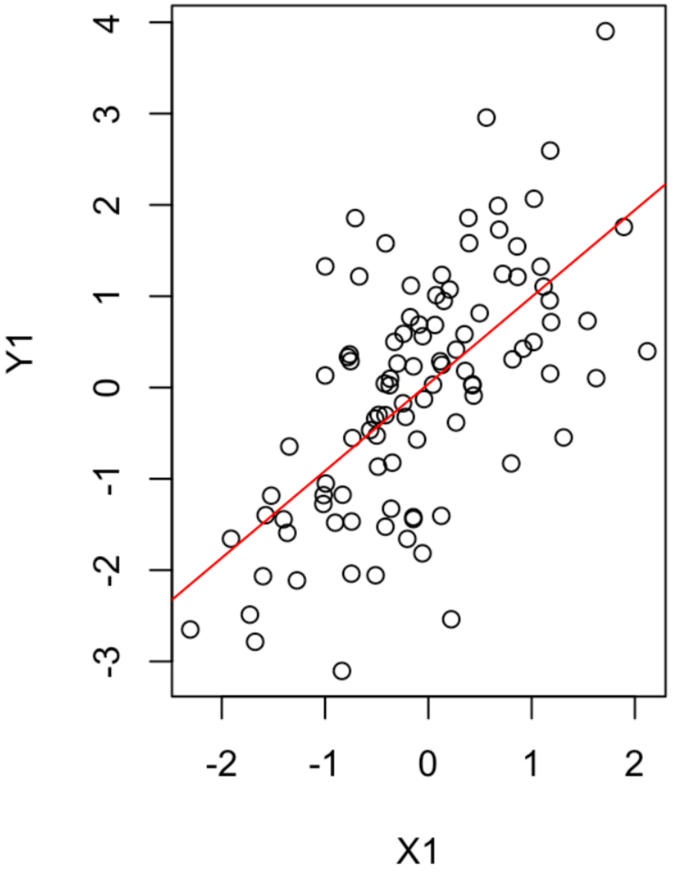
1. Conditioning on Z reduces variance in X
2. Conditioning on S reduces variance in Y



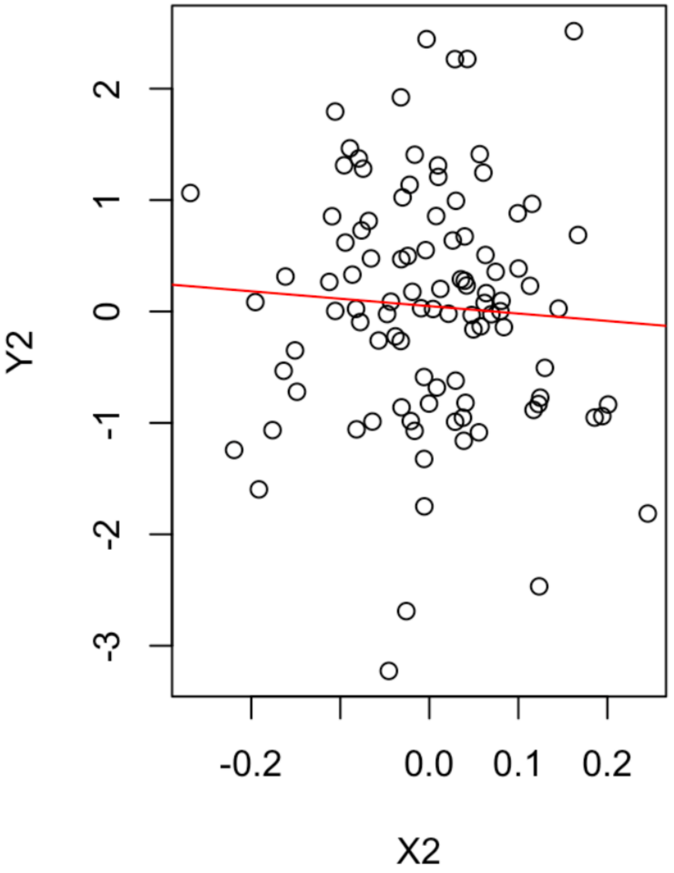
Simulation: Lower noise on source X (**condition on Z**)

For simplicity, we simulate a linear model from the causal graph below, and consider different noise on source (τ^2), and noise on target (σ^2)

$$\begin{cases} X \sim \mathcal{N}(0, \tau^2) \\ Y = X + \epsilon, \qquad \text{with } \epsilon \sim \mathcal{N}(0, \sigma^2) \end{cases}$$



```
##                Estimate Std. Error
## (Intercept)        0.04       0.10
## X1                 0.95       0.12
```

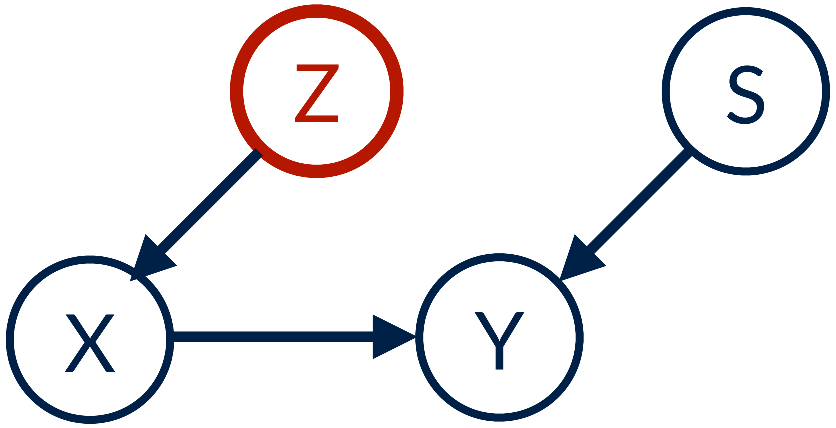


```
##                Estimate Std. Error
## (Intercept)       0.048       0.11
## X2               -0.670       1.10
```

Parameters (n=100):

$\tau_1 = 1$ vs $\tau_2 = 0.1$

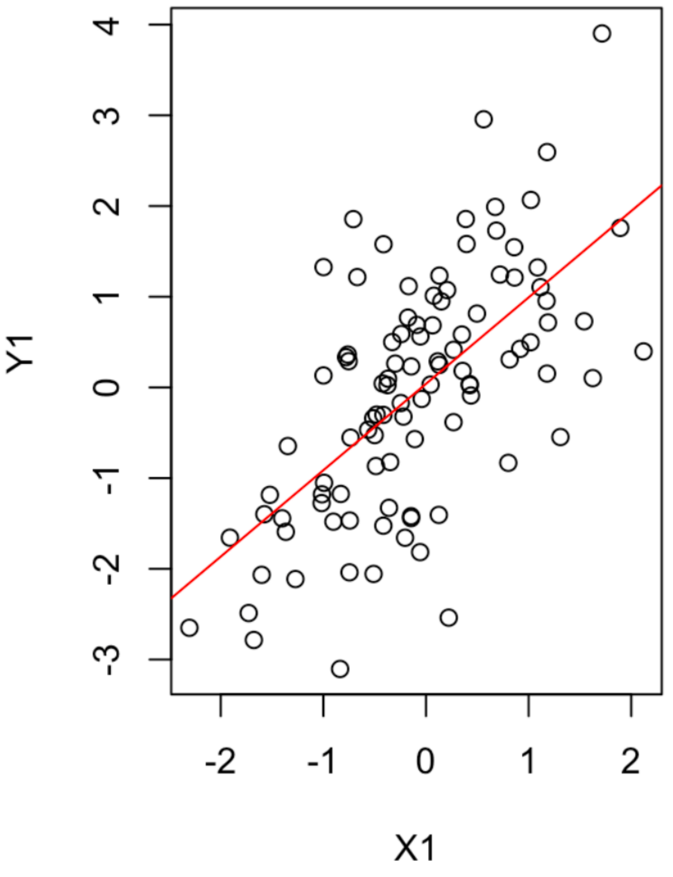
$\sigma_1 = 1$ vs $\sigma_2 = 1$



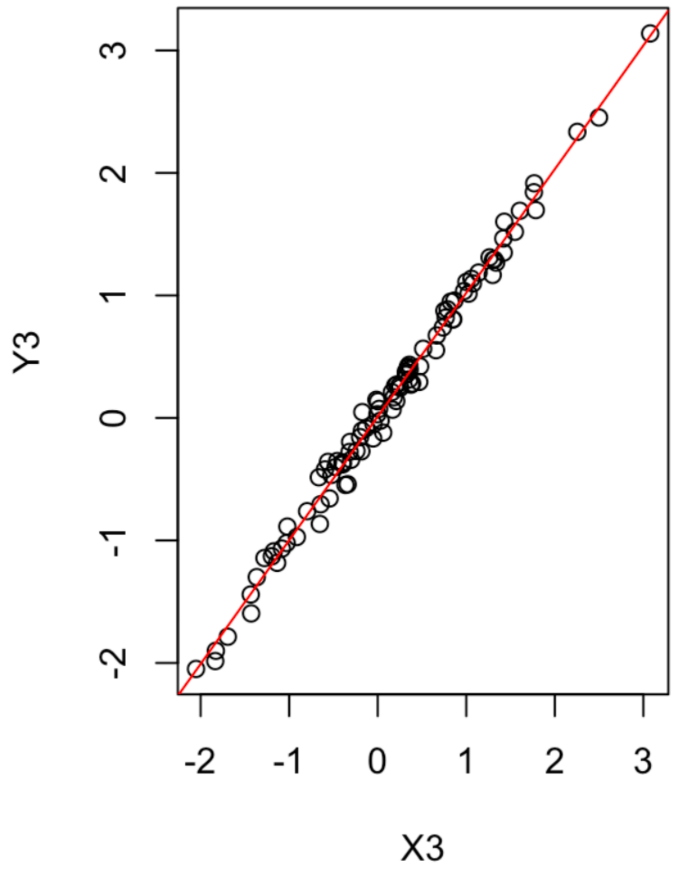
Simulation: Lower noise on target Y (**condition on S**)

For simplicity, we simulate a linear model from the causal graph below, and consider different noise on source (τ^2), and noise on target (σ^2)

$$\begin{cases} X \sim \mathcal{N}(0, \tau^2) \\ Y = X + \epsilon, \qquad \text{with } \epsilon \sim \mathcal{N}(0, \sigma^2) \end{cases}$$



```
##                Estimate Std. Error
## (Intercept)        0.04       0.10
## X1                 0.95       0.12
```

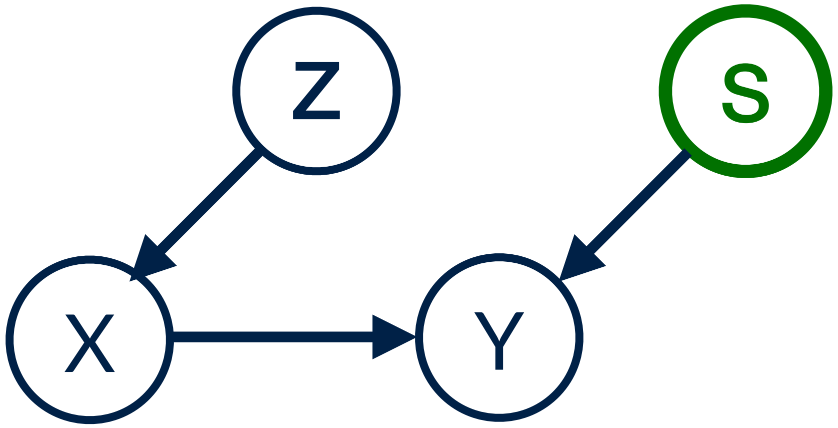


```
##              Estimate Std. Error
## (Intercept)     0.011     0.0095
## X3              1.000     0.0096
```

Parameters (n=100):

$\tau_1 = 1 \text{ vs } \tau_2 = 1$

$\sigma_1 = 1 \text{ vs } \sigma_2 = 0.1$



Optimal adjustment sets

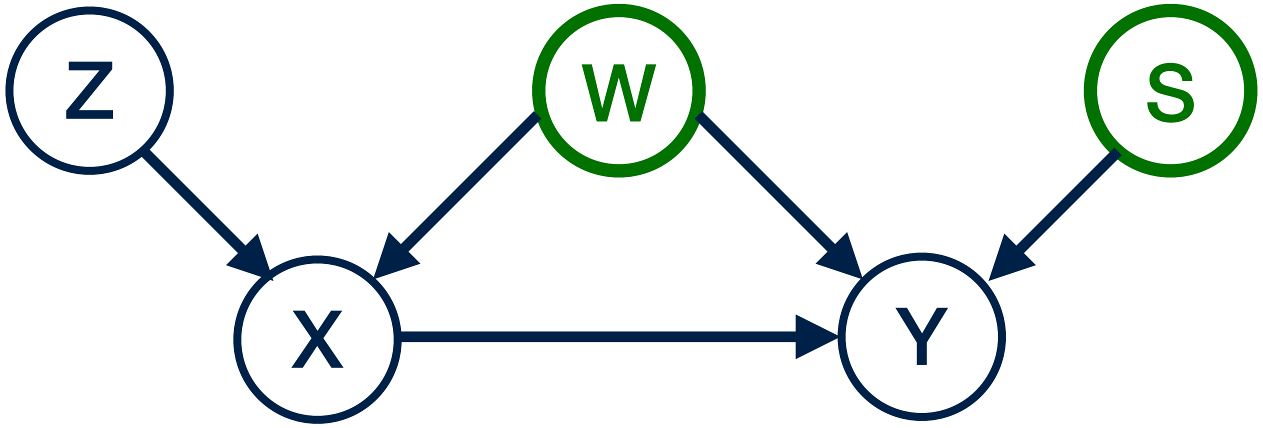
Question: Suppose we identify multiple adjustment sets, which do we choose?

Idea: We aim to estimate a causal effect, e.g., $p(Y = y | do(X) = x)$ but we do so from *observational data*. Thus, there will be some error due to finite data and the *smaller* this error, the *better* our estimate of the causal effect.

Initial guess: The more variables we condition on, the harder it is to estimate a conditional probability or conditional expectation value …

… so the *smallest* ✗ adjustment set should be the *optimal* adjustment set!

Adjustment sets:
{W}, {W, Z}, {W, S}, and {W, S, Z}



Optimal set is {W,S}!!!

Optimal adjustment sets

Theorem (Rotnitzky and Smucler, 2020)

The most efficient adjustment set to use for the effect of X on Y is

$$\mathrm{pa}_G\big(\mathrm{cn}_G(X \to Y)\big) \setminus \big(\mathrm{cn}_G(X \to Y) \cup \{X\}\big)$$

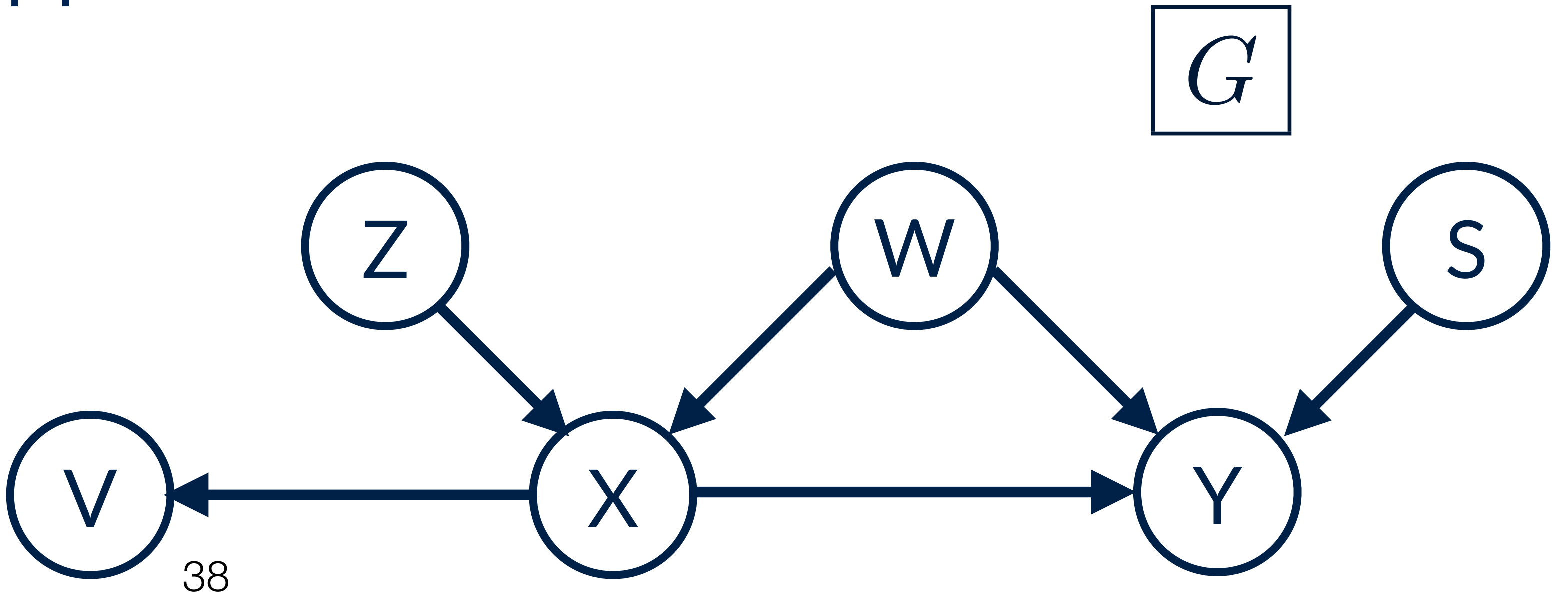
where $\mathrm{cn}_G(X \to Y)$ are all the nodes on a causal (i.e. directed) path from X to Y, but excluding X itself. (So parents of this set ***not*** on the causal path.)

Example

Here $\mathrm{cn}_G(X \to Y)$ consists only of Y

The parents of Y are X, W, and S

Thus, by the above the optimal adjustment set is {W,S}



Optimal adjustment sets

Theorem (Rotnitzky and Smucler, 2020)

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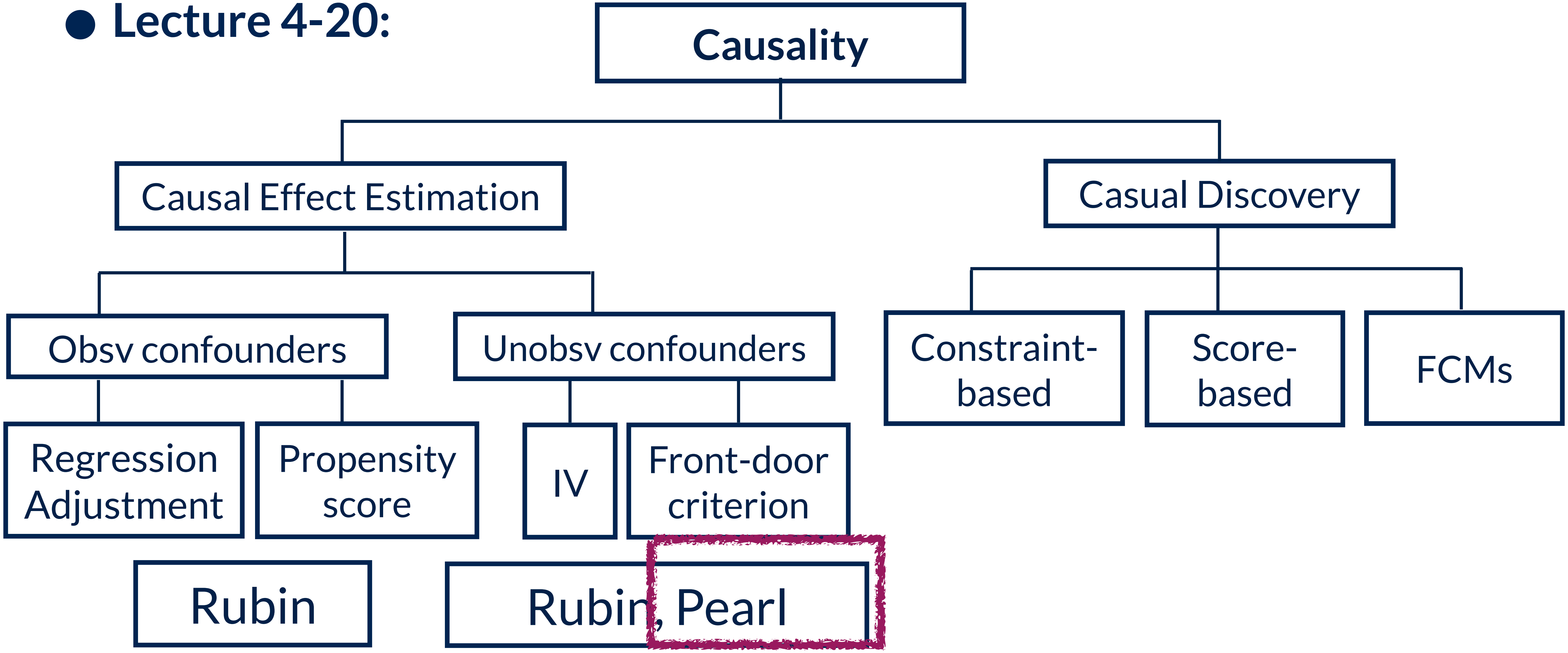
Remarks

1. Optimal set adjusts for some unnecessary variables (here, S) since these are not actually confounders
2. Optimal set does **not control for instruments** (here, Z)

The key quantity to keep as small as possible for optimality is $\dfrac{\text{variance in } Y}{\text{variance in } X}$

Overview of the course

- **Lecture 1**: Introduction & Motivation, why do we care about causality? Why deriving causality from observational data is non-trivial.
- **Lecture 2:** Recap of probability theory, variables, events, conditional probabilities, independence, law of total probability, Bayes' rule
- **Lecture 3**: Recap of regression, multiple regression, graphs, SCM
- **Lecture 4-20:**


Causality
Causal Effect Estimation
Casual Discovery
Obsv confounders
Unobsv confounders
Constraint-based
Score-based
FCMs
Regression Adjustment
Propensity score
IV
Front-door criterion
Rubin
Rubin, Pearl